Tensor products

A brief summary of the theory Assume V, W are vector spaces over $\mathbb{F}$. Their tensor product $V \otimes_{\mathbb{F}} W$ is an $\mathbb{F}$-vector space together with an $\mathbb{F}$-bilinear map $b: V \times W \to V \otimes_{\mathbb{F}} W$ such that for every $\mathbb{F}$-space X and every $\mathbb{F}$-bilinear map $b': V \times W \to X$ there exists unique $\mathbb{F}$-linear map $c: V \otimes_{\mathbb{F}} W \to X$ such that $b' = cb$. The value of $b(v, w)$ is denoted by $v \otimes w$.

The standard argument gives that $V \otimes_{\mathbb{F}} W$ is essentially unique. It can be constructed from bases: If B_V is a basis of V and B_W is a basis of W, define $V \otimes_{\mathbb{F}} W$ as a vector space with basis $B_V \times B_W$. The map b is then defined by

$$b(\sum_{b \in B_V} c_b b, \sum_{b' \in B_W} d_{b'} b') := \sum_{b \in B_V, b' \in B_W} c_b d_{b'} (b, b')\,.$$

Working with particular realization of $V \otimes_{\mathbb{F}} W$ can have more disadvantages than advantages. Usually we think about $V \otimes_{\mathbb{F}} W$ as the most general $\mathbb{F}$-vector space with the following properties:

a) every element of $V \otimes W$ is a sum (or a linear combination) of elements from the set $\{v \otimes w \mid v \in V, w \in W\}$

b) if $v_1, v_2 \in V$, $w \in W$, then $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

c) if $v \in V$, $w_1, w_2 \in W$, then $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

d) if $v \in V, w \in W$, $t \in \mathbb{F}$, then $(tv) \otimes w = t(v \otimes w) = v \otimes (tw)$.

Assume V_1, V_2, W_1, W_2 are $\mathbb{F}$-spaces and $\alpha \in \mathrm{Hom}_{\mathbb{F}}(V_1, W_1)$ and $\beta \in \mathrm{Hom}_{\mathbb{F}}(V_2, W_2)$ are given. If $b_1: V_1 \times W_1 \to V_1 \otimes_{\mathbb{F}} W_1$ and $b_2: V_2 \times W_2 \to V_2 \otimes_{\mathbb{F}} W_2$ are corresponding tensor products, then the unique $c: V_1 \otimes_{\mathbb{F}} W_1 \to V_2 \otimes_{\mathbb{F}} W_2$ such that $b_2(\alpha \times \beta) = cb_1$ is denoted by $c =: \alpha \otimes \beta$. $\alpha \otimes \beta$ is characterized by $(\alpha \otimes \beta)(v_1 \otimes w_1) = \alpha(v_1) \otimes \beta(w_1)$.

1. Assume G is a finite group, $\varphi: G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $\psi: G \to \mathrm{Aut}_{\mathbb{F}}(W)$ are representations of G over $\mathbb{F}$. We can define $\varphi \otimes \psi: G \to \mathrm{Aut}(V \otimes_{\mathbb{F}} W)$ by the obvious formula $\varphi \otimes \psi: g \mapsto \varphi(g) \otimes \psi(g)$. Verify that $\varphi \otimes \psi$ is indeed a representation of G and prove that $\chi_{\varphi \otimes \psi} = \chi_\varphi \cdot \chi_\psi$ provided φ and ψ are representations of finite degree.

Solution: First of all we should check that $\varphi(g) \otimes \psi(g)$ is an automorphism of $V \otimes_{\mathbb{F}} W$ for every $g \in G$ (actually tensor product of automorphisms is an automorphism). Note that $[(\varphi(g) \otimes \psi(g)) \circ (\varphi(h) \otimes \psi(h))](v \otimes w) = [\varphi(g) \circ \varphi(h)] \otimes [\psi(g) \circ \psi(h)](v \otimes w) = \varphi(gh)(v) \otimes \psi(gh)(w) = [\varphi(gh) \otimes \psi(gh)](v \otimes w)$. Since $\{v \otimes w \mid v \in V, w \in W\}$ generate $V \otimes W$, $(\varphi(g) \otimes \psi(g)) \circ (\varphi(h) \otimes \psi(h)) = \varphi(gh) \otimes \psi(gh)$. A similar computation shows that $\varphi(1_G) \otimes \psi(1_H) = \mathrm{id}_{V \otimes W}$. Then $\varphi(g) \otimes \psi(g)$ is an automorphism of $V \otimes_{\mathbb{F}} W$ with inverse $\varphi(g^{-1}) \otimes \psi(g^{-1})$.

