Implicitly constituted materials with fading memory

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Incompressible simple fluid

Truesdell and Noll (1965):

$$\mathbb{T} = -p\mathbb{I} + \mathfrak{F}_{s=0}^{+\infty}(\mathbb{C}_t\,(t - s))$$

Differential type models

General form, Rivlin and Ericksen (1955):

$$\mathbb{T} = -p\mathbb{I} + \mathfrak{f}(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots)$$

where

$$\mathbb{A}_1 = 2\mathbb{D}$$
$$\mathbb{A}_n = \frac{\mathrm{d}\mathbb{A}_{n-1}}{\mathrm{d}t} + \mathbb{A}_{n-1}\mathbb{L} + \mathbb{L}^\top \mathbb{A}_{n-1}$$

Coleman and Noll (1960): These models can be understood as successive approximations of the history functional

$$\mathbb{T} = -p\mathbb{I} + \mathfrak{F}_{s=0}^{+\infty}(\mathbb{C}_t(t-s))$$

with "fading memory".

Pressure dependent viscosity

$$\mathbb{T} = -p\mathbb{I} + 2\mu(p)\mathbb{D}$$

Stress dependent viscosity

$$\mathbb{T} = -p\mathbb{I} + \mu(\mathbb{T})\mathbb{D}$$

Seely (1964):

$$\mu(\mathbb{T}) = \mu_\infty + (\mu_0 - \mu_\infty)\, \mathrm{e}^{-\frac{|\mathbb{T}_\delta|}{\tau_0}}$$

Blatter (1995):

$$\mu(\mathbb{T}) = \frac{A}{\left(|\mathbb{T}_\delta|^2 + \tau_0^2\right)^{\frac{n-1}{2}}}$$

Matsuhisa and Bird (1965):

$$\mu(\mathbb{T}) = \frac{\mu_0}{1 + \alpha\, |\mathbb{T}_\delta|^{n-1}}$$

Implicit constitutive relation

Rajagopal (2003, 2006); Rajagopal and Srinivasa (2008):

$$f(\mathbb{T}, \mathbb{D}) = 0$$

Rate type models

$$\mathbb{T} = -\pi \mathbb{I} + \mathbb{S}$$

Oldroyd (1958):

$$\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \frac{\lambda_3}{2} \left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D} \right) + \frac{\lambda_5}{2} \left(\operatorname{Tr} \mathbb{S} \right) \mathbb{D} + \frac{\lambda_6}{2} \left(\mathbb{S} : \mathbb{D} \right) \mathbb{I} = -\mu \left(\mathbb{D} + \lambda_2 \overset{\triangledown}{\mathbb{D}} + \lambda_4 \mathbb{D}^2 + \frac{\lambda_7}{2} \left(\mathbb{D} : \mathbb{D} \right) \mathbb{I} \right)$$

Phan Thien (1978):

$$Y \mathbb{S} + \lambda \overset{\triangledown}{\mathbb{S}} + \frac{\lambda \xi}{2} \left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D} \right) = -\mu \mathbb{D}$$

$$Y = \mathrm{e}^{-\varepsilon \frac{\lambda}{\mu} \operatorname{Tr} \mathbb{S}}$$

Notation:

$$\overset{\triangledown}{\mathbb{b}^\flat} =_{\mathrm{def}} \frac{\mathrm{d} \mathbb{b}^\flat}{\mathrm{d} t} - \left[\nabla \mathbf{v} \right] \mathbb{b}^\flat - \mathbb{b}^\flat \left[\nabla \mathbf{v} \right]^\top$$

Materials with fading memory

Implicit algebraic relation:

$$f(\mathbb{T}, \mathbb{D}) = 0$$

Implicit relation between the histories:

$$\mathfrak{H}_{s=0}^{+\infty} \left(\mathbb{T}(t-s), \mathbb{C}_t\,(t-s) \right) = \mathbb{0}$$

Questions:

- Are rate type and differential type models (and other known models) special instances or approximations of the material with fading memory?
- Is something like the celebrated retardation theorem by Coleman and Noll (1960) available for implicit type materials with fading memory?

