Problem session March 20, 2023

Complex representations of groups of order 8

Recall that that 8-element groups up to isomorphism are $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_8 and Q. D_8 is the group of symmetries of square and Q is the quaternion group.

We know irreducible representations over $\mathbb{C}$ for the first three groups in the list. Try to find all different irreducible representations of D_8 and Q over $\mathbb{C}$. You can follow the strategy we used for A_4:

1. Compute how many irreducible representations we have to find.
2. Find all representations of degree 1.
3. Try to determine degrees of the remaining representations.
4. Guess the remaining representations and show they are irreducible and pair-wise non-equivalent.

Solution: Let us start with $D_8 = \{\mathrm{id}, r, r^2, r^3, \sigma, r, \sigma r^2, \sigma r^3\}$, where r is a rotation of the square $\pi/2$ degrees (for example counter clockwise) and σ is some reflection (for example a reflection whose axis joins opposite vertices). The multiplication in the group is determined by relations $\sigma r \sigma = r^{-1} = r^3, \sigma^2 = 1$. The conjugacy classes of D_8 are $\{1\}$, $\{r, r^3\}$, $\{r^2\}$, $\{\sigma, \sigma r^2\}$, $\{\sigma r, \sigma r^3\}$. Therefore we have to find 5 non-equivalent irreducible representations.

$N := [D_8, D_8]$ is the smallest normal subgroup of D_8 inducing abelian factor of D_8. The center $Z(D_8) = \{1, r^2\}$ has to be such a subgroup since all four element groups are commutative. Note that $D_8/N = \langle rN \rangle \times \langle \sigma N \rangle$. Homomorphisms $\mathrm{Hom}(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^*)$ were described in the lecture, we can use them to find complex representations of degree 1. There are 4 of them, their values on the cosets of N indicates the table

	N	rN	σN	$\sigma r N$
φ_1	1	1	1	1
φ_2	1	1	-1	-1
φ_3	1	-1	1	-1
φ_4	1	-1	-1	1

Recall two different representations of degree 1 are not equivalent, so we have to find one more irreducible representation of degree $d > 1$.

$|D_8| = 8 = 1^2 + 1^2 + 1^2 + 1^2 + d^2$, so $d = 2$. The natural candidate is a geometric realization of D_8 as a subgroup of $\mathrm{GL}(2, \mathbb{R})$. Imagine a square in the plane, whose vertices are $(1, 0), (0, 1), (-1, 0), (0, -1)$. Elements of D_8 are represented by matrices of the corresponding transformations, for example

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, r^2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, r^3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma r \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma r^2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma r^3 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This assignment gives a representation $\varphi_5: D_8 \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^2)$. We have to check that this representation is irreducible. If $\mathbb{C}^2$ has non-trivial φ-invariant subspace, this subspace is one-dimensional. But this means that 8 matrices representing elements of D_8 have a common eigen vector. Note that matrices representing σ and σr^2 have common eigen vectors $(1,0)^T$ and $(0,1)^T$ and matrices representing σr and σr^3 have common eigenvectors $(1,1)^T$ and $(-1,1)^T$. From this it is obvious that 4 matrices representing reflections have no eigenvector in common.

Conclusion: $\varphi_1, \ldots, \varphi_5$ is the list of all different irreducible representations of D_8 over $\mathbb{C}$ up to equivalence.

Similar approach works also for the group $Q = \{1, -1, i, -i, j, -j, k, -k\}$. The unit of the group is 1, for the Cayley table of the group see for example Wikipedia. The important properties are: -1 has order 2, $x^2 = -1$ for every $x \in \{i, -i, j, -j, k, -k\}$, -1 is a central element of Q, $ij = k$. The conjugacy classes of Q are $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$. Thus we have to find 5 irreducible representations.

The argument we used for D_8 shows that $[Q, Q] = \{1, -1\}$ and we have the following representations of degree 1

	$\{1,-1\}$	$\{i,-i\}$	$\{j,-j\}$	$k,-k$
φ_1	1	1	1	1
φ_2	1	1	-1	-1
φ_3	1	-1	1	-1
φ_4	1	-1	-1	1

Again, we are left to guess one representation of degree $d = \sqrt{8-1-1-1-1} = 2$.

