1. *Complex representations of A_4*

   a) Find conjugacy classes of A_4 (the group of even permutations on 4 elements)

   b) Find $[A_4, A_4]$.

   c) Find all representations of A_4 over $\mathbb{C}$ of degree 1 up to equivalence.

   d) Find an irreducible complex representation of A_4 of degree > 1.

   d') Find all irreducible representations of A_4 over $\mathbb{C}$ up to equivalence.

   e) Show that $\mathbb{C}$-algebras $\mathbb{C}A_4$ and $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathrm{M}_3(\mathbb{C})$ are isomorphic

   f) Prove or disprove: $\mathbb{Q}$-algebras $\mathbb{Q}A_4$ and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{M}_3(\mathbb{Q})$ are isomorphic.

a) Recall one very useful observation about symmetric groups: If $\pi \in S_n, c = (i_1, i_2, \ldots, i_k)$ is a cycle in S_n, then $\pi c \pi^{-1} = (\pi(i_1), \pi(i_2), \ldots, \pi(i_k))$ (of course a similar statement holds for arbitrary finite products of cycles). So S_4 has the following conjugacy classes consisting of even permutations: $\{\mathrm{id}\}, \{(1,2)(3,4), (1,4)(2,3), (1,3)(2,4)\}$, $\{3\text{−cycles}\}$.

If $\sigma, \tau \in \{(1,2)(3,4), (1,4)(2,3), (1,3)(2,4)\}$, there exists $\pi \in S_4$ such that $\pi\sigma\pi^{-1} = \tau$. Note that there exists an odd permutation $\sigma_0 \in S_4 \setminus A_4$ such that $\sigma_0 \sigma \sigma_0^{-1} = \sigma$ (for example, σ_0 can be chosen to be a transposition for the decomposition of σ). Therefore we see that σ and τ are conjugated already in A_4 - we can see it from relations $\pi\sigma\pi^{-1} = \tau$ and $(\pi\sigma_0)\sigma(\pi\sigma_0)^{-1} = \tau$ since either $\pi \in A_4$ or $\pi\sigma_0 \in A_4$.

On the other hand, note that $\{\pi \in S_4 \mid \pi(a,b,c)\pi^{-1} = (a,b,c)\} = \{\mathrm{id}, (a,b,c), (a,c,b)\}$. Consider 3-cycles σ and τ and $\pi, \pi_1 \in S_4$ such that $\pi\sigma\pi^{-1} = \tau, \pi_1\sigma\pi_1^{-1} = \tau$. Then $(\pi_1^{-1}\pi)\sigma(\pi_1^{-1}\pi)^{-1} = \sigma$. Hence π and π_1 have equal parity. It follows that if two 3-cycles are conjugated in S_4 by an odd permutation, they are not conjugated in A_4. Then it is easy to see that 3-cycles form 2 conjugacy classes in A_4, namely $\{(1,2,3), (2,1,4), (3,4,1), (4,3,2)\}$ and $\{(1,3,2), (2,4,1), (3,1,4), (4,2,3)\}$.

To sum up: $C_1 = \{\mathrm{id}\}$, $C_2 = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$, $C_3 = \{(1,2,3), (1,4,2), (1,3,4), (2,4,3)\}$ and $C_4 = \{(1,3,2), (1,2,4), (1,4,3), (2,3,4)\}$ are the conjugacy classes in A_4.

b) Recall that if G is a group, then $[G,G]$ is the smallest element of the set $\{H \mid H \text{ is a normal subgroup of } G \text{ such that } G/H \text{ is abelian }\}$. Note that $V = \{\mathrm{id}\} \cup \{(1,2)(3,4), (1,4)(2,3), (1,3)(2,4)\}$ is a normal subgroup of A_4 and $G/A_4 \simeq \mathbb{Z}_3$. Since the only normal subgroup of A_4 contained in V is trivial, the commutant $[A_4, A_4]$ is V.

c) Recall the general principle how to find degree one representations of a finite group G: Find degree one representations of $G/[G,G]$ and compose them with the canonical projection $\pi \colon G \to G/[G,G]$.

For $G = A_4$ we start with complex representations of $\mathbb{Z}_3 \simeq A_4/[A_4, A_4]$:

	0	1	2
φ'_1	1	1	1
φ'_2	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
φ'_3	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$

Instead of canonical projection we consider an onto homomorphism $\pi\colon A_4 \to \mathbb{Z}_3$ given by Ker $\pi = V$ and $\pi((1,2,3)) = 1$. Then we get the following degree one representations of A_4

	C_1	C_2	C_3	C_4
φ_1	1	1	1	1
φ_2	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
φ_3	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$

d),d') The representations found in c) are pair-wise non-equivalent. We are searching for one more irreducible representation of degree $d > 1$. Note that $1^2 + 1^2 + 1^2 + d^2 = 12$ gives $d = 3$. There are natural candidates for this representation: We can use that A_4 is isomorphic to a group of rotations of tetrahedron - imagine a tetrahedron in $\mathbb{R}^3$ having barycenter in the origin the we can map elements of A_4 to the corresponding rotation and obtain a representation of degree 3 of A_4. Then we have to check its irreducibility. The other possibility is to use the canonical representation of A_4 on $\mathbb{C}^4$, restrict it to a 3-dimensional invariant subspace and hope that we get an irreducible representation. We are going to use this strategy.