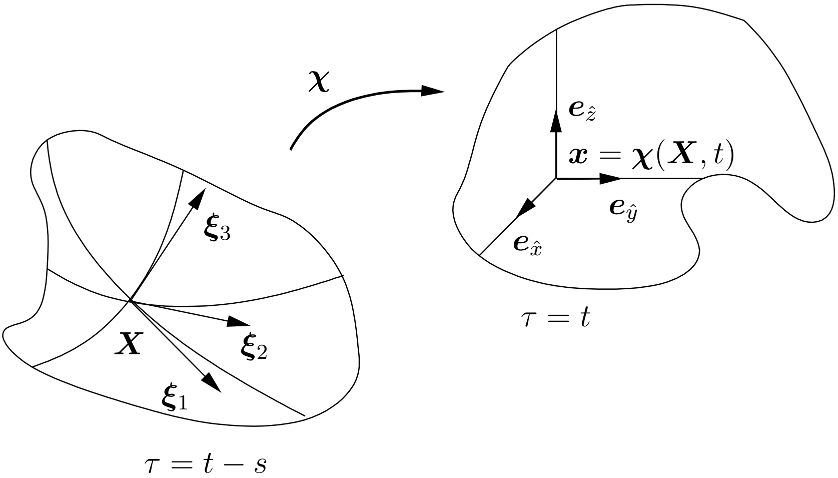
Independent variables

[...] the properties of a material element may depend upon the previous rheological states through which that element has passed, but not in any way on the states of neighbouring elements and not on the motion of the element as a whole in the space.

[...] only those tensor quantities need to be considered which have a significance for the material element independent of its motion as a whole in space.

Oldroyd (1950)

Convected coordinate system



Plan

- Formulate the constitutive relation in the *convected coordinate system*, Hencky (1925), Oldroyd (1950).
- Formulate the constitutive relation in an implicit form, Rajagopal (2003).
- Expand the functional using an analogue of the *retardation theorem*, Coleman and Noll (1960).
- Use *representation theorems* for isotropic linear and bilinear functions, Truesdell and Noll (1965).
- Transform the constitutive relation to a *fixed-in-space coordinate system*, Oldroyd (1950).

Constitutive assumptions

General relation:

$$\mathfrak{H}_{s=0}^{+\infty}\left(\mathbb{\gamma}(\xi, t-s), \mathbb{\pi}(\xi, t-s)\right) = \mathbb{0},$$

Constitutive assumptions

General relation:

$$\mathfrak{H}_{s=0}^{+\infty}\left(\mathbb{F}(\xi, t-s), \mathbb{T}(\xi, t-s)\right) = \mathbb{O},$$

Special form of the general constitutive relation:

$$\begin{aligned} \mathbb{T} &= -\pi \mathbb{I} + \mathbb{\sigma} \\ \mathbb{O} &= \mathfrak{G}_{s=0}^{+\infty}\left(\mathbb{F}(\xi, t-s) - \mathbb{I}, \mathbb{\sigma}(\xi, t-s)\right) \end{aligned}$$

Constitutive assumptions

General relation:

$$\mathfrak{H}_{s=0}^{+\infty}\left(\mathbb{b}(\xi, t-s), \mathbb{T}(\xi, t-s)\right) = \mathbb{0},$$

Special form of the general constitutive relation:

$$\begin{aligned} \mathbb{T} &= -\pi\mathbb{I} + \mathbb{\sigma} \\ \mathbb{0} &= \mathfrak{G}_{s=0}^{+\infty}\left(\mathbb{b}(\xi, t-s) - \mathbb{I}, \mathbb{\sigma}(\xi, t-s)\right) \end{aligned}$$

Constitutive assumption:

$$\left\|\mathbb{\sigma}(\xi, t-s)\right\|_S = O\left(\left\|\mathbb{b}(\xi, t-s) - \mathbb{I}\right\|_\Gamma\right) \text{ as } \left\|\mathbb{b}(\xi, t-s) - \mathbb{I}\right\|_\Gamma \to 0+,$$

Norm

[...] the deformations that occurred in the distant past should have less influence in determining the present stress than those that occurred in the recent past.

