Problem session March 6, 2023

1. (Krull-Schmidt theorem for matrix representations) Let G be a group, $\mathbb{F}$ a field. Recall that if $\psi_1\colon G \to \mathrm{GL}(m,\mathbb{F})$ and $\psi_2\colon G \to \mathrm{GL}(n,\mathbb{F})$, then $\mathrm{Hom}(\psi_1,\psi_2)$ is the set of all $n \times m$ matrices X for which $X\psi_1(g) = \psi_2(g)X, g \in G$.

a) Show that $\mathrm{Hom}(\psi_1,\psi_2)$ is a finite dimensional subspace of $\mathrm{M}_{n,m}(\mathbb{F})$.

b) Show that $\mathrm{Hom}(\psi_1,\psi_2)$ and $\mathrm{Hom}(\psi_1',\psi_2')$ are isomorphic if ψ_1 is equivalent to ψ_1' and ψ_2 is equivalent to ψ_2'.

c) Use Schur's lemma for irreducible matrix representations to show that if $\psi_1,\psi_2,\dots,\psi_k$ and $\psi_1',\psi_2',\dots,\psi_\ell'$ are irreducible representations of G over $\mathbb{F}$ then $\psi_1\oplus\cdots\oplus\psi_k$ and $\psi_1'\oplus\cdots\oplus\psi_\ell'$ are equivalent if and only if $k=\ell$ and there exists a permutation $\sigma \in S_k$ such that ψ_i and $\psi'_{\sigma(i)}$ are equivalent for every $i \in \{1,\dots,k\}$.

Solution: a) Let $X,Y \in \mathrm{M}_{n,m}(\mathbb{F})$ be such that $X\psi_1(g) = \psi_2(g)X$ and $Y\psi_1(g) = \psi_2(g)Y$ for every $g \in G$. Then $(X+Y)\psi_1(g) = \psi_2(g)(X+Y)$ for every $g \in G$. Similarly if $t \in \mathbb{F}$, then $(tX)\psi_1(g) = \psi_2(g)(tX)$. So $\mathrm{Hom}(\psi_1,\psi_2)$ is a subspace of $\mathrm{M}_{n,m}(\mathbb{F})$. In particular, the dimension $\mathrm{Hom}(\psi_1,\psi_2)$ is at most mn.

b) Let $X_1 \in \mathrm{GL}(m,\mathbb{F})$ be a witness of the equivalence of ψ_1 and ψ_1' and similarly let $X_2 \in \mathrm{GL}(n,\mathbb{F})$ be a witness of the equivalence of ψ_2 and ψ_2'. In other words, $\psi_1(g)X_1 = X_1\psi_1'(g)$ and $\psi_2(g)X_2 = X_2\psi_2'(g)$ for every $g \in G$.

Note that if $X \in \mathrm{Hom}(\psi_1,\psi_2)$, then $X_2^{-1}XX_1 \in \mathrm{Hom}(\psi_1',\psi_2')$:

$$X_2^{-1}XX_1\psi_1'(g) = X_2^{-1}X\psi_1(g)X_1 = X_2^{-1}\psi_2(g)XX_1 = \psi_2'(g)X_2^{-1}XX_1\,.$$

Similarly, if $X' \in \mathrm{Hom}(\psi_1',\psi_2')$ then $X_2X'X_1^{-1} \in \mathrm{Hom}(\psi_1,\psi_2)$. Therefore assignments $X \mapsto X_2^{-1}XX_1$ and $X' \mapsto X_2X'X_1^{-1}$ are mutually inverse isomorphisms between $\mathrm{Hom}(\psi_1,\psi_2)$ and $\mathrm{Hom}(\psi_1',\psi_2')$.

c) Fix an irreducible representation $\psi\colon G \to \mathrm{GL}(d,\mathbb{F})$. If $\psi'\colon G \to \mathrm{GL}(d',\mathbb{F})$ is another irreducible matrix representation of G over $\mathbb{F}$, then $\mathrm{Hom}(\psi,\psi') = 0$ if ψ and ψ' are not equivalent (Schur's lemma) and $\mathrm{Hom}(\psi,\psi') \neq 0$ if ψ and ψ' are equivalent (note that the matrix witnessing the equivalence is in $\mathrm{Hom}(\psi,\psi')$). Let e be the dimension of this space.