Let $V \leq \mathbb{C}^4$, $V = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{C}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$ and consider a representation $\varphi_4\colon A_4 \to \mathrm{Aut}_{\mathbb{C}}(V)$ given by $\varphi_4(\pi) = \tau_\pi$, where

$$\tau_\pi\colon (x_1, x_2, x_3, x_4)^T \mapsto (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, x_{\pi^{-1}(3)}, x_{\pi^{-1}(4)})^T .$$

If this representation is not irreducible, then V contains a φ_4-invariant subspace W of dimension 1. Asume there exists such a subspace $W = \langle (x_1, x_2, x_3, x_4)^T \rangle$. Without loss on generality we assume that $x_4 \neq 0$. Then there exists $\lambda \in \mathbb{C}$ such that

$$\tau_{(1,2,3)}(x_1, x_2, x_3, x_4)^T = (x_3, x_1, x_2, x_4)^T = \lambda(x_1, x_2, x_3, x_4)^T$$

From these equations we obtain that $\lambda = 1$ and consequently $x_1 = x_2 = x_3 \neq 0$. Similar argument with $\tau_{(1,2,4)}$ implies $x_1 = x_2 = x_4$. Altogether $x_1 = x_2 = x_3 = x_4 \neq 0$ which contradicts $x_1 + x_2 + x_3 + x_4 = 0$.

Therefore V has no one-dimensional φ_4-invariant subspace and hence it is irreducible.

e) Recall we can use representations $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ to construct concrete isomorphism between these $\mathbb{C}$-algebras (see Problem 3 from March 6). Namely, the linear extension of map

$$\delta_g \mapsto (\varphi_1(g), \varphi_2(g), \varphi_3(g), \varphi_4(g)), g \in A_4$$

gives an isomorphism between $\mathbb{C}A_4$ and $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathrm{M}_3(\mathbb{C})$.

f) Note that the algebra $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{M}_3(\mathbb{Q})$ has 3 non-isomorphic simple modules of dimension 1. If this algebra is isomorphic to $\mathbb{Q}A_4$, then A_4 would have 3 non-equivalent 1 dimensional representations over $\mathbb{Q}$. But recall $A_4/[A_4, A_4] \simeq \mathbb{Z}_3$ and $\mathbb{Q}\mathbb{Z}_3 \simeq \mathbb{Q}\times\mathbb{Q}[x]/(x^2+x+1)$. That is, $\mathbb{Z}_3$ has over $\mathbb{Q}$ up to equivalence 2 irreducible representations - the trivial one and a representation of degree 2. So there is only one representation of A_4 over $\mathbb{Q}$ of degree one. Therefore $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathrm{M}_3(\mathbb{Q})$ and $\mathbb{Q}A_4$ are not isomorphic as $\mathbb{Q}$-algebras.

2. *Wedderburn-Artin theorem encore*
In these exercises G is a finite group, $\mathbb{F}$ is algebraically closed such that $\mathrm{char}(\mathbb{F}) \nmid |G|$.

a) (Burnside) Let $\psi: G \to \mathrm{GL}(d, \mathbb{F})$ be an irreducible matrix representation of G over $\mathbb{F}$. Show that the set $\{\psi(g) \mid g \in G\}$ spans $\mathrm{M}_d(\mathbb{F})$. (hint: try to prove it for all irreducible representations of G over $\mathbb{F}$ simultaneously).

b) (Frobenius-Schur) Let $\psi_1, \cdots, \psi_k$ be pair-wise non equivalent matrix representations of G over $\mathbb{F}$, where k is the number of conjugacy classes of G. Let $\psi_t: G \to \mathrm{GL}(d_t, \mathbb{F})$ has coordinate functions $f_{i,j}^{(t)}: G \to \mathbb{F}, 1 \le i, j \le d_t$. That is,

$$\psi_t(g) = (f_{i,j}^{(t)}(g))_{1 \le i,j \le d_t}$$

Show that $f_{i,j}^{(t)}, t = 1, \ldots, k, 1 \le i, j \le d_t$ are linearly independent elements of the space of all $\mathbb{F}$-valued functions on G.

c) Let $\psi: G \to \mathrm{GL}(d, \mathbb{F})$ be an irreducible representation. Show that $\psi(g)$ is a scalar multiple of the identity matrix for every $g \in Z(G)$. Conclude $d^2 \le [G : Z(G)]$.