Truesdell and Noll (1965)

$$\left\| \mathbb{\gamma} \right\|_{L^2_h} =_{\mathrm{def}} \left(\int_{s=0}^{+\infty} \left| \mathbb{\gamma}(s) \right|^2 h(s) \, \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\left\| [\mathbb{\gamma}, \mathbb{\sigma}]^\top \right\|_{L^2_h \times L^2_h} =_{\mathrm{def}} \left(\left\| \mathbb{\gamma} \right\|^2_{L^2_h} + \left\| \mathbb{\sigma} \right\|^2_{L^2_h} \right)^{\frac{1}{2}}$$

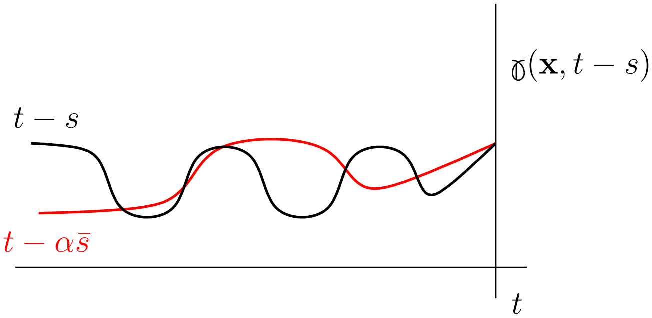
Taylor series for the functional

$$\mathbb{\pi} = -\pi \mathbb{I} + \mathbb{\sigma}$$
$$\mathbb{0} = \mathfrak{G}_{s=0}^{+\infty} \left(\mathbb{\gamma}(\xi, t-s) - \mathbb{I}, \mathbb{\sigma}(\xi, t-s) \right)$$

$$\mathfrak{G}_{s=0}^{+\infty} \left(\begin{bmatrix} \mathbb{\gamma}(\xi, t-s) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t-s) \end{bmatrix} \right) = A + B + C + o \left(\left\| \begin{bmatrix} \mathbb{\gamma}(\xi, t-s) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t-s) \end{bmatrix} \right\|_{L_h^2 \times L_h^2}^2 \right)$$

$$A = \mathfrak{G}_{s=0}^{+\infty} \left(\begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix} \right)$$
$$B = \delta \mathfrak{G}_{s=0}^{+\infty} \left(\begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix} \right) \begin{bmatrix} \mathbb{\gamma}(\xi, t-s) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t-s) \end{bmatrix}$$
$$C = \left[\mathbb{\gamma}(\xi, t-s) - \mathbb{I}, \mathbb{\sigma}(\xi, t-s) \right]^\top \delta^2 \mathfrak{G}_{s=0}^{+\infty} \left(\begin{bmatrix} \mathbb{0} \\ \mathbb{0} \end{bmatrix} \right) \begin{bmatrix} \mathbb{\gamma}(\xi, t-s) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t-s) \end{bmatrix}$$

Slow history



Taylor series for the metric tensor–stress tensor history

Informal expansion:

$$
\begin{aligned}
&\begin{bmatrix} \mathbb{g}(\xi, t - \alpha\bar{s}) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t - \alpha\bar{s}) \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{g}(\xi, t) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t) \end{bmatrix} - \alpha\bar{s} \begin{bmatrix} \frac{\mathrm{d}\mathbb{g}}{\mathrm{d}t}(\xi, t) \\ \frac{\mathrm{d}\mathbb{\sigma}}{\mathrm{d}t}(\xi, t) \end{bmatrix} + \frac{1}{2}\alpha^2\bar{s}^2 \begin{bmatrix} \frac{\mathrm{d}^2\mathbb{g}}{\mathrm{d}t^2}(\xi, t) \\ \frac{\mathrm{d}^2\mathbb{\sigma}}{\mathrm{d}t^2}(\xi, t) \end{bmatrix} + o\left(\alpha^2\right) \\
&= \underbrace{\begin{bmatrix} \mathbb{0} \\ \mathbb{\sigma}(\xi, t) \end{bmatrix}}_{\overset{(0)}{\mathbf{g}}} - \alpha \underbrace{\begin{bmatrix} \frac{\mathrm{d}\mathbb{g}}{\mathrm{d}t}(\xi, t) \\ \frac{\mathrm{d}\mathbb{\sigma}}{\mathrm{d}t}(\xi, t) \end{bmatrix}}_{\overset{(1)}{\mathbf{g}}} \bar{s} + \frac{1}{2}\alpha^2 \underbrace{\begin{bmatrix} \frac{\mathrm{d}^2\mathbb{g}}{\mathrm{d}t^2}(\xi, t) \\ \frac{\mathrm{d}^2\mathbb{\sigma}}{\mathrm{d}t^2}(\xi, t) \end{bmatrix}}_{\overset{(2)}{\mathbf{g}}} \bar{s}^2 + o\left(\alpha^2\right)
\end{aligned}
$$