Assume that $\alpha := \psi_1 \oplus \cdots \oplus \psi_k$ and $\beta := \psi_1' \oplus \cdots \oplus \psi_\ell'$ are equivalent representations. Let d_i be a degree of ψ_i, $u := d_1 + \cdots + d_k$. Consider a matrix $X \in \mathrm{M}_{u,d}(\mathbb{F})$ as a column of k blocks, the i-th block is of size $d_i \times d$. Let $X_1,\dots,X_k$ denote these blocks, i.e., $X = (X_1,X_2,\dots,X_k)^T$. Then $X \in \mathrm{Hom}(\psi,\alpha)$ if and only if $X_i \in \mathrm{Hom}(\psi,\psi_i)$ for every $1 \le i \le k$. It follows that the space $\mathrm{Hom}(\psi,\alpha)$ is isomorphic to $\oplus_{i=1}^k \mathrm{Hom}(\psi,\psi_i)$. Therefore the dimension of $\mathrm{Hom}(\psi,\alpha)$ is e times the multiplicity of ψ in α.

Similarly $\mathrm{Hom}(\psi,\beta)$ is isomorphic to e times the multiplicity of ψ in β. Since α and β are equivalent, $\mathrm{Hom}(\psi,\alpha)$ and $\mathrm{Hom}(\psi,\beta)$ have equal dimensions (part b)), we conclude that the multiplicity of ψ in α is the same as the multiplicity of ψ in β. Since this holds for every irreducible representation of G over $\mathbb{F}$, $k = \ell$ and there exists a bijection $\sigma \in S_k$ such that ψ_i is equivalent to $\psi'_{\sigma(i)}$ for every

i (σ can be constructed considering sets $\{i \in \{1, \dots, k\} \mid \psi_i$ is equivalent to $\psi\}$ and $\{i \in \{1, \dots, \ell\} \mid \psi'_i$ is equivalent to $\psi\}$ for a fixed irreducible representation ψ; since these sets are of the same size we can use some bijection between these sets to define σ on $\{i \in \{1, \dots, k\} \mid \psi_i$ is equivalent to $\psi\}$, cf. also the next exercise).

Conversely, assume that $k = \ell$ and ψ_i equivalent to ψ'_i for every $1 \leq i \leq k$. Let $X_i \in \mathrm{GL}(d_i, \mathbb{F})$ be such that $X_i\psi_i(g) = \psi'_i(g)X_i$ for every $g \in G$. Let $X := \mathrm{diag}(X_1, X_2, \dots, X_k)$ be a block diagonal matrix of size $d_1 + d_2 + \dots + d_k$. Then $X\alpha(g) = \beta(g)X$ for every $g \in G$, where $\alpha = \psi_1 \oplus \dots \oplus \psi_k$ and $\beta = \psi'_1 \oplus \dots \oplus \psi'_k$.

2. (Krull-Schmidt theorem for semisimple modules) Assume that R is a finite dimensional $\mathbb{F}$-algebra, $\mathbb{F}$ algebraically closed. Let $S_1, S_2, \dots, S_n$, $T_1, T_2, \dots, T_m$ be simple (right) R-modules such that $S := S_1 \oplus S_2 \oplus \dots \oplus S_n \simeq T_1 \oplus T_2 \oplus \dots \oplus T_m =: T$. Let X be a simple right R-module.

a) Look at dimensions of $\mathrm{Hom}_R(X, S)$ and $\mathrm{Hom}_R(X, T)$ and show that $\{1 \leq i \leq n \mid S_i \simeq X\}$ and $\{1 \leq i \leq m \mid T_i \simeq X\}$ have equal cardinalities.

b) Show that $n = m$ and there exists a permutation σ of $\{1, \dots, n\}$ such that $S_i \simeq T_{\sigma(i)}$ for every $1 \leq i \leq n$.

Solution: If $f: S \to T$ is an isomorphism, then the assignment $\mathrm{Hom}_R(X, f)$: $\mathrm{Hom}_R(X, S) \to \mathrm{Hom}_R(X, T)$

$$\mathrm{Hom}_R(X, f): g \mapsto fg$$

is not only a homomorphism of abelian groups but also of $\mathbb{F}$-spaces (recall since R is an $\mathbb{F}$-algebra, $\mathrm{Hom}_R(X, T), \mathrm{Hom}_R(X, S)$ have defined structure of an $\mathbb{F}$-space and if $t \in \mathbb{F}$, then $t(fg)$ and $f(tg)$ both send $x \in X$ to $t.[fg(x)]$). Similarly if $\pi_i: S \to S_i$ is the projection to the i-th component of the direct sum, then $\mathrm{Hom}_R(X, \pi_i): \mathrm{Hom}_R(X, S) \to \mathrm{Hom}_R(X, S_i)$ is $\mathbb{F}$-linear. The isomorphism of $\mathrm{Hom}_R(X, S) \to \oplus_{i=1}^n \mathrm{Hom}_R(X, S_i)$ given by

$$f \in \mathrm{Hom}_R(X, S) \mapsto (\pi_1 f, \pi_2 f, \dots, \pi_n f) \in \oplus_{i=1}^n \mathrm{Hom}_R(X, S_i)$$

is an isomorphism of vector spaces.

