On Sharp und Diffuse Interface Models for Viscous Two-Phase Flows

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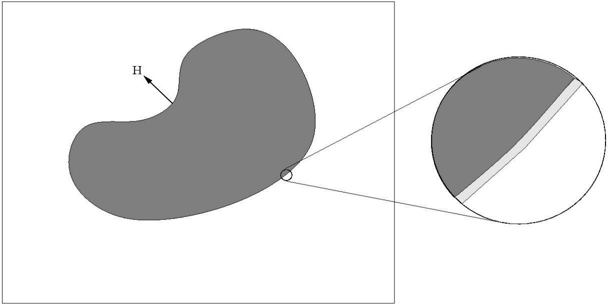
Modeling (I)

We consider two (macroscopically) immiscible incompressible, viscous fluids like oil and water.

Classical Models: Interface is a two-dimensional surface.

Surface tension is proportional to the mean curvature.



But: Sharp interface is an idealization (van der Waals).

Fluid mix in a thin interfacial region.

Overview

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Free Energy of a Two-Component Mixture

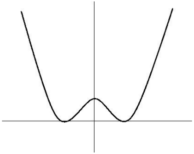
We consider a binary mixture e.g. Al-/Ni alloy, water and oil, polymeric mixture, ...

Let $c_j \colon \Omega \to \mathbb{R}$ be the concentration of the component $j = 1, 2$, $c = c_1 - c_2$, and let

$$E_\varepsilon(c) = \frac{\varepsilon}{2} \int_\Omega |\nabla c(x)|^2 \, dx + \varepsilon^{-1} \int_\Omega f(c(x)) \, dx$$

be the free energy of the mixture, where $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, $\varepsilon > 0$ and

$$f \colon \mathbb{R} \to [0, \infty) \text{ with } f(c) = 0 \Leftrightarrow c = \pm 1.$$



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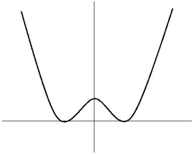

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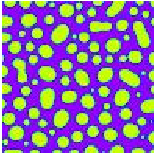
$$f : \mathbb{R} \to [0, \infty) \text{ with } f(c) = 0 \Leftrightarrow c = \pm 1.$$

Moreover, we assume

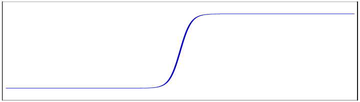
$$\frac{1}{|\Omega|} \int_\Omega c(x) \, dx = \overline{c} \in (-1, 1) \qquad \text{if } |\Omega| < \infty.$$



Remarks

- A "typical" profile of a diffuse interface is

$$c(x) = \tanh \frac{x}{2\varepsilon}, \qquad x \in \mathbb{R},$$



which minimizes E_ε in the case $\Omega = \mathbb{R}$ with constraint $c(x) \to_{x\to\pm\infty} \pm 1$.

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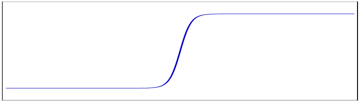

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- Modica-Mortola '77, Modica '87 proved

$$E_\varepsilon \to_{\varepsilon\to 0} \sigma P$$

in the sense of Γ-convergence (w.r.t. L^1), where

$$P(v) = \begin{cases} \mathcal{H}^{d-1}(\partial^* E) = \text{"area}(\partial E)\text{"} & \text{if } v = 2\chi_E - 1 \\ +\infty & \text{else.} \end{cases}$$

and $\sigma = \sigma(f)$.

Cahn-Hilliard Equation (I)

Let $J\colon \Omega \times (0,\infty) \to \mathbb{R}^d$ be the mass flux, i.e.

$$\frac{d}{dt}\int_V c(x,t)\,dx = -\int_{\partial V} n\cdot J(x,t)\,d\sigma(x) = -\int_V \operatorname{div} J(x,t)\,dx$$

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Assumption (Cahn-Hilliard '58): For some $m(c) > 0$ we have

$$\begin{aligned} J &= -m(c)\nabla\mu && \text{(generalized Fick's law)} \\ \mu &= \frac{\delta E_\varepsilon}{\delta c} = -\varepsilon\Delta c + \varepsilon^{-1} f'(c) && \text{(chemical potential)} \end{aligned}$$

Remark: $\mu = \frac{\delta E_\varepsilon}{\delta c} \equiv const. \Leftrightarrow J \equiv 0$

Cahn-Hilliard Equation (II)

We consider

$$\partial_t c = \operatorname{div}(m(c)\nabla\mu) \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$\mu = -\varepsilon\Delta c + \varepsilon^{-1} f'(c) \quad \text{in } \Omega \times (0,\infty) \tag{2}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$ together with

$$\mathbf{n} \cdot \nabla c|_{\partial\Omega} = \mathbf{n} \cdot m(c)\nabla\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{3}$$

$$c\big|_{t=0} = c_0 \quad \text{in } \Omega. \tag{4}$$

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Remark: For every smooth solution we have:

$$\frac{d}{dt}E_\varepsilon(c(t)) = -\int_\Omega m(c(t,x))|\nabla\mu(t,x)|^2\,dx.$$

Questions:

- Does a unique solution $c(t,x)$ exist for all $t>0$?
- Does $c(t,x)$ converge as $t\to\infty$ to a critical point of E_ε?

Well-Posedness and Convergence

If $f(c)$ is smooth, $m(c) \equiv const.$:
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One solution: Use a singular free energy density as e.g.

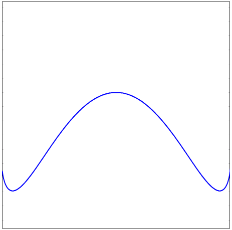
$$f(c) = \theta((1-c)\log(1-c) + (1+c)\log(1+c)) - \theta_c c^2,\ c \in [-1,1],$$

with $0 < \theta < \theta_c$, cf. Cahn & Hilliard '58.

Existence: Elliott & Luckhaus '91,
Debussche & Dettori '95, Kenmochi et al. '95

Convergence: A. & Wilke '07

Remark:
For every solution $c(t,x) \in (-1,1)$ a.e.



Other results: Existence of weak solutions for degenerate mobility (Elliott & Garcke '96) and double obstacle potential (Blowey & Elliott '91)

Overview

Monotone Operators and Subgradients (I)

Let H be a real Hilbert space with inner product $(.,.)_H$.

Definition

$\mathcal{A}\colon \mathcal{D}(A) \subseteq H \to H$ is monotone if

$$(\mathcal{A}(x) - \mathcal{A}(y), x - y)_H \geq 0 \qquad \text{for all } x, y \in \mathcal{D}(A).$$

Remark: If $E\colon H \to \mathbb{R}$ is differentiable and convex, then $DE\colon H \to H$ is monotone.

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Proof: Consider $f(t) = E(tx + (1-t)y), t \in [0,1]$.

Then $f\colon [0,1] \to \mathbb{R}$ is convex, $f'\colon [0,1] \to \mathbb{R}$ is non-decreasing and

$$f'(t) = (DE(tx + (1-t)y), x - y)_H.$$

Hence

$$f'(1) \geq f'(0) \quad \Leftrightarrow \quad (DE(x) - DE(y), x - y)_H \geq 0$$

Monotone Operators and Subgradients (II)

Definition

Let $E\colon H \to \mathbb{R} \cup \{+\infty\}$ be convex. Then the subgradient $\partial_H E\colon H \to \mathcal{P}(H)$ of E is defined by

$$w \in \partial_H E(x) \;\Leftrightarrow\; E(y) \geq E(x) + (w, y - x)_H \qquad \text{for all } y \in H.$$

Remark: $\partial_H E\colon H \to \mathcal{P}(H)$ is a multi-valued monotone operator, i.e.,

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Application: In the following let

$$E_0(c) = \frac{\varepsilon}{2}\int_\Omega |\nabla c(x)|^2\,dx + \varepsilon^{-1}\int_\Omega f_0(c(x))\,dx$$

with $f_0(c) = \theta((1-c)\log(1-c) + (1+c)\log(1+c))$ be the "convex part" of the free energy $E_\varepsilon(c)$ and

$$H \equiv L^2_{(0)}(\Omega) = \left\{u \in L^2(\Omega) : \int_\Omega u(x)\,dx = 0\right\}$$

Subgradient of the Free Energy

Let P_0 be the orthogonal projection of $L^2(\Omega)$ onto $L^2_{(0)}(\Omega) =: H$.

Theorem (A., Wilke '07)

$$\partial_{L^2_{(0)}} E_0(c) = \begin{cases} \left\{-\varepsilon\Delta c + \varepsilon^{-1} P_0 f_0'(c)\right\} & \textit{if } c \in \mathcal{D}(\partial_{L^2_{(0)}} E_0), \\ \emptyset & \textit{else} \end{cases}$$

where

$$\mathcal{D}(\partial_{L^2_{(0)}} E_0) = \left\{ c \in L^2_{(0)}(\Omega) : \nabla^2 c, f_0'(c) \in L^2(\Omega), n \cdot \nabla c|_{\partial\Omega} = 0 \right\}.$$

Moreover, we have for every $c \in \mathcal{D}(\partial_{L^2_{(0)}} E_0)$*:*

$$\|\nabla^2 c\|_{L^2(\Omega)} + \|f_0'(c)\|_{L^2(\Omega)} \leq C \left(\|\partial_{L^2_{(0)}} E_0(c)\|_{L^2(\Omega)} + 1 \right)$$

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$\Rightarrow$ $-\Delta c + P_0 f_0'(c)$ is a (maximal) monotone operator.

$\Rightarrow$ Existence of solutions of the Cahn-Hilliard equation from general theory.

Sketch of the Proof

Formal Proof: Let $c_s(x) = c(x) + s\, f_0'(c(x))$, $s > 0$.

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$$\begin{aligned}
(\partial_{L^2_{(0)}} E_0(c), f_0'(c))_{L^2(\Omega)} &= \frac{d}{ds} E_0(c_s)\Big|_{s=0} \\
&= \int_\Omega \nabla c \cdot \nabla (f_0'(c))\, dx + \int_\Omega f_0'(c) f_0'(c)\, dx \\
&= \int_\Omega \underbrace{f_0''(c)|\nabla c|^2}_{\geq 0}\, dx + \int_\Omega f_0'(c)^2\, dx
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Hence

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Hence

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\|f_0'(c)\|_{L^2(\Omega)} \leq \|\partial_{L^2_{(0)}} E_0(c)\|_{L^2(\Omega)}.
$$

To justify formal calculation:

- Approximate f_0 by non-singular $f_m \colon \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$.
- Correct mean value of c_s suitably to obtain $c_s \in L^2_{(0)}(\Omega)$.

Subgradient of the Convex Part of the Energy (II)

Now we consider E_0 as functional on $H^{-1}_{(0)}(\Omega) = (H^1(\Omega) \cap L^2_{(0)}(\Omega))'$ by setting $E_0(c) = +\infty$ if $c \notin \mathrm{dom}(E_0) \subset L^2_{(0)}(\Omega)$.