So $\varphi \otimes \psi$ is a representation of G. To compute its character we need some basis of $V \otimes W$. Let B be a basis of V, C be a basis of W. Then $D := \{b \otimes c \mid$

$b \in B, c \in C\}$ is a basis of $V \otimes_{\mathbb{F}} W$. Let $g \in G$. Write $[\varphi(g)](b) = \sum_{b' \in B} s_{b'} b'$ $[\psi(g)](c) = \sum_{c' \in C} t_{c'} c'$. Then $[\varphi(g) \otimes \psi(g)] = \sum_{b' \otimes c' \in D} s_{b'} t_{c'} b' \otimes c'$. Therefore the diagonal entry of the matrix $[\varphi(g) \otimes \psi(g)]_D$ in the position given by $b \otimes c$ is $s_b t_c$. Therefore if $[\varphi(g)]_B = (s_{i,j})_{1 \le i,j \le |B|}$ and $[\psi(g)]_C = (t_{i,j})_{1 \le i,j \le |C|}$, then

$$\chi_{\varphi}(g) = \sum_{i=1}^{|B|} s_{i,i}, \chi_{\psi}(g) = \sum_{j=1}^{|C|} t_{j,j},$$

$$\chi_{\varphi \otimes \psi}(g) = \sum_{1 \le i \le |B|, 1 \le j \le |C|} s_{i,i} t_{j,j} = \chi_{\varphi}(g) . \chi_{\psi}(g)$$

Therefore $\chi_{\varphi \otimes \psi} = \chi_{\varphi} . \chi_{\psi}$.

2. Assume that G and H are groups $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ $\psi\colon H \to \mathrm{Aut}_{\mathbb{F}}(W)$ their representations over $\mathbb{F}$. We can use tensor product to define a representation $\varphi \otimes \psi\colon G \times H \to \mathrm{Aut}_{\mathbb{F}}(V \otimes_{\mathbb{F}} W)$ by

$$(g, h) \mapsto \varphi(g) \otimes \psi(h) .$$

Prove that

a) If φ or ψ is not irreducible, then $\varphi \otimes \psi$ is not irreducible.

b) Assume that G, H are finite groups, $\varphi_1\colon G \to \mathrm{Aut}_{\mathbb{F}}(V_1)$, $\varphi_2\colon G \to \mathrm{Aut}_{\mathbb{F}}(V_2)$, $\psi_1\colon H \to \mathrm{Aut}_{\mathbb{F}}(W_1)$ $\psi_2\colon H \to \mathrm{Aut}_{\mathbb{F}}(W_2)$ their irreducible representations over $\mathbb{F}$ such that $\varphi_1 \otimes \psi_1$ and $\varphi_2 \otimes \psi_2$ are irreducible representations of $G \times H$ over $\mathbb{F}$ (we will discuss on this later). Assume $\mathbb{F}$ algebraically closed, char $\mathbb{F}$ does not divide $|G||H|$. Prove that $\varphi_1 \otimes \psi_1$ and $\varphi_2 \otimes \psi_2$ are equivalent representations of $G \times H$ if and only if φ_1 and φ_2 are equivalent representations of G and ψ_1 and ψ_2 are equivalent representations of H.

c) Consider two finite groups G, H over an algebraically closed field $\mathbb{F}$ of characteristic 0. Assume that $\varphi_1\colon G \to \mathrm{Aut}_{\mathbb{F}}(V_1), \dots, \varphi_k\colon G \to \mathrm{Aut}_{\mathbb{F}}(V_k)$ is the list of all different irreducible representations over $\mathbb{F}$ up to equivalence and $\psi_1\colon H \to \mathrm{Aut}_{\mathbb{F}}(W_1), \dots, \psi_l\colon H \to \mathrm{Aut}_{\mathbb{F}}(W_l)$ is the list of all different irreducible representations over $\mathbb{F}$ up to equivalence. Show that $\varphi_i \otimes \psi_j, 1 \le i \le k, 1 \le j \le l$ is a list of all different irreducible representations of $G \times H$ over $\mathbb{F}$ up to equivalence.

Solution: a) Assume that $0 \ne V' \subsetneq V$ is a φ-invariant subspace of V. Then the subspace $U \le V \otimes_{\mathbb{F}} W$ generated by $\{v \otimes w \mid v \in V', w \in W\}$ is easily checked to be $\varphi \otimes \psi$-invariant subspace of $V \otimes_{\mathbb{F}} W$. To see that $U \ne V \otimes_{\mathbb{F}} W$ we can check $f \otimes g$ where $f\colon V \to \mathbb{F}$ is a nonzero linear form satisfying $f(V') = 0$, $g\colon W \to \mathbb{F}$ is a nonzero linear form. Then $f \otimes g\colon V \otimes_{\mathbb{F}} W \to \mathbb{F} \otimes_{\mathbb{F}} \mathbb{F} \simeq \mathbb{F}$ is a nonzero linear form whose kernel contains U. Similarly, we check $0 \ne U$ (it is also possible to use bases of $V \otimes_{\mathbb{F}} W$).

