Problem session April 17, 2023

Character table from the structural constants

Recall the notation from the lecture: G is a finite group, $C_1, \dots, C_k$ its conjugacy classes $G = \dot{\cup}_{i=1}^k C_i$. Choose $g_i \in C_i$ for each i. Then for every $i, j, l \in \{1, \dots, k\}$

$$h_{i,j,l} := |\{(x, y) \mid x \in C_i, y \in C_j, xy = g_l\}| .$$

That is, $h_{i,j,l}$ gives the number of ways how can be a given element from C_l written as a product of an element from C_i and an element from C_j.

Let $b_i = \sum_{g \in C_i} \delta_g$, $1 \leq i \leq k$. We know, that $\{b_1, \dots, b_k\}$ is a basis of $Z(\mathbb{C}G)$ and

$$b_i * b_j := \sum_{l=1}^{k} h_{i,j,l} b_l$$

The numbers $h_{i,j,l}$ give the structure of $Z(\mathbb{C}G)$. Important point: The structure of $Z(\mathbb{C}G)$ is actually given by k. Indeed $Z(\mathbb{C}G) \simeq \mathbb{C}^k$ as $\mathbb{C}$-algebras. So when thinking about $h_{i,j,l}$ as constants describing the multiplication in $Z(\mathbb{C}G)$ we should be aware that this description is with respect to the basis $\{b_1, \dots, b_k\}$. But still, it is a bit surprising that information stored in $h_{i,j,l}$ is the same as information stored in the character table of G. In the lecture notes we showed how to compute $h_{i,j,l}$ from the character table of G over $\mathbb{C}$. The goal of this exercise is to find a way how to construct complex character table from numbers $h_{i,j,l}$.

1. Assume that the value of $h_{i,j,l}$ is known for all $1 \leq i, j, l \leq k$ but no other information about G is available.

a) Show we can find $1 \leq t \leq k$ such that $C_t = \{1_G\}$.

b) Show that for every $1 \leq i \leq k$ we can find $1 \leq j \leq t$ such that C_j contains g_i^{-1}.

c) Show that we can find $|C_1|, |C_2|, \dots, |C_k|$.

Solution: a) Assume $C_t = \{1_G\}$. Then $h_{t,l,t} = 1$ if $l = t$ and $h_{t,l,t} = 0$ if $l \neq t$. If $t' \neq t$, then $h_{t',t,t'} = 1$. So t is determined by $h_{t,l,t} \neq 0 \Rightarrow l = t$.

b), c) Assume $1 \leq t \leq k$ is given and $C_t = \{1_G\}$. Then $h_{i,j,t}$ is nonzero only if C_j contains g_i^{-1}. Indeed, if $xy = 1$ where $x \in C_i$ and $y \in C_j$, then there exists $g \in G$ such that $gxg^{-1} = g_i$. Then $(gxg^{-1})(gyg^{-1}) = 1_G$, so $g_i^{-1} = gyg^{-1} \in C_j$. On the other hand if C_j contains g_i^{-1}, then $h_{i,j,t} = |C_i|$.

2. For every $1 \leq i \leq k$ let H_i be a $k \times k$ matrix over $\mathbb{Z}$ whose element in the position (j, l) is $h_{i,j,l}$, $1 \leq j, l \leq k$. Show that $\mathbb{C}^k$ contains exactly k one-dimensional spaces which are eigenspaces to all H_i, $1 \leq i \leq k$. That is, the set

$$\{\langle u \rangle \mid 0 \neq u \in \mathbb{C}^k, H_i(u) \in \langle u \rangle \ \forall 1 \leq i \leq k\}$$

has k elements.

Solution: Recall the relation $b_i * b_j = \sum_{l=1}^{k} h_{i,j,l} b_l$. For every $b \in Z(\mathbb{C}G)$ consider $L_b \in \mathrm{End}_{\mathbb{C}}(Z(\mathbb{C}G))$ given by $L_b(x) = b * x, x \in Z(\mathbb{C}G)$. Then H_i^T is the matrix of L_{b_i} with respect to basis $\{b_1, \dots, b_k\}$. So if $\langle (u_1, \dots, u_k)^T \rangle$ is a common eigenspace of $H_1^T, \dots, H_k^T$, then $v := \sum_{i=1}^{k} u_i b_i$ is a common eigenvector of $L_{b_1}, \dots, L_{b_k}$. Note that L_b is a linear combination of $L_{b_1}, \dots, L_{b_k}$ for every $b \in Z(\mathbb{C}G)$, so v satisfies $L_b(v) \in \langle v \rangle$ for every $b \in Z(\mathbb{C}G)$. That is, $b * \langle v \rangle \subseteq \langle v \rangle$ which exactly means that $\langle v \rangle$ is an ideal of $Z(\mathbb{C}G)$.