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Corollary

$\partial_{H^{-1}_{(0)}} E_0$ is a (maximal) monotone operator on $H^{-1}_{(0)}(\Omega)$ and $\partial_{H^{-1}_{(0)}} E_0 = -\Delta_N \partial_{L^2_{(0)}} E_0$. Moreover,

$$\mathcal{D}(\partial_{H^{-1}_{(0)}} E_0) = \left\{ c \in \mathcal{D}(\partial_{L^2} E_0) : \partial_{L^2_{(0)}} E_0(c) \in H^1(\Omega) \right\}$$

Here $\Delta_N \colon H^1(\Omega) \cap L^2_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ is defined by

$$\langle -\Delta_N u, \varphi \rangle_{H^{-1},H^1} = (\nabla u, \nabla \varphi)_{L^2(\Omega)}, \quad \varphi \in H^1(\Omega) \cap L^2_{(0)}(\Omega).$$

Overview

Cahn-Hilliard Equation with Singular Free energies

We consider

$$\partial_t c = \operatorname{div}(m\nabla\mu) \quad \text{in } \Omega\times(0,\infty), \tag{5}$$

$$\mu = -\varepsilon\Delta c + \varepsilon^{-1} f'(c) \quad \text{in } \Omega\times(0,\infty) \tag{6}$$

with the initial and boundary conditions

$$\mathbf{n}\cdot\nabla c|_{\partial\Omega} = \mathbf{n}\cdot m\nabla\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{7}$$

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where f_0 is convex.

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$$f(c) = f_0(c) - \frac{\theta_c}{2} c^2,$$

where f_0 is convex. Then (5)-(6) are equivalent to

$$\partial_t c \underbrace{-\Delta(-\Delta c + f_0'(c))}_{\text{monotone operator}} = \underbrace{-\theta_c \Delta c}_{\text{“Lipschitz perturbation”}}$$

Existence of solutions:

General result on perturbations of (maximal) monotone operators.

Lipschitz Perturbations of Subgradients

Let H_j be Hilbert spaces such that $H_1 \hookrightarrow H_0$ densely. We consider

$$\frac{du}{dt}(t) + \partial_{H_0}\varphi(u(t)) \ni \mathcal{B}(u(t)) + g(t), \quad t \in (0, T), \tag{9}$$

$$u(0) = u_0. \tag{10}$$

and assume that $\mathcal{B}\colon H_1 \to H_0$ is globally Lipschitz continuous.
(In our case: $\mathcal{B} = -\frac{m}{\varepsilon}\theta_c\Delta$, $H_0 = H^{-1}_{(0)}(\Omega)$, $H_1 = H^1_{(0)}(\Omega)$.)

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Theorem (A./Wilke '07)

Let $\varphi = \varphi_1 + \varphi_2$ be a proper, l.s.c., convex functional such that

- $\varphi_2 \geq 0$ *is convex,*
- $\operatorname{dom}\varphi_1 = H_1$ *and* $\varphi_1|_{H_1}$ *is a bounded, coercive, quadratic form on* H_1.

Then for every $g \in L^2(0, T; H_0)$, $u_0 \in \operatorname{dom}(\varphi)$ *there is a unique solution* $u \in W^1_2(0, T; H_0) \cap L^\infty(0, T; H_1)$ *of (9)-(10). Moreover,* $\varphi(u) \in L^\infty(0, T)$.

Main Existence Result for Cahn-Hilliard Equation

Theorem (A./Wilke '07)

For every $c_0 \in H^1(\Omega)$ *with* $E_\varepsilon(c_0) < \infty$ *there is a unique solution* $c \in L^\infty(0,\infty; H^1(\Omega)) \cap L^2(0,\infty; H^2(\Omega))$ *of (5)-(8) with* $\partial_t c \in L^2(0,\infty; H^{-1}_{(0)}(\Omega))$, $f'(c) \in L^2((0,\infty)\times\Omega)$, $\mu \in L^2_{loc}([0,\infty); H^1(\Omega))$, *satisfying*

$$E_\varepsilon(c(T)) + \int_0^T \|\nabla\mu(t)\|^2_{L^2(\Omega)}\, dt = E_\varepsilon(c_0)$$

for all $T > 0$.

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$$E_\varepsilon(c(T)) + \int_0^T \|\nabla\mu(t)\|^2_{L^2(\Omega)}\,dt = E_\varepsilon(c_0)$$

for all $T > 0$. Furthermore, for $\delta > 0$

$$\begin{aligned} c &\in L^\infty(\delta,\infty; H^2(\Omega)), f'(c) \in L^\infty(\delta,\infty; L^2(\Omega)),\\ \mu &\in L^\infty(\delta,\infty; H^1(\Omega)),\\ \partial_t c &\in L^\infty(\delta,\infty; H^{-1}_{(0)}(\Omega)) \cap L^2(\delta,\infty; H^1(\Omega)). \end{aligned}$$

Remark: If additionally $c_0 \in \mathcal{D}(\partial E)$, then the last statement holds with $\delta = 0$.

Overview

1. Phase Separation and Cahn-Hilliard Equation
   - Free Energy and the Cahn-Hilliard Equation
   - Monotone Operators and Subgradients
   - Analysis of the Cahn-Hilliard Equation with Singular Free Energies
   - Asymptotic Behavior for Large Times
2. Model H – Diffuse Interface Model for Matched Densities
   - Basic Modeling and First Properties
   - Well-Posedness of Model H
   - Cahn-Hilliard Equation with Convection
   - Stokes Equation with Variable Viscosity
3. Diffuse Interface Models for Non-Matched Densities
   - A Model by Lowengrub and Truskinovsky
   - Modified Model H
4. Sharp Interface Limits and Analysis of a Limit Model
   - Sharp Interface Limit for the Cahn-Hilliard Equation
   - Sharp Interface Limit for Modified Model H
   - Analysis of the Navier-Stokes/Mullins-Sekerka System

Convergence to Stationary Solutions (I)

Theorem (A./Wilke '07)

Let f be analytic in $(-1,1)$. Then

$$\lim_{t\to\infty} c(t) = c_\infty \qquad \text{in } H^{2r}(\Omega), r \in (0,1),$$

for some $c_\infty \in H^2(\Omega)$ with $\overline{c_\infty(\Omega)} \subset (-1,1)$ solving the stationary system

$$-\Delta c_\infty + f'(c_\infty) = const. \qquad \text{in } \Omega, \tag{11}$$

$$\partial_\nu c_\infty|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega. \tag{12}$$

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Main ingredients:

- $c(t,x) \in [-1+\varepsilon, 1-\varepsilon]$ for all $t \geq T_1, x \in \Omega$ and some $T_1, \varepsilon > 0$.
- For $t > T_1$ replace f by smooth $\tilde{f}$ with $\tilde{f}|_{[-1+\varepsilon,1-\varepsilon]} = f|_{[-1+\varepsilon,1-\varepsilon]}$. Apply the Lojasiewicz-Simon inequality to the modified E.

Convergence to Stationary Solutions (II)

The proof is based on the Lojasiewicz-Simon gradient inequality:

$$|E_\varepsilon(c) - E_\varepsilon(c_\infty)|^{1-\theta} \leq C \|DE_\varepsilon(c)\|_{H^{-1}_{(0)}}, \quad \theta \in (0, \tfrac{1}{2}] \qquad \text{(LS)}$$

for c in a neighborhood of a critical point c_∞.

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$$\begin{aligned} H(t) &:= \left(E_\varepsilon(c(t)) - E_\varepsilon(c_\infty)\right)^\theta, \\ \Rightarrow -\frac{d}{dt}H(t) &= \theta \frac{\|\nabla\mu(t)\|_{L^2}^2}{\left(E_\varepsilon(c(t)) - E_\varepsilon(c_\infty)\right)^{\theta-1}} \\ &\overset{\text{(LS)}}{\geq} C\frac{\|\nabla\mu(t)\|_{L^2}^2}{\|DE_\varepsilon(c(t))\|_{H^{-1}_{(0)}}} \geq \|\nabla\mu(t)\|_{L^2} \end{aligned}$$

since $\frac{d}{dt}E_\varepsilon(t) = -\|\nabla\mu(t)\|_{L^2(\Omega)}^2$ and $\|DE_\varepsilon(c(t))\|_{H^{-1}_{(0)}} \leq C\|\nabla\mu(t)\|_{L^2}$.

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$$\begin{aligned} H(t) &:= \left(E_\varepsilon(c(t)) - E_\varepsilon(c_\infty)\right)^\theta, \\ \Rightarrow -\frac{d}{dt}H(t) &= \theta \frac{\|\nabla\mu(t)\|^2_{L^2}}{\left(E_\varepsilon(c(t)) - E_\varepsilon(c_\infty)\right)^{\theta-1}} \\ &\overset{\text{(LS)}}{\geq} C\frac{\|\nabla\mu(t)\|^2_{L^2}}{\|DE_\varepsilon(c(t))\|_{H^{-1}_{(0)}}} \geq \|\nabla\mu(t)\|_{L^2} \end{aligned}$$

since $\frac{d}{dt}E_\varepsilon(t) = -\|\nabla\mu(t)\|^2_{L^2(\Omega)}$ and $\|DE_\varepsilon(c(t))\|_{H^{-1}_{(0)}} \leq C\|\nabla\mu(t)\|_{L^2}$.
Hence

$$\int_0^\infty \|\partial_t c(t)\|_{H^{-1}_{(0)}}\, dt \leq C\int_0^\infty \|\nabla\mu(t)\|_2\, dt \leq C'\left(E_\varepsilon(c_0) - E_\varepsilon(c_\infty)\right)^{1-\theta}$$

$\Rightarrow \lim_{t\to\infty} c(t) = c_0 + \int_0^\infty \partial_t c(\tau)\, d\tau$ exists.

Coarse Graining/Ostwald Ripening

Question: What is the asymptotic behavior of $c(t)$ as $t \to \infty$?

Sternberg & Zumbrun '98: For every stable critical point of Ω the diffuse interface is connected.

This is related to the effect of Ostwald ripening.

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Simulation by S. Bartels

Overview

Basic Modeling (I)

Idea: Sharp interface is an idealization. (Korteweg/van der Waals)

Therefore: Introduce an interfacial region, where both fluids mix.

Moreover: Take diffusion effects of particles into account.

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Therefore: Introduce an interfacial region, where both fluids mix.

Moreover: Take diffusion effects of particles into account.

Ansatz: Let c be the concentration difference of both fluids.
Assume that the interfacial energy is given by

$$E_\varepsilon(c) = \frac{\varepsilon}{2}\int_\Omega |\nabla c(x)|^2\,dx + \varepsilon^{-1}\int_\Omega f(c(x))\,dx,$$

where the free energy density f is a suitable double well potential.

Diffusion: Assume that

$$\begin{aligned} \partial_t c + \mathbf{v}\cdot\nabla c &= \operatorname{div} J \\ J &= m\nabla\mu && \text{(Fick's law)} \\ \mu := \frac{\delta E_\varepsilon}{\delta c} &= -\varepsilon\Delta c + \varepsilon^{-1} f(c) && \text{(chemical potential)} \end{aligned}$$

Classical models: Pure transport of the interface (m=0).