b) It is a general fact that if $\varphi_1 \simeq \varphi_2$ and $\psi_1 \simeq \psi_2$, then $\varphi_1 \otimes \psi_1 \simeq \varphi_2 \otimes \psi_2$ (here $\simeq$ means equivalence of representations). Indeed, assume $\theta_1 \in$

$\mathrm{Hom}_{\mathbb{F}}(V_1, V_2)$ is an isomorphism such that $\theta_1\varphi_1(g) = \varphi_2(g)\theta_1$ for every $g \in G$. Similarly, let $\theta_2 \in \mathrm{Hom}_{\mathbb{F}}(W_1, W_2)$ be an isomorphism such that $\theta_2\psi_1(h) = \psi_2(h)\theta_2$ for every $h \in H$. Then $\theta := \theta_1 \otimes \theta_2 \in \mathrm{Hom}_{\mathbb{F}}(V_1 \otimes_{\mathbb{F}} W_1, V_2 \otimes_{\mathbb{F}} W_2)$ is an isomorphism such that

$$\theta[(\varphi_1 \otimes \psi_1)(g, h)] = [(\varphi_2 \otimes \psi_2)(g, h)]\theta, (g, h) \in G \times H$$

For the converse assume for example that φ_1 and φ_2 are not equivalent. The assumptions of the exercise imply that $\frac{1}{|G|}\sum_{g\in G} \chi_{\varphi_1}(g)\chi_{\varphi_2}(g^{-1}) = 0$. Now compute

$$\frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_{\varphi_1\otimes\psi_1}((g, h))\chi_{\varphi_2\otimes\psi_2}((g^{-1}, h^{-1})) =$$

$$\frac{1}{|G||H|} \sum_{g\in G,h\in H} \chi_{\varphi_1}(g)\chi_{\psi_1}(h)\chi_{\varphi_2}(g^{-1})\chi_{\psi_2}(h^{-1}) =$$

$$\frac{1}{|H|}\sum_{h\in H} \chi_{\psi_1}(h)\chi_{\psi_2}(h^{-1})\frac{1}{|G|}(\sum_{g\in G} \chi_{\varphi_1}(g)\chi_{\varphi_2}(g^{-1})) = 0\,.$$

The orthogonality relations (note we assume $\varphi_1 \otimes \psi_1$ and $\varphi_2 \otimes \psi_2$ irreducible) imply that $\varphi_1 \otimes \psi_1$ and $\varphi_2 \otimes \psi_2$ are not equivalent.

c) Observe that there are k conjugacy classes in G and l conjugacy classes in H. Therefore there are kl conjugacy classes in $G \times H$. The theory imply that it is sufficient to check

(i) $\varphi_i \otimes \psi_j$ is irreducible for every $1 \le i \le k, 1 \le j \le l$

(ii) If $\varphi_i \otimes \psi_j$ and $\varphi_{i'} \otimes \psi_{j'}$ are equivalent, for some $1 \le i, i' \le k$, $1 \le j, j' \le l$, then $i = i'$ and $j = j'$.

(i) Since φ_i, ψ_j are irreducible, we have $\sum_{g\in G} \chi_{\varphi_i}(g)\chi_{\varphi_i}(g^{-1}) = |G|$ and $\sum_{h\in H} \chi_{\psi_j}(h)\chi_{\psi_j}(h^{-1}) = |H|$. Then

$$\sum_{(g,h)\in G\times H} \chi_{\varphi_i\otimes\psi_j}(g, h)\chi_{\varphi_i\otimes\psi_j}(g^{-1}, h^{-1}) = \sum_{g\in G,h\in H} \chi_{\varphi_i}(g)\chi_{\psi_j}(h)\chi_{\varphi_i}(g^{-1})\chi_{\psi_j}(h^{-1}) =$$

$$\sum_{g\in G} \chi_{\varphi_i}(g)\chi_{\varphi_i}(g^{-1}) \sum_{h\in H} \chi_{\psi_j}(h)\chi_{\psi_j}(h^{-1}) = |H| \sum_{g\in G} \chi_{\varphi_i}(g)\chi_{\varphi_i}(g^{-1}) = |G||H|$$

From the lecture we know that this imply $\varphi_i \otimes \psi_j$ is irreducible.

(ii) was already proved in part b)

Remark: Note that once we know all irreducible representations of finite cyclic groups over $\mathbb{C}$, we can use tensor products in the sense of Exercise 2. to determine all irreducible complex representations of any finite commutative group.