The natural candidate is hidden in the quaternion algebra $\mathbb{H}$. This is a 4-dimensional $\mathbb{R}$-algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathsf{i} \oplus \mathbb{R}\mathsf{j} \oplus \mathbb{R}\mathsf{k}$, the relations $\mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = -1, \mathsf{ijk} = -1$[1] hold in $\mathbb{H}$. Note that $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathsf{i}$ is a subring of $\mathbb{H}$ (but it is not contained in $Z(\mathbb{H})$, therefore we cannot say that $\mathbb{H}$ is a $\mathbb{C}$-algebra). Consider $\mathbb{H}$ as a right vector space over $\mathbb{C}$, say $\mathbb{H} = \mathbb{C} \oplus \mathsf{j}\mathbb{C}$. Every $g \in Q$ induces a $\mathbb{C}$-linear map on $\mathbb{H}$, $l_g(h) = \mathsf{g}h$. The relations for computations in $\mathbb{H}$ give that the assignment $\varphi_5: g \mapsto l_g$ is a representation of Q. For example, since $\mathsf{ijk} = -1$ in $\mathbb{H}$, we have $l_i l_j l_k = l_{-1} = -\mathrm{id}$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{H})$.

We can find a matrix form of φ_5 with respect to basis $B = \{1, \mathsf{j}\}$. $l_i(1) = \mathsf{i}, l_i(j) = \mathsf{k} = \mathsf{j}(-\mathsf{i})$, so $[\varphi_5(i)]_B = \begin{pmatrix} \mathsf{i} & 0 \\ 0 & -\mathsf{i} \end{pmatrix}$. Similarly $l_j(1) = \mathsf{j} = \mathsf{j}.1, l_j(\mathsf{j}) =$

[1]Now I realized, that 1 in $\mathbb{H}$ and 1 in Q are denoted by the same symbol. I am very sorry.

$-1 = 1.(-1)$, so $[\varphi_5(\mathsf{j})]_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $l_k(1) = \mathsf{k} = \mathsf{j}(-\mathsf{i})$, $l_k(\mathsf{j}) = -\mathsf{i} = 1.(-\mathsf{i})$, so $[\varphi_5(k)]_B = \begin{pmatrix} 0 & -\mathsf{i} \\ -\mathsf{i} & 0 \end{pmatrix}$. The matrix form of φ_5 with respect to B is

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \mapsto \begin{pmatrix} \mathsf{i} & 0 \\ 0 & -\mathsf{i} \end{pmatrix}, -i \mapsto \begin{pmatrix} -\mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix},$$

$$j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} 0 & -\mathsf{i} \\ -\mathsf{i} & 0 \end{pmatrix}, -k \mapsto \begin{pmatrix} 0 & \mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix},$$

Again we check, that these matrices do not have a common eigenvector, so φ_5 is irreducible.

Conclusion: $\varphi_1, \dots, \varphi_5$ is the list of all different irreducible representations of Q over $\mathbb{C}$ up to equivalence.

Complex representations of S_4

a) Determine how many distinct (up to equivalence) irreducible representations of S_4 over $\mathbb{C}$ exist.

b) Show that S_3 is a homomorphic image of S_4. Use this fact to find an irreducible representation of S_4 over $\mathbb{C}$ of degree 2.

c) Consider a cube in $\mathbb{R}^3$ centered in the origin. Use the rotation group of the cube to find a representation $\varphi\colon S_4 \to \mathrm{GL}(3, \mathbb{R})$. Compute its character and check that φ is irreducible even if it is considered as a complex representation of S_4

d) Show that the representation $\mathrm{sgn} \otimes \varphi$ irreducible over $\mathbb{C}$ and it is not equivalent to φ.

e) Recall we already know another representation of S_4 given by canonical action of S_4 on $\mathbb{C}^4$ and we also know that the action on invariant subspace $\{(x_1, x_2, x_3, x_4)^T \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$ gives an irreducible representation of S_4 over $\mathbb{C}$. Decide whether this representation is equivalent to the representation found in c)

Solution: a) It is sufficient to count conjugacy classes in S_4 which are $C_1 := \{\mathrm{id}\}$, $C_2 := \{2-\text{cycles}\}$, $C_3 := \{3-\text{cycles}\}$, $C_4 := \{4-\text{cycles}\}$, $C_5 := \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. So there are 5 up to equivalence different irreducible complex representations of S_4.

b) Recall $V := C_1 \cup C_5$ is a normal subgroup of S_4 and the factor S_4/V has 6 elements. Recall there are only 2 groups of order 6 up to an isomorphism, namely $\mathbb{Z}_6$ and S_3. Since S_4 does not contain an element of order ≥ 4, $\mathbb{Z}_6$ cannot be a factor of S_4. Hence S_4/V has to be isomorphic to S_3.