Basic Modeling (II)

Conservation of mass and momentum yield

$$\begin{aligned} \rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{T}(c,\mathbf{v},p) &= 0 \\ \rho_t + \operatorname{div}(\rho\mathbf{v}) &= 0 \end{aligned}$$

where $\mathbf{T}(c,\mathbf{v},p)$ is the stress tensor to be specified later.
Assumption $\rho(c) \equiv const.(=1)$. Hence $\operatorname{div}\mathbf{v} = 0$.

Basic Modeling (II)

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where $\mathbf{T}(c,\mathbf{v},p)$ is the stress tensor to be specified later.
Assumption $\rho(c) \equiv const.(= 1)$. Hence $\operatorname{div}\mathbf{v} = 0$.

The kinetic energy is given by

$$
E_{\mathrm{kin}}(\mathbf{v}) = \frac{1}{2}\int_\Omega |\mathbf{v}(x)|^2\,dx
$$

and the total energy of the system is

$$
E(c,\mathbf{v}) = E_\varepsilon(c) + E_{\mathrm{kin}}(\mathbf{v}).
$$

Energy Dissipation

$$
\begin{aligned}
&\frac{d}{dt}E(c(t),\mathbf{v}(t))\\
&= -\int_\Omega \mathbf{T}(c,\nabla c,D\mathbf{v},p) : D\mathbf{v}\,dx - \int_\Omega m|\nabla\mu|^2\,dx - \int_\Omega \mu\nabla c\cdot\mathbf{v}\,dx\\
&= -\int_\Omega (\mathbf{S}(c,\nabla c,D\mathbf{v}) + \varepsilon\nabla c\otimes\nabla c) : D\mathbf{v}\,dx - \int_\Omega m|\nabla\mu|^2\,dx
\end{aligned}
$$

where $\mathbf{T}(c,\nabla c,D\mathbf{v},p) = \mathbf{S}(c,\nabla c,Dv) - p\mathbf{I}$ and

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$$\mu\nabla c = -\varepsilon\,\mathrm{div}(\nabla c\otimes\nabla c) + \nabla\left(\varepsilon^{-1}f(c) + \varepsilon\frac{|\nabla c|^2}{2}\right)$$

Constitutive Assumption:

$$\mathbf{S}(c,\nabla c, D\mathbf{v}) + \varepsilon\nabla c\otimes\nabla c = \nu(c)D\mathbf{v}$$

for some viscosity coefficient $\nu(c) > 0$.

$$\Rightarrow \frac{d}{dt}E(c(t),\mathbf{v}(t)) = -\int_\Omega \nu(c(t))|D\mathbf{v}(t)|^2\,dx - \int_\Omega m|\nabla\mu(t)|^2\,dx$$

Diffuse Interface Model in the Case of Matched Densities

We derived:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \underbrace{\operatorname{div}(\nu(c)D\mathbf{v})}_{\text{inner friction}} + \nabla p = \underbrace{-\varepsilon \operatorname{div}(\nabla c \otimes \nabla c)}_{\text{surface tension}} \tag{13}$$

$$\operatorname{div}\mathbf{v} = 0 \tag{14}$$

$$\partial_t c + \mathbf{v}\cdot\nabla c = m\Delta\mu \tag{15}$$

$$\mu = -\varepsilon\Delta c + \varepsilon^{-1} f'(c) \tag{16}$$

where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ together with

$$\mathbf{v}|_{\partial\Omega} = n\cdot\nabla c|_{\partial\Omega} = n\cdot\nabla\mu|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{17}$$

$$(\mathbf{v}, c)|_{t=0} = (\mathbf{v}_0, c_0) \quad \text{in } \Omega. \tag{18}$$

Derivation: Hohenberg & Halperin '74, Gurtin et al. '96
Analytical results:
Starovoitov '93, Boyer '03, X.Feng '06, Gal & Grasselli '09, A. '07/'09

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where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$
Remark: (13) can be replaced by:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}(\nu(c)D\mathbf{v}) + \nabla g = \mu\nabla c$$

where $g = p + \varepsilon^{-1} f(c) + \frac{\varepsilon}{2}|\nabla c|^2$. – Use (16) multiplied by ∇c and

$$-\varepsilon \operatorname{div}(\nabla c \otimes \nabla c) = -\varepsilon\Delta c\nabla c - \varepsilon\nabla\frac{|\nabla c|^2}{2}$$

Overview

Well-Posedness of Model H

Theorem (Existence, Regularity, Uniqueness, A. '07/'09)

Let $d = 2, 3$. For every $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ with $E_\varepsilon(c_0) < \infty$ there is a weak solution $(\mathbf{v}, c, \mu)$ of (13)-(16), which satisfies

$$(\mathbf{v}, \nabla c) \in L^\infty(0, \infty; L^2(\Omega)), \quad (\nabla \mathbf{v}, \nabla \mu) \in L^2(0, \infty; L^2(\Omega)),$$
$$\nabla^2 c, f'(c) \in L^2_{loc}([0, \infty); L^6(\Omega)).$$

Moreover, $c \in BUC([0, \infty); W^1_q(\Omega))$ with $q > d$. For $(\mathbf{v}_0, c_0)$ sufficiently smooth:

1. *If $d = 2$, then the weak solution is unique and regular.*
2. *If $d = 3$, there are some $0 < T_0 < T_1 < \infty$ such that the weak solution is regular and (locally) unique on $(0, T_0)$ and $[T_1, \infty)$.*
3. *There is a critical point c_∞ of E_ε s.t. $(\mathbf{v}(t), c(t)) \to_{t \to \infty} (0, c_\infty)$.*

Remark: Here $\varepsilon > 0$ and $m > 0$ are essential!

Structure of the Proof

First study the separate systems:

1. Cahn-Hilliard equation with convection and singular potential (based on $E_\varepsilon(c) = E_0(c) - \frac{\theta}{2}\|c\|_2^2$ with E_0 convex)
2. (Navier-)Stokes system with variable viscosity

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1. Cahn-Hilliard equation with convection and singular potential (based on $E_\varepsilon(c) = E_0(c) - \frac{\theta}{2}\|c\|_2^2$ with E_0 convex)
2. (Navier-)Stokes system with variable viscosity

Existence of weak solutions:

Approximation and compactness argument

Higher Regularity: Use regularity results for separate systems

Uniqueness: Gronwall's inequality once $c \in L^\infty(0, T; C^1(\overline{\Omega}))$ and $\mathbf{v} \in L^\infty(0, T; W_s^1(\Omega))$, $s > d$.

Crucial ingredient for higher regularity:

A priori estimate for $c \in BUC([0, \infty); W_q^1(\Omega))$, $q > d$!

Convergence to stationary solutions: Based on regularity for large times and the Lojasiewicz-Simon inequality.

Overview

Cahn-Hilliard Equation with Convection – Existence

We consider

$$\begin{aligned} \partial_t c + \mathbf{v} \cdot \nabla c &= m\Delta\mu && \text{in } \Omega \times (0,\infty), && (17) \\ \mu &= -\varepsilon\Delta c + \varepsilon^{-1} f'(c) && \text{in } \Omega \times (0,\infty) && (18) \\ \mathbf{n} \cdot \nabla c|_{\partial\Omega} = \mathbf{n} \cdot \nabla\mu|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,\infty), && (19) \\ c|_{t=0} &= c_0 && \text{in } \Omega. && (20) \end{aligned}$$

where $m \equiv const., \varepsilon > 0$ for a given $\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma) \cap L^2(0,\infty; H^1)$

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where $m \equiv const., \varepsilon > 0$ for a given $\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma) \cap L^2(0,\infty; H^1)$

Theorem (A. '07/'09)

For every $c_0 \in H^1(\Omega)$ with $E_\varepsilon(c_0) < \infty$ there is a unique solution $c \in L^\infty(0,\infty; H^1(\Omega)) \cap L^2_{\text{uloc}}([0,\infty); W^2_6(\Omega))$ of (17)-(20) with $\partial_t c \in L^2(0,\infty; H^{-1}_{(0)}(\Omega))$, $f'(c) \in L^2_{\text{uloc}}([0,\infty); L^6(\Omega))$, $\mu \in L^2_{\text{uloc}}([0,\infty); H^1(\Omega))$. Moreover, for every $T > 0$

$$E_\varepsilon(c(T)) + \int_0^T \|\nabla\mu(t)\|^2_{L^2(\Omega)}\,dt = E_\varepsilon(c_0) - \int_0^T\int_\Omega \mathbf{v}\cdot\mu\nabla c\,dx\,dt$$

A priori Estimates for c

W^2_r-estimate for c: Formally multiply

$$\mu(x,t) = -\Delta c(x,t) + f'(c(x,t))$$

by $f'(c(x,t)) = f_0'(c(x,t)) - \theta_c c(x,t)$ to obtain

$$\int_\Omega f_0'(c(t))^2\,dx + \int_\Omega \underbrace{f_0''(c(t))}_{\geq 0} |\nabla c(t)|^2\,dx \leq C(\|\mu(t)\|_2^2 + \|\nabla c\|_2^2).$$

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Similarly, multiplying with $f_0'(c)|f_0'(c)|^{r-2}$ for $2 \leq r < \infty$ yields

$$\|f_0'(c(t))\|_r + \|c(t)\|_{W^2_r} \leq C_r \left(\|\mu(t)\|_r + \|\nabla c(t)\|_2\right).$$

$$\Rightarrow c \in L^2_{\text{uloc}}([0,\infty); W^2_6(\Omega))$$

where

$$\|c\|_{L^2_{\text{uloc}}([0,\infty);X)} = \sup_{t\geq 0} \|c\|_{L^2(t,t+1;X)}.$$

Cahn-Hilliard Equation with Convection – Regularity

Lemma

Let (c, μ) be the solution above, $c_0 \in \mathcal{D}(\partial_{H^{-1}_{(0)}} E_0)$, and let $0 < T < \infty$.

1. *If $\partial_t \mathbf{v} \in L^1(0, T; L^2(\Omega))$, then (c, μ) satisfy*

$$\partial_t c \in L^\infty(0, T; H^{-1}_{(0)}(\Omega)) \cap L^2(0, T; H^1(\Omega)),$$
$$c \in L^\infty(0, T; W^2_6(\Omega)), \ f'(c) \in L^\infty(0, T; L^6(\Omega)),$$
$$\mu \in L^\infty(0, T; H^1(\Omega)).$$

2. *If $\mathbf{v} \in B^\alpha_{\frac{4}{3}\infty}(0, T; H^s(\Omega))$ for some $-\frac{1}{2} < s \leq 0$ and $\alpha \in (0, 1)$, then*

$$\kappa c \in C^\alpha([0, T]; H^{-1}_{(0)}(\Omega)) \cap B^\alpha_{2\infty}(0, T; H^1(\Omega)).$$

Remark: In general we only have $\partial_t \mathbf{v} \in L^{\frac{4}{3}}_{\text{uloc}}(0, \infty; H^{-1}(\Omega)^d)$ and the first part cannot be applied; but the second part.