Rigorous result:

$$
\lim_{\alpha \to 0+} \frac{1}{\alpha^2} \left\| \begin{bmatrix} \mathbb{g}(\xi, t - \alpha\bar{s}) - \mathbb{I} \\ \mathbb{\sigma}(\xi, t - \alpha\bar{s}) \end{bmatrix} - \left(\overset{(0)}{\mathbf{g}} - \alpha \overset{(1)}{\mathbf{g}} \bar{s} + \frac{1}{2}\alpha^2 \overset{(2)}{\mathbf{g}} \bar{s}^2 \right) \right\|_{L^2_h \times L^2_h} = 0
$$

Approximation for slow histories

Use constitutive assumption

$$\|\mathfrak{o}(\xi, t-s)\|_S = O\left(\|\mathfrak{d}(\xi, t-s) - \mathbb{I}\|_\Gamma\right) \text{ as } \|\mathfrak{d}(\xi, t-s) - \mathbb{I}\|_\Gamma \to 0+,$$

and substitute to the Taylor formula for the functional.
First order:

$$\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix} \mathfrak{d}(\xi, t-\alpha\bar{s}) - \mathbb{I} \\ \mathfrak{o}(\xi, t-\alpha\bar{s}) - \mathfrak{o}(\xi, t) \end{bmatrix}\right) = \mathfrak{f}_0\left(\overset{(0)}{\mathbf{g}}\right) + \mathfrak{f}_1\left(\overset{(1)}{\mathbf{g}}\right) + o(\alpha)$$

Second order:

$$\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix} \mathfrak{d}(\xi, t-\alpha\bar{s}) - \mathbb{I} \\ \mathfrak{o}(\xi, t-\alpha\bar{s}) - \mathfrak{o}(\xi, t) \end{bmatrix}\right) = \mathfrak{f}_0\left(\overset{(0)}{\mathbf{g}}\right) + \mathfrak{f}_1\left(\overset{(1)}{\mathbf{g}}\right) + \mathfrak{f}_2\left(\overset{(2)}{\mathbf{g}}\right)$$
$$+ \mathfrak{g}_{00}\left(\overset{(0)}{\mathbf{g}}, \overset{(0)}{\mathbf{g}}\right) + \mathfrak{g}_{10}\left(\overset{(1)}{\mathbf{g}}, \overset{(0)}{\mathbf{g}}\right) + \mathfrak{g}_{01}\left(\overset{(0)}{\mathbf{g}}, \overset{(1)}{\mathbf{g}}\right) + \mathfrak{g}_{11}\left(\overset{(1)}{\mathbf{g}}, \overset{(1)}{\mathbf{g}}\right) + o\left(\alpha^2\right)$$

Linear and bilinear tensor functions

Representation for isotropic linear functions:

$$\mathfrak{h}(\mathbb{A}) = a_1 \left(\operatorname{Tr} \mathbb{A}\right) \mathbb{I} + a_2 \mathbb{A}$$

Representation for isotropic bilinear functions:

$$\begin{aligned} \mathfrak{h}\left(\mathbb{A}, \mathbb{B}\right) = &\left(c_1 \operatorname{Tr} \mathbb{A} \operatorname{Tr} \mathbb{B} + c_2 \operatorname{Tr}\left(\mathbb{A}\mathbb{B}\right)\right) \mathbb{I} \\ &+ c_3 \left(\operatorname{Tr} \mathbb{A}\right) \mathbb{B} + c_4 \left(\operatorname{Tr} \mathbb{B}\right) \mathbb{A} + c_5 \left(\mathbb{A}\mathbb{B} + \mathbb{B}\mathbb{A}\right) \end{aligned}$$