Solution: 2. a) Let $\psi_1, \ldots, \psi_k$ be a list of all different irreducible representations of G over $\mathbb{F}$ up to equivalence, say $\psi_i: G \to \mathrm{M}_{d_i}(\mathbb{F})$. Again we are going to use the isomorphism between $\mathbb{F}G$ and $\mathrm{M}_{d_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{d_k}(\mathbb{F})$ given by

$$\delta_g \mapsto (\psi_1(g), \psi_2(g), \ldots, \psi_k(g)), g \in G\,.$$

Let us call this isomorphism Ψ. Note that if π_i is the projection to the i-th component of the product $\mathrm{M}_{d_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{d_k}(\mathbb{F})$, then $\pi_i\Psi: \mathbb{F}G \to \mathrm{M}_{d_i}(\mathbb{F})$ is a surjective linear map. Since $\mathbb{F}G$ is an $\mathbb{F}$-vector space spanned by $\{\delta_g \mid g \in G\}$, $\mathrm{M}_{d_i}(\mathbb{F})$ is spanned by $\{\pi_i\Psi(\delta_g) \mid g \in G\} = \{\psi_i(g) \mid g \in G\}$. Now ψ is equivalent to ψ_i for some $1 \le i \le k$. That is, $d_i = d$ and there exists $X \in \mathrm{GL}(d, \mathbb{F})$ such that $\psi(g) = X\psi_i(g)X^{-1}$ for every $g \in G$. Then $\{\psi(g) \mid g \in G\}$ is easily seen to span $\mathrm{M}_d(\mathbb{F})$. This concludes the proof of the statement.

b) Assume that there are $a_{i,j}^{(t)} \in \mathbb{F}$ such that $\sum_{i,j,t} a_{i,j}^{(t)} f_{i,j}^{(t)} = 0$ in the space of all functions from G to $\mathbb{F}$, that is, $\sum_{i,j,t} a_{i,j}^{(t)} f_{i,j}^{(t)}(g) = 0$ for every $g \in G$ (we omit the bounds for i, j, t in the sums, they should be $1 \le t \le k$ and $1 \le i, j \le d_t$, where k and $d_1, \ldots, d_t$ have the same meaning as in the previous exercise).

Consider the homomorphism Ψ from the previous exercise, then $f_{i,j}^{(t)}(g)$ is the value in the i-th row and the j-th column of the t-th component of $\Psi(\delta_g)$ (in other words $\Psi(\delta_g)$ is a k-tuple of matrices, we look at the matrix in position t and to the element in the position (i, j) in this matrix).

Fix some $i_0, j_0, t_0,\ t_0 \in \{1, \dots, k\}$ and $1 \le i_0, j_0 \le d_{t_0}$. Our aim is to show that $a_{i_0,j_0}^{(t_0)} = 0$. Consider an element $\alpha = \sum_{g\in G} c_g \delta_g \in \mathbb{F}G$ such that $\psi_(\alpha) = (X_1, X_2, \dots, X_k)$, where $X_t = 0$ unless $t = t_0$ and X_{t_0} is a matrix having 1 in the position (i_0, j_0) and zeros everywhere else. Since $\sum_{i,j,t} a_{i,j}^{(t)} f_{i,j}^{(t)}(g) = 0$ for every $g \in G$, we obtain

$$\sum_{g\in G} c_g \sum_{i,j,t} a_{i,j}^{(t)} f_{i,j}^{(t)}(g) = 0\,.$$

Changing the order of summation,

$$\sum_{i,j,t} a_{i,j}^{(t)} \sum_{g\in G} c_g f_{i,j}^{(t)}(g) = 0\,.$$

The inner sum is the value the entry of $\Psi(\alpha)$ has in the position given by i, j, t. Hence the value of $\sum_{g\in G} c_g f_{i,j}^{(t)}(g)$ is zero unless $i = i_0, j = j_0, t = t_0$ and in this case $\sum_{g\in G} c_g f_{i_0,j_0}^{(t_0)}(g) = 1$. Then $a_{i_0,j_0}^{(t_0)} = 0$ as we wanted to show.

c) This as easy application of the Schur's lemma for matrix represenations: If $h \in Z(G)$ and $\psi\colon G \to \mathrm{GL}(d, \mathbb{F})$ is irreducible, then $gh = hg, g \in G$ implies $\psi(g)\psi(h) = \psi(h)\psi(g)$ for every $g \in G$. By the lemma of Schur, $\psi(h) = \lambda_h E$ for some $\lambda_h \in \mathbb{F}$. Apply a) to see that $\mathrm{M}_d(\mathbb{F})$ is spanned by $\{\psi(g) \mid g \in G\}$. But if $g_1 Z(G) = g_2 Z(G), g_1, g_2 \in G$ then $\psi(g_1)$ is a scalar multiple of $\psi(g_2)$. Therefore if $g_1, \dots, g_m$ are representatives of cosets in $G/Z(G)$, then $\{\psi(g_i) \mid i = 1, \dots, m\}$ still spans $\mathrm{M}_d(\mathbb{F})$ and hence $d^2 \le m = [G : Z(G)]$.