Higher Time Regularity for c

First part: $L^\infty(0,\infty;H^{-1}_{(0)})$-estimate of $\partial_t c$ follows from: Multiplying

$$\partial_t^2 c + \Delta(\Delta\partial_t c - \underbrace{f_0''(c)}_{\geq 0}\partial_t c) = -\partial_t(\mathbf{v}\cdot\nabla c) - \theta_c\Delta\partial_t c$$

by $-\Delta_N^{-1}\partial_t c$ yields

$$\|\partial_t c\|_{L^\infty(0,\infty;H^{-1}_{(0)})} + \|\nabla\partial_t c\|_{L^2(Q)} \leq C(c_0)\left(1 + \|\partial_t \mathbf{v}\|_{L^{\frac{4}{3}}_{\mathrm{uloc}}(0,\infty;V_n')}\right)$$

where $V_n(\Omega) = \{\varphi \in H^1(\Omega)^d : n\cdot\varphi|_{\partial\Omega} = 0\}$.

$\Rightarrow \mu \in L^\infty(0,\infty;H^1(\Omega))$

$\Rightarrow c \in L^\infty(0,\infty;W^2_r(\Omega))$, $r = 6$ if $d = 3$ and $1 < r < \infty$ if $d = 2$.

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$\Rightarrow c \in L^\infty(0,\infty;W^2_r(\Omega))$, $r = 6$ if $d = 3$ and $1 < r < \infty$ if $d = 2$.

Second part: Replace $\partial_t c$ by $h^{-\alpha}\Delta_h c$. Use $\mathbf{v} \in B^\alpha_{\frac{4}{3}\infty;\mathrm{uloc}}([0,\infty);H^{-s}(\Omega))$ with $0 < s < \frac{1}{2}$ as well as $H^s_0(\Omega) = H^s(\Omega)$ and $H^{-s}(\Omega) = H^s(\Omega)'$.

Overview

Maximal Regularity for the Stokes Equation

We consider the Stokes equation with variable viscosity

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{div}(\nu(x,t)D\mathbf{v}) + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0,T), && (21)\\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega \times (0,T), && (22)\\ \mathbf{v}|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,T), && (23)\\ \mathbf{v}|_{t=0} &= 0 && \text{in } \Omega && (24) \end{aligned}$$

where $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ in a suitable domain $\Omega \subseteq \mathbb{R}^d$ with $\partial\Omega \in W_r^{2-\frac{1}{r}}$, $\nu \in BUC([0,T]; W_r^1(\Omega))$, where $2 \leq d < r \leq \infty$.

Theorem (A. & Terasawa '09, A '10/ A '07 (q=2))

Let $1 < q < \infty$ with $q, q' \leq r$, $\nu(x) \geq \nu_0 > 0$, and $0 < T < \infty$. Then for every $\mathbf{f} \in L^q(\Omega \times (0,T))^d$ there is a unique solution of v of (21)-(24) s.t.

$$\|(\partial_t \mathbf{v}, \nabla^2 \mathbf{v}, \nabla p)\|_{L^q(\Omega\times(0,T))} \leq C_T \|\mathbf{f}\|_{L^q(\Omega\times(0,T))}.$$

NB: $fg \in W_q^1(\Omega)$ if $f \in W_q^1(\Omega), g \in W_r^1(\Omega)$, $1 < q \leq r$, and $r > d$.

Maximal Regularity for the Stokes Equation

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where $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ in a suitable domain $\Omega \subseteq \mathbb{R}^d$ with $\partial\Omega \in W_r^{2-\frac{1}{r}}$, $\nu \in BUC([0,T]; W_r^1(\Omega))$, where $2 \leq d < r \leq \infty$.

Remark: If $\nu(x,t) = \nu_0(x)$, (21)-(24) can be written as X-valued ODE:

$$\begin{aligned} \frac{d}{dt}\mathbf{v}(t) + A_q \mathbf{v}(t) &= P_q \mathbf{f}(t), && t \in (0,\infty),\\ \mathbf{v}|_{t=0} &= 0 \end{aligned}$$

where $A_q \mathbf{v} = -P_q \operatorname{div}(\nu_0(x) D\mathbf{v})$, P_q is the Helmholtz projection, and $X = L^q_\sigma(\Omega) = \{f \in L^q(\Omega)^d : \operatorname{div} \mathbf{f} = 0, \mathbf{n} \cdot \mathbf{f}|_{\partial\Omega} = 0\}$.

Maximal Regularity for the Stokes Equation

We consider the Stokes equation with variable viscosity

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\end{aligned}$$

where $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ in a suitable domain $\Omega \subseteq \mathbb{R}^d$ with $\partial\Omega \in W_r^{2-\frac{1}{r}}$, $\nu \in BUC([0,T]; W_r^1(\Omega))$, where $2 \leq d < r \leq \infty$.

If $q = 2$, $\nu(x,t) = \nu_0(x)$, the results follows from the fact that $A_2 \colon \mathcal{D}(A_2) \subseteq L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ is a positive self-adjoint operator, where

$$\mathcal{D}(A_2) = H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega).$$

Maximal Regularity for the Stokes Equation

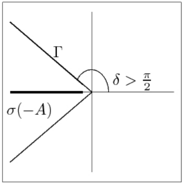
We consider the Stokes equation with variable viscosity

$$\begin{aligned}
\partial_t \mathbf{v} - \operatorname{div}(\nu(x,t)D\mathbf{v}) + \nabla p &= \mathbf{f} && \text{in } \Omega \times (0,T), && (21)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega \times (0,T), && (22)\\
\mathbf{v}|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,T), && (23)\\
\mathbf{v}|_{t=0} &= 0 && \text{in } \Omega && (24)
\end{aligned}$$

where $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ in a suitable domain $\Omega \subseteq \mathbb{R}^d$ with $\partial\Omega \in W_r^{2-\frac{1}{r}}$, $\nu \in BUC([0,T]; W_r^1(\Omega))$, where $2 \leq d < r \leq \infty$.

If $1 < q < \infty$, Dore & Venni '87 implies the result if A_q possesses bounded imaginary powers, i.e.,

$$A_q^{iy} := \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{iy} (\lambda + A_q)^{-1} d\lambda, \qquad y \in \mathbb{R},$$



is bounded on $L^q_\sigma(\Omega)$, where $(\lambda + A_q)^{-1} = O(|\lambda|^{-1})$.

Proof: Approximation of $(\lambda + A_q)^{-1}$ with pseudodifferential operators.

Overview

Quasi-Incompressible Model

Lowengrub &Truskinovsky'98 derived:

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}(\nu(c)D\mathbf{v}) + \nabla p = \underbrace{-\varepsilon\operatorname{div}(\nabla c\otimes\nabla c)}_{\text{surface tension}} \qquad (25)$$

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{v}) = 0 \qquad (26)$$

$$\rho\partial_t c + \rho\mathbf{v}\cdot\nabla c = m\Delta\mu \qquad (27)$$

$$\mu = -\rho^{-2}\frac{\partial\rho}{\partial c}\left(p + \frac{|\nabla c|^2}{2}\right) + \varepsilon^{-1}f'(c) - \varepsilon\rho^{-1}\Delta c \qquad (28)$$

in $\Omega\times(0,T)$, where $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$, together with suitable initial and boundary conditions.

- $\mathbf{v}, p$ are the velocity and pressure of the fluid mixture.
- $\rho = \hat\rho(c)$ is the density given as a constitutive function.
- $c = c_1 - c_2$ is the difference of the (mass) concentrations of the fluids.
- μ is the chemical potential and $m > 0$ the (constant) mobility.
- $f\colon \mathbb{R}\to[0,\infty)$ is a (homogeneous) free energy density

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New difficulties:

- $\operatorname{div} v \neq 0$ and p enters equation for chemical potential (28).
- (25)-(26) and (27)-(28) are coupled in highest order if $\rho \not\equiv const.$!

Analytic results:

A. '09: Existence of weak solutions for modified free energy/system

$$E_\varepsilon(c) = \varepsilon^{q-1}\int_\Omega \frac{|\nabla c|^q}{q}\,dx + \varepsilon^{-1}\int_\Omega \rho f(c(x))\,dx \qquad \text{with } q > d!$$

A. '12: Strong well-posedness locally in time in L^2-Sobolev spaces.

Overview

New Diffuse Interface Model (A., Garcke, Grün '12)

In the case of non-matched densities one can derive

$$\rho\partial_t \mathbf{v} + \left(\rho\mathbf{v} + \frac{\partial\rho}{\partial c}\mathbf{J}_\varphi\right)\cdot\nabla\mathbf{v} - \operatorname{div}(2\nu(\varphi)D\mathbf{v}) + \nabla p = -\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi) \quad (29)$$

$$\operatorname{div}\mathbf{v} = 0 \quad (30)$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) \quad (31)$$

where $\mathbf{J}_\varphi = -m(\varphi)\nabla\mu$ together with

$$\mu = \varepsilon^{-1}f'(\varphi) - \varepsilon\Delta\varphi \quad (32)$$

Here

- $\mathbf{v} = \varphi_1\mathbf{v}_1 + \varphi_2\mathbf{v}_2$ – volume averaged velocity.
- $\mathbf{v}_j$ – velocity of fluid j.
- φ_j – volume fraction of fluid j, $\varphi = \varphi_2 - \varphi_1$.
- $\rho = \rho(\varphi) = \frac{1-\varphi}{2}\tilde{\rho}_1 + \frac{1+\varphi}{2}\tilde{\rho}_2$ and $\tilde{\rho}_j$ are the specific densities.

Lowengrub,Truskinovsky: $\mathbf{v}$ is the mass averaged velocity $\rho\mathbf{v} = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2$

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Conservation of mass:

$$\partial_t\rho + \operatorname{div}\left(\rho\mathbf{v} - \underbrace{\frac{\tilde\rho_2 - \tilde\rho_1}{2}m(\varphi)\nabla\mu}_{=-\frac{\partial\rho}{\partial c}\mathbf{J}_\varphi}\right) = 0$$

Here $\frac{\tilde\rho_2-\tilde\rho_1}{2}m(\varphi)\nabla\mu = \frac{\partial\rho}{\partial\varphi}m(\varphi)\nabla\mu$ is a flux relative to $\rho\mathbf{v}$ related to diffusion of the particles.

Modeling: Conservation of Linear Momentum

Starting point:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) + \operatorname{div}\left(\mathbf{v}\otimes\frac{\partial\rho}{\partial\varphi}\mathbf{J}_\varphi\right) = \operatorname{div}\mathbf{T}$$

where $\mathbf{J}_\varphi = m(\varphi)\nabla\mu$, cf. Alt '09.

