Problem session April 3, 2023

The aim of the series is to construct the character table of irreducible complex representations of A_5.

1. Show A_5 has 5 conjugacy classes. We denote them C_1, C_2, C_3, C_4, C_5, where $g_1 := \mathrm{id} \in C_1$, $g_2 := (1,2)(3,4) \in C_2$, $g_3 := (1,2,3) \in C_3$, $g_4 := (1,2,3,4,5) \in C_4$, $g_5 = (1,3,5,2,4) \in C_5$. Determine $|C_1|, \ldots, |C_5|$. Show that the formula $d_1^2+d_2^2+d_3^5+d_4^2+d_5^2 = 60$ determines degrees d_1, d_2, d_3, d_4, d_5 of irreducible representations of A_5 over $\mathbb{C}$ - the irreducible representations have degrees $1, 3, 3, 4, 5$. (We will not need this, but in general it can help with searching for irreducible representations.)

Solution: Consider the action of S_5 on A_5 by conjugation: $\pi * x = \pi x \pi^{-1}, \pi \in S_5, x \in A_5$. There are four orbits of this action each consists of all even permutations having the same type of cyclic decomposition. Note that if $x \in A_5$ and the stabilizer of x contains an odd permutation, then the orbit of x is also the conjugacy class of A_5. It is easy to check that the stabilizers of elements g_1, g_2, g_3 contain an odd permutation. Therefore $C_1 = \{g_1\}$, C_2 contains all permutations which are products of two independent transpositions, C_3 contains all 3-cycles. With 5-cycles the situation is different: The stabilizer $\{\pi \in S_5 \mid \pi * g_4 = g_4\}$ has size 5, so the stabilizer of g_4 contains only powers of g_4. Then $\{\pi \in S_5 \mid \pi * g_4 = g_5\}$ contains only odd permutations. So the orbit of 5-cycles splits as a union of 2 conjugacy classes of A_5.

We compute $|C_1| = 1$, $|C_2| = 5*3 = 15$, $|C_3| = 10*2 = 20$, $|C_4| = 60/5 = 12$ and $|C_5| = |C_4|$.

For searching positive integers $d_1 \le d_2 \le d_3 \le d_4 \le d_5$ satisfying $d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 60$, notice that at least one d_i has to be even and considering the equation modulo 4 we conclude that exactly one d_i has to be even or all d_i's are even. The later can be relatively easily eliminated by the brute force. So assume that exactly one d_i is even. Then by a brute force argument we check $d_5 \le 5$. Now consider the equation modulo 6 and conclude that exactly two d_i's are 3. Then necessarily $d_5 = 5$, consequently $d_4 = 4$ and $d_1 = 1$.

2. A group is perfect, if $G = [G,G]$ (so a simple group is perfect if it is not commutative). Show that a finite perfect group has only 1 matrix representation of degree 1 over any field.

Solution: If $\varphi\colon G \to \mathbb{F}^*$ is a degree one representation, then Im φ is a commutative group and Ker φ has to contain $[G,G]$. If G is perfect, then $\varphi(G) = 1$, so φ has to be trivial.

3. Construct an irreducible complex representation of A_5 of degree four. Compute its character.

Solution: After many instances of the idea we met so far only very briefly: A_5 naturally acts on $\{1,2,3,4,5\}$ and this action induces a representaion $\varphi\colon A_5 \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^5)$. In $\mathbb{C}^5$ there are two interesting φ-invariant subspaces $V = \{(x,x,x,x,x)^T \mid x \in \mathbb{C}\}$ and $W = \{(x_1,x_2,x_3,x_4,x_5)^T \in \mathbb{C}^5 \mid x_1+x_2+x_3+x_4+x_5 = 0\}$. The corresponding representations φ_W and φ_V have degrees 4 and 1.