It can be useful to find such an isomorphism: Consider the canonical embedding $\iota\colon S_3 \to S_4$ (based on $\{1,2,3\} \subseteq \{1,2,3,4\}$) and the canonical projection $\pi\colon S_4 \to S_4/V$. Observe that $\iota(S_3) \cap V = \{\mathrm{id}\}$, hence the composition

$\pi\iota: S_3 \to S_4/V$ is a monomorphism and, since both groups are of order 6, $\pi\iota$ is even an isomorphism. Therefore if $\varphi: S_3 \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a complex representation of S_3, then $\varphi' := \varphi \circ (\pi\iota)^{-1} \circ \pi: S_4 \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a complex representation of S_4. It follows easily from the definition that φ is irreducible if and only if φ' is irreducible (note that irreducibility of a representation actually depends only on the image of the representation).

So if $\varphi_2: S_3 \to \mathrm{Aut}_{\mathbb{C}}(V)$ is an irreducible representation of degree 2 (see for example the lecture), then $\varphi_2': S_4 \to \mathrm{Aut}_{\mathbb{C}}(V)$ is an irreducible complex representation of S_4 of degree 2. If $\sigma \in S_4$, the value of $\varphi_2'(\sigma)$ is computed as follows: Find $i := \sigma(4) \in \{1,2,3,4\}$. Let $\tau \in V$ be such that $\tau(4) = (i)$ (so $\tau = \{\mathrm{id}\}$ if $i = 4$ and $\tau = (i,4)(k,l) \in C_5, \{i,k,l\} = \{1,2,3\}$ if $i \neq 4$). Then $\tau\sigma(4) = 4$, so we may consider $\tau\sigma$ as a permutation of S_3. Hence $\varphi_2'(\sigma) = \varphi_2'(\tau\sigma) = \varphi_2(\tau\sigma)$.

c) Let A, B, C, D, E, F, G, H be the 'usual' labeling of vertices of the cube, i.e., $X := \{AG, BH, CE, DF\}$ is the set of diagonals of the cube. Recall the rotations of the cube, i.e. isometries of $\mathbb{R}^3$ preserving the position of the cube, are the following:

(a) identity

(b) rotations by π around axes connecting centers of opposite eges of the cube (in the considered labeling of the vertices, the opposite edges are AB and GH,BC and EH, CD and EF, AD and FG, BF and DH, CG and AE).

(c) rotations by $\pm 2\pi/3$ around the diagonals

(d) rotations by $\pm\pi/4$ around the axes connecting centers of the opposite faces

(e) rotations by $\pi/2$ arround the axes of connecting centers of the opposite faces

Let Γ be the set of these rotations. Since these are all the rotations of the cube, Γ is closed under composition and inverses and hence is a subgroup of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3)$. The group Γ naturally acts on the set X of diagonals of the cube and by an inspection we see that rotations form parts (a), (b), (c), (d), (e) act like identity, 2-cycles, 3-cycles, 4-cycles and product of 2 transpositions. Actually it is easy to see that the homomorphism $\alpha: \Gamma \to S(X)$ given by the action of Γ on X is an isomorphism. Of course $S(X) \simeq S_4$. Therefore the composition of this isomorphism and the inclusion of Γ to $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3)$ gives a representation of S_4 over $\mathbb{R}$.