3. *Wedderburn-Artin theorem hardcore*

(You can ignore this if you are not interested in ring theory.) Let R be a semisimple artinian ring and let S a simple (right) R-module. Schur's lemma part b) implies that $D = \mathrm{End}_R(S)$ is a division ring. The module S has a canonical structure as a left module over D:

$$ds := d(s), d \in D, s \in S$$

By the Wedderburn-Artin theory applied for D every left D-module is isomorphic to a direct sum of simple left D-modules and the only one simple left D-module up to an isomorphism is ${}_D D$. Then ${}_D S \simeq {}_D D^{(\kappa)}$ for some cardinal κ. The Krull-Schmidt theorem (in a bit more general form than it was introduced in the lecture) implies that the cardinal κ is uniquely determined by S. So we could define $\dim_D(S) := \kappa$.

Recall that the multiplicity of S in R_R is defined as follows: Write R_R as a direct sum of simple right modules, say $R_R = \oplus_{i=1}^{n} S_i$, where each S_i is simple. The multiplicity of S in R_R is then $|\{i \in \{1, \dots, n\} \mid S_i \simeq S\}|$.

Show that the multiplicity of S in R_R is $\dim_D(S)$

Solution: Recall that if X, Y are right R-modules then $\operatorname{Hom}_R(X, Y)$ has a canonical structure of a left $\operatorname{End}_R(Y)$-module given by

$$\alpha f : x \mapsto \alpha(f(x)); x \in X, f \in \operatorname{Hom}_R(X, Y), \alpha \in \operatorname{End}_R(Y) .$$

It is easy to check that if $X \simeq X'$ are isomorphic R-modules, then the $\operatorname{End}_R(Y)$-modules $\operatorname{Hom}_R(X, Y)$ and $\operatorname{Hom}_R(X', Y)$ are isomorphic. (This is a general fact, does not need any extra assumptions on R.)

We start with the isomorphism $i: S \simeq \operatorname{Hom}_R(R_R, S_R)$ given by

$$i(s): r \mapsto sr$$

(the inverse of i maps an element from $\operatorname{Hom}_R(R, S)$ to its value at 1). Observe that it is also an isomorphism of left D-modules, where $D = \operatorname{End}_R(S)$. Indeed, if $\alpha \in D$ and $s \in S$, then

$$i(\alpha s) = i(\alpha(s)): r \to \alpha(s)r$$

Since $\alpha(s)r = \alpha(sr)$ for every $s \in S$ and every $r \in R$, the map $i(\alpha(s))$ can be understood as $\alpha \circ i(s)$. In the language of left D-modules we proved $i(\alpha s) = \alpha i(s)$, so $i \in \operatorname{Hom}_D({}_D S, {}_D \operatorname{Hom}_R(R_R, S_R))$.

Let $S_1, S_2, \ldots, S_k$ be a representative set of simple right R-modules, and assume $S = S_1$. Let n_i be the multiplicity of S_i in R_i, that is, there exists an isomorphism of right R-modules

$$R_R \simeq S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k} .$$

As explained above, we have the folowing isomorphisms of left D-modules

$$S \simeq \operatorname{Hom}_R(R, S) \simeq \operatorname{Hom}_R(S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}, S)$$

Schur's lemma (part a)) gives an isomorphism

$$\operatorname{Hom}_R(S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}, S) \simeq \operatorname{Hom}_R(S_1^{n_1}, S)$$

given by the rule $f \mapsto f|_{S_1^{n_1}}$ (note this is also an isomorphism of D-modules).

Recall that $S_1 = S$, so we are interested in left D-module $\operatorname{Hom}_R(S^{n_1}, S)$. For $i = 1, 2, \ldots, n_1$ let $\iota_i: S \to S^{n_1}$ be the canonical embedding of S into the i-th component of S^{n_1}. Then the assignment $f \mapsto (f\iota_1, f\iota_2, \ldots, f\iota_{n_1})$ gives an isomorphism of left modules

$$\operatorname{Hom}_R(S^{n_1}, S) \to \operatorname{Hom}_R(S, S)^{n_1}$$

Overall we checked that ${}_D S \simeq {}_D D^{n_1}$, that is the multiplicity of S in R is indeed the dimension of ${}_D S$.