The degree theorem

We prove the following result on irreducible reps of finite groups over $\mathbb{C}$

<u>Theorem</u> Let $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ be irreducible and G finite. Then ~~do~~ the degree of φ divides $|G|$.

We present two proofs. The first one uses some very basic commutative algebra

Let $\mathbb{A} = \{ \alpha \in \mathbb{C} \mid \exists h \in \mathbb{Z}[x]$ monic, such that $h(\alpha) = 0 \}$

<u>Fact</u>: (a) $\mathbb{A}$ is a subring of $\mathbb{C}$

(b) $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$

The ring $\mathbb{A}$ is called the <u>ring of algebraic integers</u>.

Probably you know (b) in more general form as Every Gauss domain is integrally closed.

A quick proof of (b): let $q = a/b$, where $a, b \in \mathbb{Z}$, $b > 0$ $GCD(a,b) = 1$. If $q \in \mathbb{A}$ there are $c_{d-1}, \dots, c_0 \in \mathbb{Z}$ such that $q^d + c_{d-1} q^{d-1} + \dots + c_1 q + c_0 = 0$, thus

$$a^d = -b \cdot (c_{d-1} a^{d-1} b^0 + c_{d-2} a^{d-2} \cdot b + \dots + c_1 a b^{d-2} + c_0 b^{d-1}),$$ in particular,

$b \mid a^d$ and since $GCD(a,b) = 1$, we get $b = 1$, $q \in \mathbb{Z}$ □

The strategy of the proof of the Theorem is this:

Let $C_1, C_2, \dots, C_k$ be the conjugacy classes of G, $G = C_1 \dot\cup C_2 \dot\cup \cdots \dot\cup C_k$; In each C_i choose some $g_i \in C_i$

Since φ is irreducible, the orthogonality relations of characters (the first one) give $\sum_{g \in G} \chi_\varphi(g)\chi_\varphi(g^{-1}) = |G|$

This can be rewritten as $\sum_{i=1}^{k} |C_i| \chi_\varphi(g_i)\chi_\varphi(g_i^{-1}) = |G|$

Therefore $\dfrac{|G|}{d} = \sum_{i=1}^{k} \dfrac{|C_i|\chi_\varphi(g_i)}{d} \chi_\varphi(g_i^{-1})$, where $d = \chi_\varphi(1_G)$ is the degree of φ. Our aim is to prove that the sum on the RHS is in $\mathbb{A}$. Then $\dfrac{|G|}{d} \in \mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

<u>Lemma</u> If G is a finite group, $\psi \in \mathrm{Rep}_{\mathbb{C}}(G)$ of finite degree, then $\chi_\psi(g) \in \mathbb{A}$ $\forall g \in G$

Proof: If $d = \chi_\psi(1_G)$ is the degree of ψ, then we know $\chi_\psi(g) = \lambda_1 + \lambda_2 + \dots + \lambda_d$, where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $\psi(g)$; moreover $\lambda_i^{|G|} = 1$ $\forall i \in \{1, \dots, d\}$

Hence $\lambda_1, \dots, \lambda_d \in \mathbb{A}$ and since $\mathbb{A}$ is closed under $+$, $\chi_\psi(g) \in \mathbb{A}$

$\square$

The quantity $\frac{|C_i| \chi_\varphi(g_i)}{d}$ looks familiar, see the part about structural constants of $Z(\mathbb{C}G)$ from last notes.

We know that $\forall 1 \leq i \leq k$ $\sum_{g \in C_i} \varphi(g) = \lambda_i \cdot \mathrm{Id}$ for some $\lambda_i \in \mathbb{C}$ (to be consistent with the previous notation we should write λ_i^φ, but now we care only about one rep), where $\lambda_i = \frac{|C_i| \chi_\varphi(g_i)}{d}$, where $d = \chi_\varphi(1_G)$ is the degree of φ. All we have to prove is $\lambda_i \in \mathbb{A}$

<u>Lemma</u> Using the introduced notation $\lambda_i \in \mathbb{A}$ $\forall 1 \leq i \leq k$

Proof: The point is that every eigenvalue of a matrix with coefficients in $\mathbb{Z}$ is in $\mathbb{A}$. Indeed, if $M \in M_n(\mathbb{Z})$ then $\det(M - x \cdot E) \in \mathbb{Z}[x]$ has the leading coefficient ± 1, so every eigenvalue of M is in $\mathbb{A}$.

If $h_{i,j,l}$ are the structural constants of $Z(\mathbb{C}G)$ introduced last time, then $h_{i,j,l} = |\{(x,y) \mid x \in C_i, y \in C_j, xy = g_l\}| \in \mathbb{Z}$

$$\lambda_i \cdot \lambda_j = \sum_{l=1}^{k} h_{i,j,l} \lambda_l \quad \forall 1 \leq i, j \leq k$$

$$\vec{\lambda} := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}, \quad H_i := \begin{array}{c c} j & \begin{pmatrix} \quad h_{i,j,l} \quad \end{pmatrix}_{1 \le j,l \le k} \\ & l \end{array} \in M_k(\mathbb{Z})$$

Note that relations $\lambda_i \cdot \lambda_j = \sum_{l=1}^{k} h_{i,j,l} \lambda_l \quad 1 \le i,j \le k$

can be written in the matrix form

$$H_i \cdot \vec{\lambda} = \lambda_i \cdot \vec{\lambda},$$ Note that if $g_t = 1_G$, then

$$\lambda_t = \frac{|C_t| \cdot \chi_\varphi(g_t)}{d} = 1$$

Since $\vec{\lambda} \neq 0$, λ_i is an eigenvalue of H_i.

In particular $\lambda_1, \dots, \lambda_k \in \mathbb{A}$ $\square$

<u>Remark</u>: Note that $\vec{\lambda}$ is a common eigenvector ~~to~~ of $H_1, \dots, H_k$. This is a key observation if we want to show that the character table of G can be reconstructed from the structural constants $h_{i,j,l}$. In fact, we do not want to show it here but in the exercises since it is not going to be a subject of the exam.

<u>Proof of the Theorem (summary)</u>:

Start with $\displaystyle \frac{|G|}{d} = \sum_{i=1}^{k} \frac{|C_i| \chi_\varphi(g_i)}{d} \chi_\varphi(g_i^{-1}) = \sum_{i=1}^{k} \lambda_i \chi_\varphi(g_i^{-1})$

By lemmas $\lambda_i, \chi_\varphi(g_i^{-1}) \in \mathbb{A}$, so by Fact ⓐ $\frac{|G|}{d} \in \mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$ $\square$

The second proof uses elementary lemma on abelian groups – next time.