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where $\mathbf{J}_\varphi = m(\varphi)\nabla\mu$, cf. Alt '09. This is equivalent to

$$\rho \partial_t \mathbf{v} + \left(\rho \mathbf{v} + \frac{\partial \rho}{\partial \varphi} \mathbf{J}_\varphi\right) \cdot \nabla \mathbf{v} = \operatorname{div} \mathbf{T} \tag{33}$$

and, if $V(t)$ is transported by $\rho\tilde{\mathbf{v}} = \rho \mathbf{v} - \frac{\partial \rho}{\partial \varphi} m(\varphi)\nabla\mu$,

$$\frac{d}{dt}\int_{V(t)} \frac{\rho|\mathbf{v}|^2}{2}\, dx = \int_{\partial V(t)} \mathbf{n} \cdot \mathbf{T}\, dx$$

Note:

- The left-hand side of (33) is objective in contrast to $\rho\partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v}$ and $\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v})$ in our situation.
- Therefore $\mathbf{T}$ is objective too.

Derivation of the Model

Starting Point

$$\rho\partial_t \mathbf{v} + (\rho\mathbf{v} + \tfrac{\partial\rho}{\partial\varphi}\mathbf{J}_\varphi)\cdot\nabla\mathbf{v} = \operatorname{div}\mathbf{T} \quad \text{(conservation of momentum)}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{(conservation law for components, I)}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi + \operatorname{div}\mathbf{J}_\varphi = 0 \quad \text{(conservation law for components, II)}$$

$$\partial_t e + \mathbf{v}\cdot\nabla e + \operatorname{div}\mathbf{J}_e \leq 0 \quad \text{(local energy inequality)}$$

Here $\mathbf{T}$ is the stress tensor, $\mathbf{J}_\varphi, \mathbf{J}_e$ are fluxes, and

$$e = e(\mathbf{v}, \varphi, \nabla\varphi) = \hat{\rho}(\varphi)\frac{|\mathbf{v}|^2}{2} + \varepsilon^{-1}f(\varphi) + \varepsilon\frac{|\nabla\varphi|^2}{2}.$$

Lagrange multiplier approach:

- Exploiting the energy inequality and the conservation laws give restrictions for the constitutive assumptions on $\mathbf{T}$, $\mathbf{J}_\varphi, \mathbf{J}_e$.
- The chemical potential μ and the pressure p arise as Lagrange multipliers to the constraints given by the conservation laws.

Existence of Weak Solutions: Assumptions

We consider

$$\rho\partial_t \mathbf{v} + (\rho\mathbf{v} + \tfrac{\partial\rho}{\partial c}\mathbf{J}_\varphi)\cdot\nabla\mathbf{v} - \operatorname{div}(2\nu(\varphi)D\mathbf{v}) + \nabla p = -\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi) \tag{34}$$

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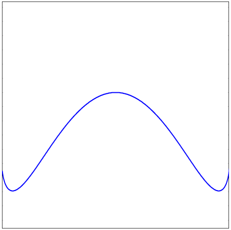
$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) \tag{36}$$

$$\mu = \varepsilon^{-1}f'(\varphi) - \varepsilon\Delta\varphi \tag{37}$$

where $\mathbf{J}_\varphi = m(\varphi)\nabla\mu$ with $0 < m_0 \leq m(\varphi) \leq M_0$ in $\Omega\times(0,T)$, where $\Omega\subseteq\mathbb{R}^n$ is a bounded smooth domain, together with

$$\mathbf{v}|_{\partial\Omega} = \mathbf{n}\cdot\nabla\varphi|_{\partial\Omega} = \mathbf{n}\cdot\nabla\mu|_{\partial\Omega} = 0 \tag{38}$$

$$(\mathbf{v},\varphi)|_{t=0} = (\mathbf{v}_0,\varphi_0) \tag{39}$$



For f we choose e.g.:

$$f(\varphi) = \begin{cases} \theta((1-\varphi)\log(1-\varphi) + (1+\varphi)\log(1+\varphi))\varphi - \theta_c\varphi^2, & \varphi\in[-1,1], \\ +\infty & \text{else.} \end{cases}$$

Theorem (A., Depner, Garcke '11)

Let $d = 2, 3$. For every $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $\varphi_0 \in H^1(\Omega)$ with $E_\varepsilon(\varphi_0) < \infty$ there is a weak solution (v, φ, μ) of (34)-(39), which satisfies

$$(\mathbf{v}, \nabla\varphi) \in L^\infty(0,\infty; L^2(\Omega)), \quad (\nabla\mathbf{v}, \nabla\mu) \in L^2(0,\infty; L^2(\Omega)),$$
$$\nabla^2\varphi, f'(\varphi) \in L^2_{loc}([0,\infty); L^2(\Omega)).$$

In particular, $\varphi(t,x) \in (-1,1)$ almost everywhere.

Energy dissipation: Proof is based on a priori estimates deduced from

$$\begin{aligned} \frac{d}{dt}E(\varphi(t),\mathbf{v}(t)) &= -\int_\Omega \nu(\varphi)|D\mathbf{v}|^2\,dx - \int_\Omega m(\varphi)|\nabla\mu|^2\,dx \quad \text{with} \\ E(\varphi(t),\mathbf{v}(t)) &= \int_\Omega \left(\varepsilon\frac{|\nabla\varphi|^2}{2} + \frac{1}{\varepsilon}f(\varphi)\right)dx + \int_\Omega \frac{\rho|\mathbf{v}(t)|^2}{2}\,dx \end{aligned}$$

Structure of the Proof

- We approximate (34)-(39) by an implicit time discretization for which we have an analogous discrete energy estimate.
- In order to deal with the singular logarithmic terms, we use again that

$$f(\varphi) = f_0(\varphi) - \frac{\theta_c}{2}\varphi^2,$$

where f_0 is convex. Then

$$\mu = -\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}f'(\varphi) = \underbrace{-\varepsilon\Delta\varphi + \tfrac{1}{\varepsilon}f_0'(\varphi)}_{=\partial E_0(\varphi)} - \theta_c\varphi$$

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- Essential step: Use regularity result for ∂E_0:

$$\begin{aligned} \|\varphi\|_{H^2(\Omega)} + \|f_0'(\varphi)\|_{L^2(\Omega)} &\leq C\left(\|\partial E_0(\varphi)\|_{L^2(\Omega)} + 1\right) \\ \Rightarrow \quad \Delta\varphi, f'(\varphi) &\in L^2(0,T;L^2(\Omega)) \end{aligned}$$

Strong Compactness of Velocity Field

Let $(\varphi_k, \mathbf{v}_k, p_k)$ be a sequence of solutions with bounded energies.

In order to pass to the limit in

$$\partial_t(\rho_k \mathbf{v}_k) + \operatorname{div}\left(\mathbf{v}_k \otimes \left(\rho_k \mathbf{v}_k + \frac{\partial\rho}{\partial\varphi}\mathbf{J}_k\right)\right) - \operatorname{div}(2\nu(\varphi_k) D\mathbf{v}_k) + \nabla p_k = -\varepsilon \operatorname{div}(\nabla\varphi_k \otimes \nabla\varphi_k)$$

we use that this equation implies (for a subsequence)

$$P_\sigma(\rho_k \mathbf{v}_k) \to_{k\to\infty} P_\sigma(\rho \mathbf{v}) \quad \text{in } L^2(\Omega \times (0, T))$$

by the Lemma of Aubin-Lions. Here P_σ is the Helmholtz projection.

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by the Lemma of Aubin-Lions. Here P_σ is the Helmholtz projection. Hence

$$\int_0^T \int_\Omega \rho_k |\mathbf{v}_k|^2 \, dx\, dt = \int_0^T \int_\Omega P_\sigma(\rho_k \mathbf{v}_k) \mathbf{v}_k \, dx\, dt \to_{k\to\infty} \int_0^T \int_\Omega \rho |\mathbf{v}|^2 \, dx\, dt$$

and therefore $\mathbf{v}_k \to_{k\to\infty} \mathbf{v}$ in $L^2(\Omega \times (0,T))$ since $\rho_k \to_{k\to\infty} \rho$ a.e.

Weak Continuity of the Velocity

Goal: Show $\mathbf{v}\colon [0,\infty) \to L^2(\Omega)$ is weakly continuous and $\mathbf{v}|_{t=0} = \mathbf{v}_0$

Problem: Weak formulation of moment equation (34) only gives control of

$$\partial_t P_\sigma(\rho \mathbf{v}) \in L^2(0,T;H^{-s}(\Omega)) \qquad \text{for some } s<0 \text{ and all } T<\infty.$$

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$$P_\sigma(\rho\mathbf{v}) \in C([0,\infty);H^{-1}) \cap L^\infty(0,\infty;L^2) \hookrightarrow C_w([0,\infty);L^2)$$

Hence $P_\sigma(\rho\mathbf{v}|_{t=0}) = P_\sigma(\rho_0\mathbf{v}_0)$.

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Hence $P_\sigma(\rho\mathbf{v}|_{t=0}) = P_\sigma(\rho_0\mathbf{v}_0)$. To conclude $\rho\mathbf{v}|_{t=0} = \rho_0\mathbf{v}_0$, we use:

Lemma

Let $\mathbf{v}_j \in L^2_\sigma(\Omega)$, $j=1,2$ such that

$$\int_\Omega \rho\mathbf{v}_1 \cdot \varphi\, dx = \int_\Omega \rho\mathbf{v}_2 \cdot \varphi\, dx \quad \text{for all } \varphi \in C^\infty_{0,\sigma}(\Omega).$$

Then $\mathbf{v}_1 = \mathbf{v}_2$.

Using this lemma, one can also show $\mathbf{v} \in C_w([0,\infty);L^2(\Omega))$.

Overview

Sharp Interface Limit of Cahn-Hilliard Equation

We consider

$$\partial_t c = m\Delta\mu, \tag{40}$$

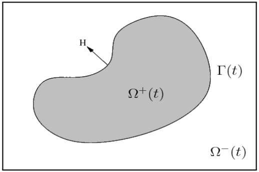
$$\mu = -\varepsilon\Delta c + \varepsilon^{-1} f'(c) \tag{41}$$

together with suitable boundary and initial conditions. Then (40)-(41) converges to the Mullins-Sekerka equation if $m = m(\varepsilon) \equiv const. > 0$:

$$V = -\frac{m}{2}[\mathbf{n}_{\Gamma(t)} \cdot \nabla\mu] \quad \text{on } \Gamma(t)$$

$$\mu|_{\Gamma(t)} = \sigma H \quad \text{on } \Gamma(t)$$

$$\Delta\mu = 0 \quad \text{on } \Omega^{\pm}(t)$$



due to

- Alikakos et al. '94 (local strong solutions)
- X. Chen '96 (global varifold solutions).

Theorem (X. Chen '96)

Let $(c_\varepsilon, \mu_\varepsilon)_{0<\varepsilon\leq 1}$ *be solutions of* (40)-(41). *Then for a suitable subsequence*

$$\begin{aligned} c_\varepsilon &\to_{\varepsilon\to 0} -1 + 2\chi_E && \text{in } C^{\frac{1}{9}}_{loc}([0,\infty); L^2(\Omega)) \text{ and a.e.} \\ \mu_\varepsilon &\rightharpoonup_{\varepsilon\to 0} \mu && \text{in } L^2_{loc}([0,\infty); H^1(\Omega)), \end{aligned}$$

where $\chi_E \in L^\infty(0,\infty; BV(\Omega))$ *and*

$$\begin{aligned} \partial_t \chi_E &= \frac{m}{2}\Delta\mu && \text{in } \mathcal{D}'(\Omega\times(0,\infty)), \\ -\mu\nabla\chi_E &= \frac{1}{2}\delta V_t && \text{in } \mathcal{D}'(\Omega\times(0,\infty)), \end{aligned}$$

where

$$\langle \delta V^t, \psi\rangle = \int_\Omega \nabla\psi : \left(\mathbf{I}\, d\nu - d(\nu_{ij})_{i,j=1}^d\right)$$

for all $C^1_0(\Omega)^d$ *and* $0 \leq (\nu_{ij})_{i,j=1}^d \leq \mathbf{I}\nu$ *in* $\mathcal{M}(\Omega)^{d\times d}$.