The representation φ_V is trivial and it is a general fact that the character of φ is determined by the number of fixed points of the permutation. If the notation of 1. is used, then

$$\chi_\varphi(g_1) = 5, \chi_\varphi(g_2) = 1, \chi_\varphi(g_3) = 2, \chi_\varphi(g_4) = \chi_\varphi(g_5) = 0\,.$$

Since φ is a direct sum of φ_W and φ_V, $\chi_{\varphi_W} = \chi_\varphi - \chi_{\varphi_V}$.

$$\chi_{\varphi_W}(g_1) = 4, \chi_{\varphi_W}(g_2) = 0, \chi_{\varphi_W}(g_3) = 1, \chi_{\varphi_W}(g_4) = \chi_{\varphi_W}(g_5) = -1\,.$$

The irreducibility of φ_W can be tested via its character:

$$\sum_{i=1}^{5} |C_i|\chi_{\varphi_W}(g_i)\overline{\chi_{\varphi_W}(g_i)} = 1*16+15*0+20*1+12*1+12*1 = 60 = |A_5|\,.$$

4. Consider X to be the set of 2-elements subsets of $\{1,\ldots,5\}$. In particular $|X| = 10$. A_5 has a canonical action on X: $\pi * \{i,j\} := \{\pi(i),\pi(j)\}$, where $\pi \in A_5$ and $\{i,j\} \in X$. This action induces a representation of A_5 of degree 10, let $\varphi\colon A_5 \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^{10})$ be this representation. Recall, we can compute $\chi_\varphi(\pi)$ as the cardinality of $\{x \in X \mid \pi * x = x\}$.

a) Compute χ_φ.

b) Check $\sum_{g\in A_5} \chi_\varphi(g)\chi_\varphi(g^{-1}) = 180$.

c) Show that φ is equivalent to a direct sum of 3 irreducible representations of degrees $1, 4, 5$.

d) Compute the character of the 5-dimensional irreducible representation of A_5.

Solution: a) For each of g_1, g_2, g_3, g_4, g_5 we have to find the number of elements of X fixed by this elements. Of course, every $x \in X$ is fixed by g_1. Further $g_2 * \{i,j\} = \{i,j\}$ only for $\{i,j\} = \{1,2\}$ or $\{i,j\} = \{3,4\}$ and $g_3 * \{i,j\} = \{i,j\}$ only for $\{i,j\} = \{4,5\}$. No elements of X are fixed by the action of g_4 or g_5. Therefore

$$\chi_\varphi(g_1) = 10, \chi_\varphi(g_2) = 2, \chi_\varphi(g_3) = 1, \chi_\varphi(g_4) = 0, \chi_\varphi(g_5) = 0\,.$$

b) Just a computation

$$\sum_{i=1}^{5} |C_i|\chi_\varphi(g_i)\overline{\chi_\varphi(g_i)} = 100 + 15*4 + 20*1 + 12*0 + 12*0 = 180\,.$$

c) Recall how the multiplicities are detected by characters. Assume that $\varphi_1, \ldots, \varphi_5$ are pair-wise non-equivalent irreducible representations of A_5 over $\mathbb{C}$

and assume φ is equivalent to a direct sum of $\overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{n_1} \oplus \overbrace{\varphi_2 \oplus \cdots \oplus \varphi_2}^{n_2} \oplus \cdots \oplus \overbrace{\varphi_5 \oplus \cdots \oplus \varphi_5}^{n_5}$, where $n_1, n_2, n_3, n_4, n_5 \in \mathbb{N}_0$. Then

$$n_i = \frac{1}{|A_5|} \sum_{g \in A_5} \chi_\varphi(g) \overline{\chi_{\varphi_i}(g)}$$

$$\sum_{i=1}^{5} n_i^2 = \frac{1}{|A_5|} \sum_{g \in A_5} \chi_\varphi(g) \overline{\chi_\varphi(g)}$$

From b) we know that $\sum_{i=1}^{5} n_i^2 = 3$. Therefore $n_1, n_2, n_3, n_4, n_5 \in \{0, 1\}$ and φ is a direct sum of three non-equivalent irreducible representations. Let φ_1 be the trivial representation.

$$\sum_{g \in A_5} \chi_\varphi(g) \overline{\chi_{\varphi_1}(g)} = 10 + 15 * 2 + 20 * 1 + 12 * 0 + 12 * 0 = 60$$

It follows that $n_1 = 1$. Assume φ_2 is the representation of degree 4 found in exercise 3. Then

$$\sum_{g \in A_5} \chi_\varphi(g) \overline{\chi_{\varphi_2}(g)} = 40 + 15 * 0 + 20 * 1 + 12 * 0 + 12 * 0 = 60 \,.$$