Time derivatives with respect to fixed-in-space coordinate system

Oldroyd (1950):

$$\frac{\mathfrak{d} \mathrm{b}^k{}_i}{\mathfrak{d} t} =_{\mathrm{def}} \left(\frac{\partial \mathrm{b}^k{}_i}{\partial t} + \mathrm{v}^m \mathrm{b}^k{}_i|_m \right) - \mathrm{v}^k|_m \mathrm{b}^m{}_i + \mathrm{b}^k{}_m \mathrm{v}^m|_i,$$

$$\frac{\mathfrak{d} \mathrm{b}_{ki}}{\mathfrak{d} t} =_{\mathrm{def}} \left(\frac{\partial \mathrm{b}_{ki}}{\partial t} + \mathrm{v}^m \mathrm{b}_{ki}|_m \right) + \mathrm{v}^m|_k \mathrm{b}_{mi} + \mathrm{b}_{km} \mathrm{v}^m|_i,$$

$$\frac{\mathfrak{d} \mathrm{b}^{ki}}{\mathfrak{d} t} =_{\mathrm{def}} \left(\frac{\partial \mathrm{b}^{ki}}{\partial t} + \mathrm{v}^m \mathrm{b}^{ki}|_m \right) - \mathrm{v}^k|_m \mathrm{b}^{mi} - \mathrm{b}^{km} \mathrm{v}^i|_m,$$

Time derivatives with respect to fixed-in-space coordinate system

$$
\begin{aligned}
\frac{\mathfrak{d}\mathbb{b}}{\mathfrak{d}t} &= \frac{\mathrm{d}\mathbb{b}}{\mathrm{d}t} - [\nabla \mathbf{v}]\,\mathbb{b} + \mathbb{b}\,[\nabla \mathbf{v}]\,, \\
\frac{\mathfrak{d}\mathbb{b}^{\sharp}}{\mathfrak{d}t} &= \frac{\mathrm{d}\mathbb{b}^{\sharp}}{\mathrm{d}t} + [\nabla \mathbf{v}]^{\top}\,\mathbb{b}^{\sharp} + \mathbb{b}^{\sharp}\,[\nabla \mathbf{v}]\,, \\
\frac{\mathfrak{d}\mathbb{b}^{\flat}}{\mathfrak{d}t} &= \frac{\mathrm{d}\mathbb{b}^{\flat}}{\mathrm{d}t} - [\nabla \mathbf{v}]\,\mathbb{b}^{\flat} - \mathbb{b}^{\flat}\,[\nabla \mathbf{v}]^{\top}\,,
\end{aligned}
$$

Identification of the derivatives

$$\mathbb{\sigma}(\xi, t) \mapsto \mathbb{S}(\mathbf{x}, t)$$
$$\frac{\mathrm{d}\mathbb{\Phi}}{\mathrm{d}t}(\xi, t) \mapsto \mathbb{D}(\mathbf{x}, t)$$
$$\frac{\mathrm{d}^2\mathbb{\Phi}}{\mathrm{d}t^2}(\xi, t) \mapsto \overset{\triangledown}{\mathbb{D}}(\mathbf{x}, t)$$
$$\frac{\mathrm{d}\mathbb{\sigma}}{\mathrm{d}t}(\xi, t) \mapsto \overset{\triangledown}{\mathbb{S}}(\mathbf{x}, t)$$
$$\frac{\mathrm{d}^2\mathbb{\sigma}}{\mathrm{d}t^2}(\xi, t) \mapsto \overset{\triangledown\triangledown}{\mathbb{S}}(\mathbf{x}, t)$$

Approximation formulae

First order:

$$\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix}\mathfrak{F}(\xi, t-\alpha\bar{s})-\mathbb{I}\\ \sigma(\xi, t-\alpha\bar{s})-\sigma(\xi, t)\end{bmatrix}\right)$$

$$\mapsto b_0\left(\operatorname{Tr}\mathbb{S}\right)\mathbb{I}+b_1\mathbb{S}+2b_3\mathbb{D}+b_4\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)\mathbb{I}+b_5\overset{\triangledown}{\mathbb{S}}+o\left(\alpha\right)$$

Approximation formulae

Second order:

$$
\mathfrak{G}_{s=0}^{+\infty}\left(\begin{bmatrix}\mathfrak{b}(\xi, t-\alpha\bar{s}) - \mathbb{I} \\ \mathfrak{s}(\xi, t-\alpha\bar{s}) - \mathfrak{s}(\xi, t)\end{bmatrix}\right)
$$

$$
\mapsto \Big[b_0 \left(\operatorname{Tr}\mathbb{S}\right) + b_4 \left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_6 \left(\operatorname{Tr}\overset{\triangledown\triangledown}{\mathbb{S}}\right) + \left(b_{15} - 2b_8\right)\operatorname{Tr}\left(\mathbb{D}\right)^2
$$