Sketch of the Proof (due to X. Chen) (I)

Energy estimate: For every $0 < T < \infty$:

$$E_\varepsilon(c_\varepsilon(\cdot, T)) + m \int_0^T \int_\Omega |\nabla \mu|^2 \, dx \, dt \le E_\varepsilon(c_{0,\varepsilon}) \le M.$$

Moreover, $\partial_t c_\varepsilon = m\Delta\mu_\varepsilon$ is bounded in $L^2(0,\infty; H^{-1}(\Omega))$. Arguments by Modica and Mortola and embeddings give

$$c_\varepsilon \to_{\varepsilon\to 0} -1 + 2\chi_E \quad \text{in } C^{\frac{1}{9}}_{loc}([0,\infty); L^2(\Omega)) \text{ and a.e.,}$$

where $\|\nabla\chi_E(t)\|_{\mathcal{M}(\Omega)} \le \frac{1}{\sigma} M$ for a.e. $0 < t < \infty$.

Sketch of the Proof (due to X. Chen) (II)

Let $e_\varepsilon = \varepsilon\frac{|\nabla c_\varepsilon|^2}{2} + \frac{f(c_\varepsilon)}{\varepsilon}$. Then $(e_\varepsilon)_{0<\varepsilon\leq 1} \subseteq L^\infty(0,\infty; L^1(\Omega))$. Hence

$$\begin{aligned} e_\varepsilon &\rightharpoonup^*_{\varepsilon\to 0} \nu && \text{in } L^\infty_{w*}(0,\infty;\mathcal{M}(\Omega)) \\ \varepsilon\nabla c_\varepsilon \otimes \nabla c_\varepsilon &\rightharpoonup^*_{\varepsilon\to 0} (\nu_{i,j})_{i,j=1}^d && \text{in } L^\infty_{w*}(0,\infty;\mathcal{M}(\Omega)^{d\times d}) \end{aligned}$$

Using

$$\mu_\varepsilon \nabla c_\varepsilon = \operatorname{div}\left(e_\varepsilon \mathbf{I} - \varepsilon\nabla c_\varepsilon \otimes \nabla c_\varepsilon\right)$$

yields in the limit $\varepsilon \to 0$

$$2\mu\nabla\chi_E = \operatorname{div}\left(\nu\mathbf{I} - (\nu_{i,j})_{i,j=1}^d\right) = \delta V$$

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Essential step: To show $0 \leq (\nu_{i,j})_{i,j=1}^d \leq \mathbf{I}\nu$ in $\mathcal{M}(\Omega)^{d\times d}$ one uses that

$$(\xi_\varepsilon(c_\varepsilon))^+ \, dx\, dt \rightharpoonup^*_{\varepsilon\to 0} 0 \qquad \text{in } \mathcal{M}(\Omega\times(0,\infty)),$$

where $\xi(c_\varepsilon) := \varepsilon\frac{|\nabla c_\varepsilon|^2}{2} - \frac{f(c_\varepsilon)}{\varepsilon}$ (discrepancy measure).

Overview

Diffuse Interface Model (A., Garcke, Grün '12)

We consider

$$\rho\partial_t \mathbf{v} + (\rho\mathbf{v} + \frac{\partial\rho}{\partial\varphi}\mathbf{J}_\varphi)\cdot\nabla\mathbf{v} - \operatorname{div}(2\nu(\varphi)D\mathbf{v}) + \nabla p = -\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi) \tag{42}$$

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where $\mathbf{J}_\varphi = -m(\varphi)\nabla\mu$ together with

$$\mu = \varepsilon^{-1}f'(\varphi) - \varepsilon\Delta\varphi \tag{45}$$

Here

- $\mathbf{v} = \varphi_1\mathbf{v}_1 + \varphi_2\mathbf{v}_2$ – volume averaged velocity.
- $\mathbf{v}_j$ – velocity of fluid j.
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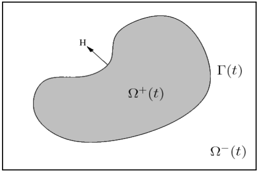
- $\rho = \rho(\varphi) = \frac{1-\varphi}{2}\tilde{\rho}_1 + \frac{1+\varphi}{2}\tilde{\rho}_2$.

Sharp Interface Limits via Matched Asymptotics (AGG '12)

Bulk equations: In $\Omega^\pm(t)$ we have



$$\rho\partial_t \mathbf{v} + \left(\rho\mathbf{v} + \tfrac{\rho_1-\rho_2}{2}\mathbf{J}\right)\cdot\nabla\mathbf{v} - \operatorname{div}(\nu^\pm D\mathbf{v}) + \nabla p = 0$$
$$\operatorname{div}\mathbf{v} = 0$$

Interface equations:

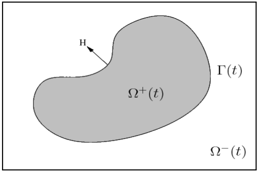
Case I: $m = \varepsilon m_0$: On $\Gamma(t)$ we have

$$-\left[\mathbf{n}\cdot(\nu^\pm D\mathbf{v} - pI)\right] = \sigma H\mathbf{n}$$
$$V = \mathbf{n}\cdot\mathbf{v}|_{\Gamma(t)}$$

V is the normal velocity, H is the mean curvature, $\mathbf{n}$ is a normal. $\mathbf{J}\equiv 0$

Sharp Interface Limits via Matched Asymptotics (AGG '12)

Bulk equations: In $\Omega^{\pm}(t)$ we have



$$\rho\partial_t \mathbf{v} + \left(\rho\mathbf{v} + \tfrac{\rho_1-\rho_2}{2}\mathbf{J}\right)\cdot\nabla\mathbf{v} - \operatorname{div}(\nu^{\pm} D\mathbf{v}) + \nabla p = 0$$
$$\operatorname{div}\mathbf{v} = 0$$

Interface equations:

Case I: $m = \varepsilon m_0$: On $\Gamma(t)$ we have

$$-\left[\mathbf{n}\cdot(\nu^{\pm} D\mathbf{v} - pI)\right] = \sigma H\mathbf{n}$$
$$V = \mathbf{n}\cdot\mathbf{v}|_{\Gamma(t)}$$

V is the normal velocity, H is the mean curvature, $\mathbf{n}$ is a normal. $\mathbf{J}\equiv 0$

Case II: $m = m_0 > 0$: On $\Gamma(t)$ we have

$$-\left[n\cdot(\nu^{\pm} D\mathbf{v} - pI)\right] = \sigma H\mathbf{n}$$
$$V = \mathbf{n}\cdot\mathbf{v}|_{\Gamma(t)} - \tfrac{m_0}{2}[\mathbf{n}\cdot\nabla\mu]$$
$$2\mu|_{\Gamma(t)} = \sigma H$$

together with $\Delta\mu = 0$ in $\Omega^{\pm}(t)$, $\mathbf{J} = \frac{m_0}{2}\nabla\mu$.

Theorem (Sharp Interface Limit in Varifold Sense (A. forthcoming))

Let $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)_{0<\varepsilon\leq 1}$ *be weak solutions of* (13)-(16) *with* $m = m(\varepsilon) \to m_0 \geq 0$ *such that* $\lim_{\varepsilon\to 0} \varepsilon m(\varepsilon)^{-1} = 0$. *Then for a suitable subsequence*

$$(\mathbf{v}_\varepsilon, m(\varepsilon)\mu_\varepsilon) \rightharpoonup_{\varepsilon\to 0} (\mathbf{v}, m_0\mu) \qquad \text{in } L^2_{loc}([0,\infty); H^1(\Omega))$$

$$\varphi_\varepsilon \to_{\varepsilon\to 0} -1 + 2\chi_E \quad \text{in } C^{\frac{1}{9}}_{loc}([0,\infty); L^2(\Omega)) \text{ and a.e.}$$

where $\chi_{E_t} \in L^\infty(0,\infty; BV(\Omega))$ *and*

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes (\rho\mathbf{v} + m_0 \tfrac{\rho_1-\rho_2}{2}\nabla\mu)) - \operatorname{div}(\nu(\chi_{E_t})D\mathbf{v}) + \nabla q = -\delta V$$

$$\partial_t\chi_{E_t} + v\cdot\nabla\chi_{E_t} = \frac{m_0}{2}\Delta\mu$$

$$\text{If } m_0 > 0: -\mu\nabla\chi_{E_t} = \frac{1}{2}\delta V_t$$

in $\mathcal{D}(\Omega\times(0,\infty))$, *where* δV *is as in X. Chen '96 and* $\rho = \rho(\chi_E)$.

Case $\rho_1 = \rho_2$, $m_0 > 0$: See A. Röger '09.

Sketch of the Proof (I)

Energy estimate: For every $0 < T < \infty$:

$$E_\varepsilon(\varphi_\varepsilon(T)) + \int_\Omega \rho(c_\varepsilon(T)) \frac{|\mathbf{v}(T)|^2}{2}\, dx$$

$$+ \int_0^T \int_\Omega \left(\nu(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2 + m_\varepsilon |\nabla \mu|^2 \right) \, dx\, dt \leq E_\varepsilon(\varphi_{0,\varepsilon}) + \int_\Omega \rho(c_{0,\varepsilon}) \frac{|\mathbf{v}_0|^2}{2}\, dx.$$

Adapting the arguments of X. Chen/Modica and Mortola one shows

$$c_\varepsilon \to_{\varepsilon \to 0} -1 + 2\chi_E \quad \text{in } C^{\frac{1}{9}}_{loc}([0,\infty); L^2(\Omega)) \text{ and a.e.,}$$

where $\|\nabla \chi_E(t)\|_{\mathcal{M}(\Omega)} \leq \frac{1}{\sigma} M$ for a.e. $0 < t < \infty$.

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where $\|\nabla\chi_E(t)\|_{\mathcal{M}(\Omega)} \leq \frac{1}{\sigma} M$ for a.e. $0 < t < \infty$.

Strong convergence of $\mathbf{v}_\varepsilon$: First one shows

$$P_\sigma(\rho(\varphi_\varepsilon)\mathbf{v}_\varepsilon) \to_{\varepsilon\to 0} P_\sigma(\rho(\chi_E)\mathbf{v}) \qquad \text{in } L^2(\Omega \times (0,T))$$

for all $0 < T < \infty$ using the Lemma of Aubin-Lions. This implies

$$\mathbf{v}_\varepsilon \to_{\varepsilon\to 0} \mathbf{v} \qquad \text{in } L^2(\Omega \times (0,T)) \text{ for all } 0 < T < \infty$$

similarly as in A., Depner, Garcke '11 since $\operatorname{div} \mathbf{v}_\varepsilon = \operatorname{div} \mathbf{v} = 0$.