Then $n_2 = 1$. Therefore φ is equivalent to $\varphi_1 \oplus \varphi_2 \oplus \psi$, where ψ is an irreducible representation of degree 5. To compute its character we use $\chi_\varphi = \chi_{\varphi_1} + \chi_{\varphi_2} + \chi_\psi$. Therefore

$$\chi_\psi(g_1) = 5, \chi_\psi(g_2) = 1, \chi_\psi(g_3) = -1, \chi_\psi(g_4) = \chi_\psi(g_5) = 0 \,.$$

Let us summarize the results. The part of the character table of A_5 we know is

	C_1	C_2	C_3	C_4	C_5
φ_1	1	1	1	1	1
φ_4	4	0	1	-1	-1
φ_5	5	1	-1	0	0
φ_3	3	x_2	x_3	x_4	x_5
φ_3'	3	y_2	y_3	y_4	y_5

5. It remains to determine $x_2, \ldots, x_5, y_2, \ldots, y_5$. There are various ways to do it. Recall what we have - orthogonality relations (two types), also we know $\mathrm{reg} \simeq \varphi_1 \oplus \varphi_3 \oplus \varphi_3 \oplus \varphi_3 \oplus \varphi_3' \oplus \varphi_3' \oplus \varphi_3' \oplus \varphi_4 \oplus \varphi_4 \oplus \varphi_4 \oplus \varphi_4 \oplus \varphi_5 \oplus \varphi_5 \oplus \varphi_5 \oplus \varphi_5 \oplus \varphi_5$.

Solution: Recall that the character of the regular representation of A_5 satisfies

$$\chi_{\mathrm{reg}}(g_1) = 60, \chi_{\mathrm{reg}}(g_2) = 0, \chi_{\mathrm{reg}}(g_3) = 0, \chi_{\mathrm{reg}}(g_4) = 0, \chi_{\mathrm{reg}}(g_5) = 0 \,.$$

Therefore

$$1*1+4*0+5*1+3*x_2+3*y_2=0, 1*1+4*1+5*(-1)+3*x_3+3*y_3=0$$

$$1*1+4*(-1)+5*0+3*x_4+3*y_4=0, 1*1+4*(-1)+5*0+3*x_5+3*y_5=0$$

We get $x_2+y_2=-2$, $x_3+y_3=0$, $x_4+y_4=1$, $x_5+y_5=1$.

The second orthogonality relations imply that $||c_i||^2=|G|/|C_i|$, where c_i is the i-th column of the character table. Therefore

$$2+x_2^2+y_2^2=60/15, 3+x_3^2+y_3^2=60/20, 2+x_4^2+y_4^2=60/12, 2+x_5^2+y_5^2=60/12\,.$$

Start with $x_2+y_2=-2$, $x_2^2+y_2^2=2$. This system has only one solution $x_2=y_2=-1$.

Consider $x_3+y_3=0$, $x_3^2+y_3^2=0$. This system has only one solution $x_3=y_3=0$.

The system $x_4+y_4=-1$, $x_4^2+y_4^2=3$. This system has has two solutions $x_4=\frac{1+\sqrt{5}}{2}, y_4=\frac{1-\sqrt{5}}{2}$ and $x_4=\frac{1-\sqrt{5}}{2}, y_4=\frac{1+\sqrt{5}}{2}$.

Similarly, $x_5+y_5=-1$, $x_5^2+y_5^2=3$ has two solutions. Since the rows of the character table have to be different if we set $x_4=\frac{1+\sqrt{5}}{2}, y_4=\frac{1-\sqrt{5}}{2}$ we have to put $x_5=\frac{1-\sqrt{5}}{2}, y_5=\frac{1+\sqrt{5}}{2}$. If we set $x_4=\frac{1-\sqrt{5}}{2}, y_4=\frac{1+\sqrt{5}}{2}$, we have to put $x_5=\frac{1+\sqrt{5}}{2}, y_5=\frac{1-\sqrt{5}}{2}$. Note that both possibilities give essentially the same character table (the last two rows are of the table are switched). With the former choice of x_4, y_4, the resulting table is

	C_1	C_2	C_3	C_4	C_5
φ_1	1	1	1	1	1
φ_4	4	0	1	-1	-1
φ_5	5	1	-1	0	0
φ_3	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
φ_3'	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$