Conversely if $\langle v' \rangle$ is an ideal of $Z(\mathbb{C}G)$ of dimension 1, then $[v']_B \in \mathbb{C}^k$ is a common eigenvector of $H_1^T, \dots, H_k^T$.

Since $Z(\mathbb{C}G) \simeq \mathbb{C}^k$ as $\mathbb{C}$-algebras we can easily compute the number of 1-dimensional ideals of $\mathbb{C}^k$. Recall that every ideal of $\mathbb{C}^k$ is of the form $I_1 \times I_2 \times \cdots \times I_k$, where $I_i = 0$ or $I_i = \mathbb{C}$ for every $1 \le i \le k$. Therefore there are exactly k ideals of $\mathbb{C}^k$ of dimension 1 (those having only one I_i in the product nonzero).

So far we have proved that there exists a basis $v_1, v_2, \dots, v_k$ of $\mathbb{C}^k$ such that

- $H_i^T v_j \in \langle v_j \rangle$ for every $1 \le i, j \le k$.
- Each common eigenvector of $H_1^T, \dots, H_k^T$ is in $\cup_{j=1}^{k} \langle v_j \rangle$.

Let $V \in \mathrm{GL}(k, \mathbb{C})$ be the matrix with columns $v_1, \dots, v_k$. Then there are diagonal matrices $D_1, \dots, D_k$ such that $H_i^T V = V D_i$ for every $1 \le i \le k$. Then $V^{-1} H_i^T V = D_i$ and also $V^T H_i (V^{-1})^T = D_i$. If $w_1, \dots, w_k$ are the columns of $(V^{-1})^T$ then $\{w_1, \dots, w_k\}$ is a basis of $\mathbb{C}^k$ and $H_i w_j \in \langle w_j \rangle$ for every $1 \le i, j \le k$.

Finally if $w \in \mathbb{C}^k$ is a common eigenvector of $H_1, \dots, H_k$, there exists a matrix W whose first column is w and for each $1 \le i \le k$ there exists a diagonal matrix D_i' such that $W^{-1} H_i W = D_i'$. Then each column of $(W^{-1})^T$ is a common eigenvector of $H_1^T, \dots, H_k^T$. Therefore there exists a permutation matrix P and a diagonal matrix D such that $(W^{-1})^T = VPD$. Therefore $W = (V^{-1})^T (P^{-1})^T D^{-1}$. Observe that columns of $(V^{-1})^T (P^{-1})^T$ are just $w_1, \dots, w_k$ possibly written in a different order. Therefore $w \in \cup_{i=1}^{k} \langle w_i \rangle$.

3. Look at the lecture notes and find formulae for common eigen vectors of $H_1, \dots, H_k$. Conclude we can find a character table of G over $\mathbb{C}$ provided only the value of $h_{i,j,l}$ is known for every $1 \le i, j, l \le k$.

Solution: Let $\varphi_1, \dots, \varphi_k \in \mathrm{Rep}_{\mathbb{C}}(G)$ be a complete list of different irreducible representations of G over $\mathbb{C}$. For each $1 \le i \le k$ let $\chi_i := \chi_{\varphi_i}$ and $d_i := \chi_i(1_G)$. That is, d_i is the degree of φ_i.

For $1 \le j \le k$ define $\vec{\lambda}_j := (\lambda_1^{\varphi_j}, \dots, \lambda_k^{\varphi_j})^T \in \mathbb{C}^k$, where $\lambda_i^{\varphi_j} := \frac{|C_i| \chi_j(g_i)}{d_j}$. As written on page 4 of the lecture notes with title *The degree theorem* for every $1 \le i, j \le k$, $\vec{\lambda}_j$ is an eigenvector of H_i with eigenvalue $\lambda_i^{\varphi_j}$.

The regularity of the character table implies that $\vec{\lambda}_1, \vec{\lambda}_2, \dots, \vec{\lambda}_k$ are linearly independent.