$$
+ \left(b_{10}\left(\operatorname{Tr}\mathbb{S}\right)^2 + b_{11}\operatorname{Tr}\left(\mathbb{S}\right)^2\right) + \left(b_{18}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)^2 + b_{19}\operatorname{Tr}\left(\overset{\triangledown}{\mathbb{S}}\right)^2\right)
$$

$$
+ b_{23}\operatorname{Tr}\left(\mathbb{D}\overset{\triangledown}{\mathbb{S}}\right) + \left(b_{27}\operatorname{Tr}\mathbb{S}\operatorname{Tr}\overset{\triangledown}{\mathbb{S}} + b_{28}\operatorname{Tr}\left(\mathbb{S}\overset{\triangledown}{\mathbb{S}}\right)\right) + b_{33}\operatorname{Tr}\left(\mathbb{S}\mathbb{D}\right)\Big]\,\mathbb{I}
$$

$$
+ \left[b_1 + b_{12}\left(\operatorname{Tr}\mathbb{S}\right) + b_{30}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right)\right]\mathbb{S} + \left[b_3 + b_{25}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_{34}\left(\operatorname{Tr}\mathbb{S}\right)\right]\mathbb{D}
$$

$$
+ b_{13}\left(\mathbb{S}\right)^2 + b_{17}\left(\mathbb{D}\right)^2 + b_{36}\left(\mathbb{S}\mathbb{D} + \mathbb{D}\mathbb{S}\right)
$$

$$
+ b_9\overset{\triangledown}{\mathbb{D}} + \left[b_5 + b_{20}\left(\operatorname{Tr}\overset{\triangledown}{\mathbb{S}}\right) + b_{29}\left(\operatorname{Tr}\mathbb{S}\right)\right]\overset{\triangledown}{\mathbb{S}} + b_{21}\left(\overset{\triangledown}{\mathbb{S}}\right)^2
$$

$$
+ b_{26}\left(\mathbb{D}\overset{\triangledown}{\mathbb{S}} + \overset{\triangledown}{\mathbb{S}}\mathbb{D}\right) + b_{31}\left(\mathbb{S}\overset{\triangledown}{\mathbb{S}} + \overset{\triangledown}{\mathbb{S}}\mathbb{S}\right) + b_7\overset{\triangledown\triangledown}{\mathbb{S}} + o\left(\alpha^2\right)
$$

Rate type models

$$\mathbb{T} = -\pi \mathbb{I} + \mathbb{S}$$

Oldroyd (1958):

$$\mathbb{S} + \lambda_1 \overset{\triangledown}{\mathbb{S}} + \frac{\lambda_3}{2} \left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D} \right) + \frac{\lambda_5}{2} \left(\operatorname{Tr} \mathbb{S} \right) \mathbb{D} + \frac{\lambda_6}{2} \left(\mathbb{S} : \mathbb{D} \right) \mathbb{I}$$
$$= -\mu \left(\mathbb{D} + \lambda_2 \overset{\triangledown}{\mathbb{D}} + \lambda_4 \mathbb{D}^2 + \frac{\lambda_7}{2} \left(\mathbb{D} : \mathbb{D} \right) \mathbb{I} \right)$$

Phan Thien (1978):

$$Y \mathbb{S} + \lambda \overset{\triangledown}{\mathbb{S}} + \frac{\lambda \xi}{2} \left(\mathbb{D}\mathbb{S} + \mathbb{S}\mathbb{D} \right) = -\mu \mathbb{D}$$
$$Y = \mathrm{e}^{-\varepsilon \frac{\lambda}{\mu} \operatorname{Tr} \mathbb{S}}$$

Notation:

$$\overset{\triangledown}{\mathbb{b}^\flat} =_{\mathrm{def}} \frac{\mathrm{d}\mathbb{b}^\flat}{\mathrm{d}t} - \left[\nabla \mathbf{v} \right] \mathbb{b}^\flat - \mathbb{b}^\flat \left[\nabla \mathbf{v} \right]^\top$$

Conclusion

- Implicit constitutive relations provide a general concept that can accommodate both differential and rate type models. Generalization of the concept of simple fluid.
- Rate type models for viscoelastic materials can be seen as special instances of a general material with fading memory. (After a rigorous approximation procedure.)
- This is only a proof of concept—there are better ways how to derive implicit type constitutive relations that are consistent with the laws of thermodynamics.

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