Sketch of the Proof (II)

As before let $e_\varepsilon = \varepsilon\frac{|\nabla c_\varepsilon|^2}{2} + \frac{f(c_\varepsilon)}{\varepsilon}$. Then $(e_\varepsilon)_{0<\varepsilon\leq 1} \subseteq L^\infty(0,\infty;L^1(\Omega))$.
Hence

$$
\begin{aligned}
e_\varepsilon &\rightharpoonup^*_{\varepsilon\to 0} \nu && \text{in } L^\infty_{w*}(0,\infty;\mathcal{M}(\Omega))\\
\varepsilon\nabla c_\varepsilon \otimes \nabla c_\varepsilon &\rightharpoonup^*_{\varepsilon\to 0} (\nu_{i,j})_{i,j=1}^d && \text{in } L^\infty_{w*}(0,\infty;\mathcal{M}(\Omega)^{d\times d})
\end{aligned}
$$

Using

$$\mu_\varepsilon \nabla c_\varepsilon = \operatorname{div}\left(e_\varepsilon \mathbf{I} - \varepsilon\nabla c_\varepsilon \otimes \nabla c_\varepsilon\right)$$

yields in the limit $\varepsilon \to 0$

$$2\mu\nabla\chi_E = \operatorname{div}\left(\nu\mathbf{I} - (\nu_{i,j})_{i,j=1}^d\right) = \delta V$$

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Essential step: To show $0 \leq (\nu_{i,j})_{i,j=1}^d \leq \mathbf{I}\nu$ in $\mathcal{M}(\Omega)^{d\times d}$ one uses that

$$(\xi_\varepsilon(c_\varepsilon))^+ \, dx\, dt \rightharpoonup^*_{\varepsilon\to 0} 0 \qquad \text{in } \mathcal{M}(\Omega\times(0,\infty)),$$

where $\xi(c_\varepsilon) := \varepsilon\frac{|\nabla c_\varepsilon|^2}{2} - \frac{f(c_\varepsilon)}{\varepsilon}$ (discrepancy measure), cf. X. Chen '96.

To this end one needs: $\varepsilon m(\varepsilon)^{-1} \to_{\varepsilon\to 0} 0$!

Overview

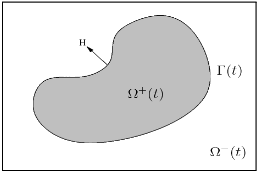
Analytic Results for the Mullins-Sekerka Equation:

We consider

$$V = m[\mathbf{n}_{\Gamma(t)} \cdot \nabla \mu] \quad \text{on } \Gamma(t) \qquad (46)$$

$$\mu|_{\Gamma(t)} = \sigma H \quad \text{on } \Gamma(t) \qquad (47)$$

$$\Delta \mu = 0 \quad \text{on } \Omega^{\pm}(t) \qquad (48)$$



together with $\Gamma(0) = \Gamma_0 \subset\subset \Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$.

Existence of local, classical solutions:
X. Chen, Hong & Yi '93, ($d = 2$),
Escher & Simonett '96/'97 ($d \geq 2$).

Stability of spheres: X. Chen '93, ($d = 2$), Escher & Simonett '98, Prüß, Simonett, & Zacher '09, Köhne, Prüß & Wilke '10 ($d \geq 2$).

Existence of weak solutions: Röger '05

Existence of Strong Solutions (Escher & Simonett '96/'97)

Basic idea: Write $\Gamma(t)$ as a graph over a smooth reference manifold Σ:

$$\Gamma(t) = \left\{x \in \Omega : x = s + \mathbf{n}_\Sigma h(t,s) =: \theta_{h(t)} s \text{ for } s \in \Sigma\right\} = \theta_{h(t)}(\Sigma)$$

where $h(t) \in C^2(\Sigma)$. Extend $\theta_{h(t)}$ to a diffeomorphism

$$\Theta_{h(t)} \colon \Omega \to \Omega \qquad \text{(Hansawa transformation)}$$

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Then (46)-(48) is equivalent to

$$\partial_t h + G(h) = 0 \quad \text{on } \Sigma \times (0,T), \quad h(0) = h_0, \tag{49}$$

where $G(h) = D(h)H(h)$ and

- $H(h)$ is the transformed mean curvature of $\Gamma(t)$.
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Here $DG(0) \approx (-\Delta_\Sigma)^{\frac{1}{2}}(-\Delta_\Sigma)$ generates an analytic semigroup e.g. on $h^\alpha(\Sigma) = \overline{C^\infty(\Sigma)}^{C^\alpha(\Sigma)}$

Local existence: Theory of abstract quasi-linear parabolic equations

Local Existence of Strong Solutions

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div} T(\mathbf{v},q) = 0 \qquad \text{in } \Omega^\pm(t) \tag{50}$$
$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } \Omega^\pm(t) \tag{51}$$
$$\Delta\mu = 0 \qquad \text{in } \Omega^\pm(t) \tag{52}$$
$$-\left[\mathbf{n}_{\Gamma(t)}\cdot T(\mathbf{v},q)\right] = \sigma H \mathbf{n}_{\Gamma(t)} \qquad \text{on } \Gamma(t), \tag{53}$$
$$V = \mathbf{n}_{\Gamma(t)}\cdot\mathbf{v}|_{\Gamma(t)} - m[\mathbf{n}_{\Gamma(t)}\cdot\nabla\mu] \qquad \text{on } \Gamma(t), \tag{54}$$
$$\mu|_{\Gamma(t)} = \sigma H \qquad \text{on } \Gamma(t). \tag{55}$$

Theorem (A. & Wilke '11)

Let $\mathbf{v}_0 \in H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $\Gamma_0 = \theta_{h_0}$ *with* $h_0 \in W_p^{4-\frac{4}{p}}(\Sigma)$, $p \in (3, \frac{2(d+2)}{d}]$, $d = 2,3$. *Then there is some* $T > 0$ *such that (50)-(55) has a unique solution* $(v(t), \Gamma(t))$ *for* $t \in (0,T)$, *where* $\Gamma(t) = \theta_{h(t)}\Sigma$

$$\mathbf{v} \in L^2(0,T; H^2(\Omega\setminus\Gamma(t))) \cap H^1(0,T;L^2(\Omega))$$
$$h \in L^p(0,T; W_p^{4-\frac{1}{p}}(\Sigma)) \cap W^1_p(0,T; W_p^{1-\frac{1}{p}}(\Sigma))$$

Solving the Navier-Stokes-Part

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div} T(\mathbf{v},q) = 0 \quad \text{in } \Omega^\pm(t), t\in(0,T), \tag{56}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega^\pm(t), t\in(0,T), \tag{57}$$

$$[\mathbf{v}] = 0 \quad \text{on } \Gamma(t), t\in(0,T), \tag{58}$$

$$-\left[n_{\Gamma(t)}\cdot T(\mathbf{v},q)\right] = \sigma H n_{\Gamma(t)} \quad \text{on } \Gamma(t), t\in(0,T), \tag{59}$$

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{60}$$

$$\mathbf{v}|_{t=0} = v_0 \quad \text{in } \Omega \tag{61}$$

where $T(\mathbf{v},p) = \mu^\pm D\mathbf{v} - pI$. Here $\Gamma(t) = \theta_{h(t)}\Sigma$ is given!

Theorem (A. & Wilke '11)

Let $h\in L^p(0,T_0;W_p^{4-\frac{1}{p}})\cap W_p^1(0,T_0;W_p^{1-\frac{1}{p}})$, $\mathbf{v}_0\in H_0^1(\Omega)^d\cap L_\sigma^2(\Omega)$. *Then there is some* $0<T\le T_0$ *such that* (56)-(61) *has a unique solution*

$$\mathbf{v}\in L^2(0,T;H^2(\Omega\setminus\Gamma(t)))\cap H^1(0,T;L^2(\Omega))$$

Moreover, the mapping $h\mapsto\mathbf{v}$ *is* C^1 *w.r.t. to the corresponding norms.*

Solving the Navier-Stokes-Part – Sketch of Proof

Let $F_h(t) = \Theta_{h(t)} \circ \Theta_{h_0}^{-1} : \Omega \to \Omega$. Then $F_h(t)(\Gamma_0) = \Gamma(t)$ for all $t \in (0, T)$ and $F_h(0) = \mathrm{Id}$. Defining $\mathbf{u}(x,t) = \mathbf{v}(F_h(t)(x), t)$ (56)-(59) can be transformed to

$$\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{h,t} \mathbf{u} - \mathrm{div}_{h,t}\, \mathbf{T}_{h,t}(\mathbf{u}, \tilde{q}) &= \partial_t F_h \cdot \nabla_{h,t} \mathbf{u} && \text{in } (\Omega \setminus \Gamma_0) \times (0,T) \\
\mathrm{div}_{h,t}\, \mathbf{u} &= 0 && \text{in } (\Omega \setminus \Gamma_0) \times (0,T) \\
[\mathbf{u}] &= 0 && \text{on } \Gamma_0 \\
-[A_{h,t} n_{\Gamma_0} \cdot \mathbf{T}_{h,t}(\mathbf{u}, \tilde{q})] &= \sigma \tilde{H}_{h,t} A_{h,t} \mathbf{n}_{\Gamma_0} && \text{on } \Gamma_0 \times (0,T) \\
\mathbf{u}|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,T)
\end{aligned}$$

Here

$$A_{h,t} \approx \mathrm{I},\ \nabla_{h,t} \approx \nabla,\ \mathbf{T}_{h,t}(\mathbf{u}, \tilde{q}) \approx \mathbf{T}(\mathbf{u}, \tilde{q}), \ldots \quad \text{if } t \in (0,T), 0 < T \ll 1$$

Moreover, since $p > 3$,

$$\begin{aligned}
\tilde{H}_{h,t} &\in L^p(0,T; W_p^{2-\frac{1}{p}}(\Gamma_0)) \cap W_p^{\frac{1}{3}}(0,T; W_p^{1-\frac{1}{p}}(\Gamma_0)) \\
&\hookrightarrow\hookrightarrow L^2(0,T; H^{\frac{1}{2}}(\Gamma_0)) \cap H^{\frac{1}{4}}(0,T; L^2(\Gamma_0))
\end{aligned}$$

Sketch of Proof: Local Well-Posedness

Again we write $\Gamma(t)$ as a graph over a smooth reference manifold Σ:

$$\Gamma(t) = \left\{x \in \Omega : x = s + \mathbf{n}_\Sigma h(t,s) =: \theta_{h(t)} s \text{ for } s \in \Sigma\right\} = \theta_{h(t)}(\Sigma)$$

where $h(t) \in C^2(\Sigma)$ and use the Hansawa transformation $\Theta_{h(t)} : \Omega \to \Omega$. Then (46)-(48) is equivalent to

$$\begin{aligned} \partial_t h(t) + G(h(t)) + F_T(h)(t) &= 0, \qquad t \in (0,T), &(62)\\ h(0) &= h_0, &(63)\end{aligned}$$

where $G(h) = D(h)H(h)$ and

- $H(h)$ is the transformed mean curvature of $\Gamma(t)$.
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Here $F_T(h)$ is a non-local Volterra-type operator and a lower order perturbation. Therefore local existence can be proved similarly as for the Mullins-Sekerka system.