As noted in Exercise 2., $\mathbb{C}^k$ contains exactly k one-dimensional spaces $\langle u_1 \rangle, \dots, \langle u_k \rangle$ such that $H_i \langle u_j \rangle \subseteq \langle u_j \rangle$ for every $1 \le i, j \le k$. Therefore when a suitable labeling of $u_1, \dots, u_k$ is chosen, we may assume $\langle u_j \rangle = \langle \vec{\lambda}_j \rangle$ for every $1 \le j \le k$.

It follows that the set $\{\langle\lambda_1\rangle, \dots, \langle\lambda_k\rangle\}$ is determined by matrices $H_1, \dots, H_k$. Therefore $\vec{\lambda}_1, \dots, \vec{\lambda}_k$ are determined by $H_1, \dots, H_k$ up to a scalar multiple. (In theory, if we find a basis of $\mathbb{C}^k$ such that every element of this basis is a common eigenvector of $H_1, \dots, H_k$, then the set $\{\vec{\lambda}_1, \dots, \vec{\lambda}_k\}$ can be obtained by adjusting elements of the basis by suitable scalars.) If $t \in \{1, \dots, k\}$ is such that $C_t = \{1_G\}$ then $\lambda_t^{\varphi_j} = 1$. It follows that $\{\vec{\lambda}_1, \dots, \vec{\lambda}_k\}$ is determined by $H_1, \dots, H_k$. Since $|C_1|, \dots, |C_k|$ are known, we can divide the i-th row of each $\vec{\lambda}_j$ by $|C_i|$ for every $1 \le i \le t$. Therefore if $A = (a_{i,j})_{1 \le i,j \le k}$ is a complex character table of G such that $a_{i,j} = \chi_{\varphi_i}(g_j)$, the matrix $A' := \mathrm{diag}(\frac{1}{d_1}, \dots, \frac{1}{d_k})A$ is determined by matrices $H_1, \dots, H_k$. To get A it is sufficient to compute degrees $d_1, \dots, d_k$ from entries of $A' = (a'_{i,j})_{1 \le i,j \le k}$. Recall, that the irreducibility of φ_i implies $\sum_{g \in G} \chi_i(g)\overline{\chi_i(g)} = |G|$, or

$$\sum_{j=1}^{k} |C_j| d_i^2 a'_{i,j} \overline{a'_{i,j}} = |G|, d_i^2 = \frac{|G|}{\sum_{j=1}^{k} |C_j||a'_{i,j}|^2}$$

Remark: Most of the arguments works for an algebraically closed field whose characteristic does not divide $|G|$. The last two equations of Exercise 3. have to be changed. For example, we may use Exercise 1b) instead.

4. Show that there exists an algorithm which for a given group produces its character table over $\mathbb{C}$ (do not care about the efficiency of the algorithm, just show the existence).

Solution: For a given group, structural constants $h_{i,j,l}$ can be found. Note that it is enough to find the set of vectors $\{\vec{\lambda}_1, \dots, \vec{\lambda}_k\}$ from the previous exercise. The coordinates of these vectors are eigenvalues of matrices $H_1, \dots, H_k$. So if we can find the set $S := \{\lambda \in \mathbb{C} \mid \exists\, 1 \le i \le k$ such that λ is an eigenvalue of $H_i\}$, then we can find the set $\{\vec{\lambda}_1, \dots, \vec{\lambda}_k\}$ (for this exercise for example check all $|S|^k$ vectors by brute force; of course, much better attitudes can be found). Note that if $\zeta = e^{2\pi \mathrm{i}/|G|}$, then $\mathbb{Q}[\zeta]$ is a splitting field of characteristic polynomials of $H_1, \dots, H_k$. The Galois group $\mathrm{Gal}(\mathbb{Q}[\zeta] \mid \mathbb{Q})$ is commutative, so solvable, and the set S can be found. There is more elementary way to find S: $\lambda_i^{\varphi_j} = \frac{|C_i|\chi_j(g_i)}{d_j}$, where $|C_i|$ is known, d_j divides $|G|$, so there are only finitely many possibilities for d_j, and $\chi_j(g_i)$ is a sum of d_j elements from the set $\{\zeta^t \mid 0 \le t \le |G|\}$. So there are only finitely many possible values for $\lambda_i^{\varphi_j}$, so in theory, we can find all eigenvalues of H_i by brute force.