Let $\varphi_3: S_4 \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^3)$ be this representation, let us compute its character $\chi_3 := \chi_{\varphi_3}$

Recall characters are constant of conjugacy classes. Keeping the notation of part a) the conjugacy classes C_1, C_2, C_3, C_4, C_5 correspond to rotations of type (a),(b),(c),(d),(e). Note that the trace of the rotation actually depends only of the angle, since if b_1, b_2, b_3 a an orthogonal bases such that $\langle b_1 \rangle$ is the axis of the rotation then the matrix of the rotation with respect to this basis is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Hence the trace of this homomorphism is the $1 + 2\cos(\alpha)$. So the character of φ_3 can be easily computed

	C_1	C_2	C_3	C_4	C_5
α	0	π	$\pm\frac{2}{3}\pi$	$\pm\frac{\pi}{2}$	π
χ_3	3	-1	0	1	-1

Let $\varphi\colon S_4 \to \mathrm{GL}(3,\mathbb{R})$ be a matrix form of φ (for example with respect to the canonical basis of $\mathbb{R}^3$ but the choice of the basis is not important here). Using the inclusion $\mathrm{GL}(3,\mathbb{R}) \subseteq \mathrm{GL}(3,\mathbb{C})$ it is possible to consider φ also as a complex representation of S_4. Note that $\chi_\varphi = \chi_{\varphi_3}$

There are at least two possibilities how to check the irreducibility of φ: We can verify the formula

$$\frac{1}{|S_4|}\sum_{g\in S_4} \chi_\varphi(g)\overline{\chi_\varphi(g)} = 1\,.$$

Indeed, if $g_i \in C_i$, then $\sum_{g\in S_4} \chi_\varphi(g)\overline{\chi_\varphi(g)} = \sum_{i=1}^{5} |C_i|\chi_3(g_i)\overline{\chi_3(g_i)} = 1 * 3^2 + 6 * (-1)^2 + 8 * 0^2 + 6 * 1^2 + 3 * (-1)^2 = 9 + 6 + 0 + 6 + 3 = 24$, so φ is irreducible even when considered as a complex representation of S_4.

The other possibility is to consider characters of irreducible representations of degree 1 and 2 and check that φ cannot be a direct sum of these representations.

Regarding the representation of degree 1 there are two representations: trivial and the sign. Then we have representation φ_2' of degree 2 found in part b). The remaining two irreducible representations of S_4 over $\mathbb{C}$ have degree 3 (use the formula that the sum of squares of the degrees has to be $|S_4| = 24$.)

Write characters for these representations

	C_1	C_2	C_3	C_4	C_5
χ_{triv}	1	1	1	1	1
χ_{sign}	1	-1	1	-1	1
$\chi_{\varphi_2'}$	2	0	-1	0	2

Note that characters of all these representations has positive values on elements from C_5. Hence the character of any representation which is a finite direct sum of representations of degree 1 and 2 has positive values on C_5 which is not the case of φ_3.

d) Consider any matrix representation $\psi\colon S_4 \to \mathrm{GL}(m,\mathbb{C})$. Note that $[\mathrm{sgn}\otimes\psi](g)$ is $\mathrm{sgn}(g)\psi(g)$, so ψ and $\mathrm{sgn}\otimes\psi$ are equivalent if and only if $\chi_\psi(g) = 0$ for every $g \in S_4 \setminus A_4$. This is not the case of representation φ found in part

c) since for example $\chi_3(g_2) = -1$ for every $g_2 \in C_2$. Both arguments proving irreducibility of the representation φ_3 in part c) apply to $\text{sgn} \otimes \varphi$ as well, since $|\chi_{\text{sgn}\otimes\varphi}(g)| = |\chi_3(g)|$ for every $g \in S_4$ and also $\chi_{\text{sgn}\otimes\varphi}(g) = -1$ for every $g \in C_5$.

e) Let θ be the representation of S_4 acting on $\{(x_1, x_2, x_3, x_4)^T \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$. If ε is the trivial representation of S_4 acting on $\{(t,t,t,t)^T \in \mathbb{C}^4 \mid t \in \mathbb{C}\}$, then $\theta \oplus \varepsilon$ is the permutation representation of S_4 induced by the action of S_4 on $\{1,2,3,4\}$. The character of permutation representation is computed as the number of fixed points, i.e., $\chi_{\theta\oplus\varepsilon}(g) = |\{i \in \{1,2,3,4\} \mid g(i) = i\}|$. Of course $\chi_\varepsilon(g) = 1$ for every $g \in S_4$ and therefore $\chi_\theta(g) = |\{i \in \{1,2,3,4\} \mid g(i) = i\}| - 1$. Written in the table:

	C_1	C_2	C_3	C_4	C_5
χ_θ	3	1	0	-1	-1

Comparing characters, we see that θ is equivalent to $\text{sgn} \otimes \varphi$, where φ is the representation found in part c).