Stability of Spheres

Theorem (A. & Wilke '11)

Let $\Sigma = \partial B_R(x) \subset \Omega$. Then there is some $\delta > 0$ such that for any $v_0 \in H^1(\Omega)^d \cap L^2_\sigma(\Omega)$ and $\Gamma_0 = \theta_{h_0}$ with

$$\|v_0\|_{H^1} + \|h_0\|_{W_p^{4-\frac{4}{p}}(\Sigma)} \leq \delta,$$

such that the unique solution $(v(t), \Gamma(t))$ of (50)-(55) exists for all $t \in (0, \infty)$, where

$$v \in L^2(0,T; H^2(\Omega \setminus \Gamma(t))^d) \cap H^1(0,T; L^2(\Omega)^d)$$

$$h \in L^p(0,T; W_p^{4-\frac{1}{p}}(\Sigma)) \cap W_p^1(0,T; W_p^{1-\frac{1}{p}}(\Sigma))$$

for every $T < \infty$ and there is some h_∞ such that $\theta_{h_\infty}\Sigma$ is a sphere and $(v(t), h(t)) \to_{t\to\infty} (0, h_\infty)$ exponentially in $H^1(\Omega)^d \times W_p^{4-\frac{4}{p}}(\Sigma)$.

Proof: Based on the "Generalized Principle of Linearized Stability"

Generalized Principle of Linearized Stability

Alternative approach to stability: We consider

$$\frac{d}{dt}u(t) + A(u(t))u(t) = F(u(t)), t > 0 \quad u(0) = u_0$$

such that $(A, F) \in C^1(V, \mathcal{L}(X_1, X_0) \times X_0)$, where $X_1 \hookrightarrow X_0$ densely, $V \subset X_\gamma := (X_0, X_1)_{1-\frac{1}{p},p}$ open $1 < p < \infty$, $A(0)$ has maximal L^p-regularity, and $F(0) = 0$. Let

$$\mathcal{E} = \{u \in V \cap X_1 : A(u)u = F(u)\}.$$

Theorem (Prüß, Simonett, Zacher '09)

Assume that

- $\mathcal{E}$ *is a* C^1*-manifold of dimension* $m \in N_0$, $T_0\mathcal{E} = \mathcal{N}(A(0))$
- 0 *is a semi-simple eigenvalue, i.e.,* $\mathcal{N}(A(0)) \oplus \mathcal{R}(A(0)) = X_0$
- $\sigma(A(0)) \setminus \{0\} \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

Then 0 *is stable in* X_γ *and there is some* $\delta > 0$ *such that for every* $\|u_0\|_{X_\gamma} < \delta$ *there is some* $u_\infty \in \mathcal{E}$ *such that* $u(t) \to_{t\to\infty} u_\infty$ *exponentially.*

Remarks on the Proof

- Here $\Sigma = \partial B_{R_0}(x_0) \subset \Omega$ the set of equilibria

$$\mathcal{E} = \left\{(0,h) : h \in C^2(\Sigma), \theta_h(\Sigma) = \partial B_R(x) \subset \Omega, x \in \Omega, R > 0\right\}$$

is an $(d+1)$-dimensional manifold and $T_0\mathcal{E} = \mathcal{N}(A(0)) \cong \mathcal{N}(\mathcal{A}_\Sigma)$, $\mathcal{A}_\Sigma = \Delta_\Sigma + \frac{d-1}{R_0^2}$. Proofs: Similar to Escher & Simonett '98.

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- Since the linearized operators are defined on

$$L^2(0,\infty; L^2_\sigma(\Omega)) \times L^q(0,\infty; W_q^{1-\frac{1}{q}}(\Sigma))$$

  we do not apply the theorem directly, but modify its proof.
- The phase manifold for the evolution is given by

$$\mathcal{PM} = \left\{(u,h) \in H^1_0(\Omega)^d \times W_p^{4-\frac{4}{p}}(\Sigma) \colon \operatorname{div} u = F_d(u,h)\right\}$$

Weak Solutions – Definition

$$(v, \chi, \mu) \in L^2(0, T; H^1(\Omega)^d) \times L^\infty_{w*}(0, T; BV(\Omega)) \times L^2(0, T; H^1(\Omega))$$

is a weak solution of the Navier-Stokes/Mullins-Sekerka system if

$$\begin{aligned} \partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(\chi) Dv) + \nabla p = \mu \nabla \chi \qquad & \text{in } \mathcal{D}'(\Omega \times (0,\infty)), \\ \operatorname{div} v = 0 \qquad & \text{in } \mathcal{D}'(\Omega \times (0,\infty)), \\ \partial_t \chi + v \cdot \nabla \chi = m_0 \Delta \mu, \quad & \text{in } \mathcal{D}'(\Omega \times (0,\infty)), \end{aligned}$$

and $\frac{1}{\sigma}\mu|_{\partial^*\{\chi=1\}}$ is the generalized mean curvature of $\partial^*\{\chi = 1\}$, which is defined with the aid of inner variations.

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and $\frac{1}{\sigma}\mu|_{\partial^*\{\chi=1\}}$ is the generalized mean curvature of $\partial^*\{\chi=1\}$, which is defined with the aid of inner variations.

Theorem (A. & Röger '09)

Let $v_0 \in L^2_\sigma(\mathbb{T}^d), \chi_0 \in BV(\mathbb{T}^d;\{0,1\})$, $d = 2,3$, $T > 0$. Then there exists a weak solutions (v,χ,μ) of the Navier-Stokes/Mullins-Sekerka system with $\Omega = \mathbb{T}^d$. Moreover, $\mu|_{\partial^*\{\chi(t,.)=1\}} \in L^4(\mathbb{T}^d, d|\nabla\chi(t)|)$ and $\partial^*\{\chi(t,.)=1\}$ has generalized mean curvature $\frac{1}{\sigma}\mu$.

Note: If $m = 0$, existence of weak solution is open, cf. A. '07.

Proof: Semi-Implicit Time Discretization

Let $\chi_{k+1} = \chi_{E_{k+1}}$ be the minimizer of $F^h \colon BV(\mathbb{T}^d; \{0,1\}) \to \mathbb{R}$

$$F^h(\chi_E) = \sigma \mathcal{H}^{d-1}(\partial^* E) + \frac{1}{2h} \|\chi - \chi_k + h v_k \cdot \nabla \chi_k\|^2_{H^{-1}(\mathbb{T}^d)}$$

under the constraint $\int_\Omega \chi_E \, dx = |\Omega_0|$.

Moreover, let $v_{k+1} \in H^1_\sigma(\mathbb{T}^d)$ solve

$$\frac{1}{h}(v - v_k, \varphi)_{\mathbb{T}^d} + (v_k \cdot \nabla v, \varphi)_{\mathbb{T}^d} + (\nu(\chi_k) Dv, D\varphi)_{\mathbb{T}^d} = -(\chi_k \nabla \mu_k, \varphi)_{\mathbb{T}^d}$$

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Consequences:

1. Curvature equation:

$$\sigma H_{k+1} = \mu^0_{k+1} + \lambda_{k+1} \quad \text{on } \partial^* E_{k+1}, \tag{64}$$

   where $\mu^0_{k+1} := \Delta^{-1} \left(\frac{1}{h}(\chi_{k+1} - \chi_k) + v_k \cdot \nabla \chi_k \right)$.

2. Discrete (perturbed) energy estimate

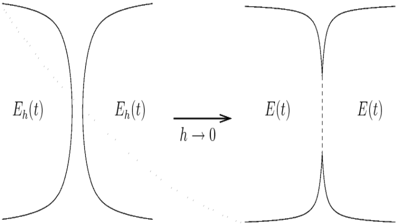
Main problem: Passing to the limit in mean curvature equation (64).

Proof: Passing to the Limit in Mean Curvature

Fundamental problem:

$$\nabla \chi_{E_h(t)} \rightharpoonup_{h\to\infty} \nabla \chi_{E(t)} \text{ in } \mathcal{D}'(\mathbb{T}^d)$$
$$|\nabla \chi_{E_h(t)}| \rightharpoonup^*_{h\to\infty} \theta(t) \quad \text{ in } \mathcal{M}(\mathbb{T}^d)$$

Then $|\nabla \chi_{E(t)}| \leq \theta(t)$.
But in general $|\nabla \chi_{E(t)}| \neq \theta(t)$!



Proof: Passing to the Limit in Mean Curvature

Fundamental problem:

$$\nabla \chi_{E_h(t)} \rightharpoonup_{h\to\infty} \nabla \chi_{E(t)} \text{ in } \mathcal{D}'(\mathbb{T}^d)$$
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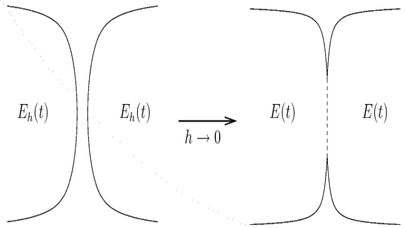

Schätzle '01 $\Rightarrow$ Since $\mu_h(t) \rightharpoonup_{h\to 0} \mu(t)$ in $H^1(\Omega)$, $\theta(t)$ is an integral varifold with weak mean curvature $H_{\theta(t)} \in L^4(d\theta(t))$ and $H_{\theta(t)} = \mu(t)\nu(t)$ holds $\theta(t)$-almost everywhere, with

$$\nu(t,\cdot) = \begin{cases} \frac{\nabla \chi_{E(t)}}{|\nabla \chi_{E(t)}|} & \text{on } \partial^* E(t), \\ 0 & \text{elsewhere.} \end{cases}$$

Overview of Analytic Results (Case of Same Densities)

Existence of local strong/global weak solutions:

	$m = 0$	$m > 0$
$\varepsilon = 0$	Classical Sharp Interface Model local strong solutions	Navier-Stokes/Mullins-Sekerka local strong & global weak sol.
$\varepsilon > 0$	Diffuse Interface Model local strong solutions	Diffuse Interface Model local strong & global weak sol.

NB: If $m = 0$, then existence of global weak solutions is open independent of $\varepsilon = 0$ or $\varepsilon > 0$! – So far only solutions in sense of general varifolds if $\varepsilon = 0$, cf. Plotnikov '93, A. '07 (Interfaces Free Bound.).

References:

$\varepsilon = m = 0$: Denisova & Solonnikov '91, Tanaka '93

$\varepsilon > 0, m > 0$: Starovoitov '93, Boyer '03, Feng '06, A. '07/'09

$\varepsilon = 0, m > 0$: A. & Röger '09, A. & Wilke '11

$\varepsilon > 0, m = 0$: A. & Terasawa '09

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