Properties determined by the complex character table

5. Let G be a finite group, N its normal subgroup and let $X = \{\theta \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \theta(N) = 1\}$. Show that $N = \cap_{\theta \in X} \mathrm{Ker}\ \theta$. Conclude that the lattice of normal subgroups is determined by the complex character table of G.

Solution:

Recall a result from the lecture: If G is a finite group, then $\cap_{\theta \in \mathrm{Hom}(G,\mathbb{C}^*)} \mathrm{Ker}\ \theta = 1$. Apply this result to G/N to get $\cap_{\theta \in \mathrm{Hom}(G/N,\mathbb{C}^*)} \mathrm{Ker}\ \theta = 1$. The homomorphism theorem gives that $X = \{\theta\pi \mid \theta \in \mathrm{Hom}(G/N, \mathbb{C}^*)\}$, where $\pi\colon G \to G/N$

is the canonical projection. Hence if $g \in \cap_{\theta \in X} \mathrm{Ker}\ \theta$, then $\pi(g) \in N$ and consequently $\cap_{\theta \in X} \mathrm{Ker}\ \theta \subseteq N$. The opposite inclusion is obvious.

6. Use the character table of the quaternion group to find its normal subgroups.

Solution: Recall the complex character table of the quaternion group

Q	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

where $\chi_1, \ldots, \chi_5$ are characters of irreducible representations $\varphi_1, \ldots, \varphi_5$. Recall that the kernel of an irreducible complex representation consists of those conjugacy classes on which the character gives the value equal to the degree of the representation. In our notation

$$\mathrm{Ker}\ \varphi_1 = Q, \mathrm{Ker}\ \varphi_2 = \{1, -1, i, -i\}, \mathrm{Ker}\ \varphi_3 = \{1, -1, j, -j\}$$

$$\mathrm{Ker}\ \varphi_4 = \{1, -1, k, -k\}, \mathrm{Ker}\ \varphi_5 = \{1\}$$

Now we have to consider all possible intersections of these groups. Note that $\mathrm{Ker}\ \varphi_i \cap \mathrm{Ker}\ \varphi_j = \{1, -1\}$ whenever $2 \le i \ne j \le 4$. By the previous exercise, any normal subgroup of Q is an intersection of kernels of irreducible complex representations of Q. Since the set of subgroups

$$\{1\}, \{1, -1\}, \{1, -1, i, -i\}, \{1, -1, j, -j\}, \{1, -1, k, -k\}, Q$$

is closed under intersection these are all the normal subgroups of Q.

7. Show that it is possible to decide whether a finite group G is nilpotent from the complex character table of G.

Solution: Recall that G is nilpotent if and only if $G/Z(G)$ is nilpotent. Hence it is enough to show that the complex character table of G determines the complex character table of $G/Z(G)$. Let $\pi \colon G \to G/Z(G)$ be canonical projection. If φ' is an irreducible complex representation of $G/Z(G)$, then $\varphi = \varphi'\pi$ is an irreducible complex representation of G. Conversely, if φ is an irreducible representation of G over $\mathbb{C}$ containing $Z(G)$ in the kernel, then there exists an irreducible representation φ' of $G/Z(G)$ such that $\varphi = \varphi'\pi$.

Let A be the character table of G. Assume that $\varphi_1, \ldots, \varphi_k$ are the complex irreducible representations of G used to create this table, i.e., the i'th row of A describes χ_{φ_i}. Also $C_1, \ldots, C_k$ are the conjugacy classes of G such that j-th column of A shows character values on C_j.

Since we can detect $Z(G)$ from the character table of G (as the union of conjugacy classes of size 1) and also the kernels of $\varphi_1, \ldots, \varphi_k$ are determined by this character table, we can find set $I := \{i \in \{1, \ldots, k\} \mid Z(G) \subseteq \mathrm{Ker}\ \varphi_i\}$.

Let A' be the matrix of size $|I| \times k$ created from A by deleting the rows which are not indexed by elements from I. We have created a table describing characters of complex irreducible representations of $G/Z(G)$. If $g \in C_i$, $h \in C_j$ then A' have equal columns indexed by i and j, then $gZ(G)$ and $hZ(G)$ are conjugated in $G/Z(G)$.

Therefore if we delete repeating columns from A' we obtain complex character table of $G/Z(G)$.