Now it is the time for Schur's lemma (for modules). If X and S_i are not isomorphic modules, $\mathrm{Hom}_R(X, S_i) = 0$. If X and S_i are isomorphic, then $\mathrm{Hom}_R(X, S_i)$ has over $\mathbb{F}$ dimension 1. Therefore $\dim_{\mathbb{F}}(\mathrm{Hom}_R(X, S)) = |\{1 \leq i \leq n \mid X \simeq S_i\}|$.

Similarly we show $\dim_{\mathbb{F}}(\mathrm{Hom}_R(X, T)) = |\{1 \leq i \leq m \mid X \simeq T_i\}|$. Now a) follows from $\mathrm{Hom}_R(X, S) \simeq \mathrm{Hom}_R(X, T)$.

b) follows from a): Set $I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$. We consider I as a set indexing $S_1, \dots, S_n$ and J as a set indexing $T_1, \dots, T_m$. Consider an equivalence relation $\simeq$ on $I \dot\cup J$: Indices from this set are equivalent if the simple modules indexed by this indices are isomorphic. Apply a) to see that for each equivalence class C of $\sim$ is $|C \cap I| = |C \cap J|$. Then $n = m$ and there is a bijection $\sigma\colon I \to J$ such that $i \sim \sigma(i)$ for every $i \in I$.

Remark: This proof of b) can be extended to (even infinite) direct sums of modules with local endomorphism rings, but it is rather categorical.

3. Assume that $\mathbb{F}$ is algebraically closed G a finite group such that char $\mathbb{F}$ does not divide $|G|$. Then we know that $\mathbb{F}$-algebras $\mathbb{F}G$ and $\mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$ are isomorphic for some $k, n_1, \ldots, n_k \in \mathbb{N}$. Observe that if we know such an isomorphism we also know all irreducible representations of G over $\mathbb{F}$.

Show that the converse is true: Assume that $\psi_1, \ldots, \psi_k$ are all different irreducible matrix representations of G over $\mathbb{F}$ up to equivalence, say $\psi_i\colon G \to \mathrm{GL}(n_i, \mathbb{F})$. Then the $\mathbb{F}$-linear map $\varphi\colon \mathbb{F}G \to \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$ defined by

$$\varphi(\delta_g) = (\psi_1(g), \psi_2(g), \ldots, \psi_k(g)), g \in G$$

is an isomorphism of $\mathbb{F}$-algebras.

Solution: Let $R := \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$. Since φ is $\mathbb{F}$-linear, $\varphi(\delta_1) = 1_R$, $\varphi(\delta_g * \delta_h) = \varphi(\delta_g)\varphi(\delta_h), g, h \in G$, φ is a homomorphism of $\mathbb{F}$-algebras. Note that $\dim_{\mathbb{F}}(R) = n_1^2 + \cdots + n_k^2 = |G|$, because $\psi_1, \ldots, \psi_k$ is the list of all irreducible representations up to equivalence.

Since $\dim_{\mathbb{F}}(\mathbb{F}G) = |G| = \dim_{\mathbb{F}} R$, it is sufficient to prove that φ is a monomorphism. Asume, that $\varphi(\sum_{g \in G} t_g \delta_g) = 0$. This literally means that $\sum_{g \in G} t_g \psi_i(g) = 0$ for every $1 \leq i \leq k$. Every irreducible matrix representation of G over $\mathbb{F}$ is equivalent to exactly one ψ_i. So we can conclude that for every irreducible representation $\omega\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ of G over $\mathbb{F}$ the relation $\sum_{g \in G} t_g \omega(g) = 0$ holds in $\mathrm{End}_{\mathbb{F}}(V)$. By the theorem of Maschke, every representation is equivalent to a direct sum of irreducible representations, so $\sum_{g \in G} t_g \theta(g) = 0$ is true for every $\theta \in \mathrm{Rep}_{\mathbb{F}}(G)$. Apply this observation to the regular representation of G over $\mathbb{F}$, that is, the representation $\mathrm{reg}\colon G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}G)$ corresponding to the module $\mathbb{F}G$. Recall $\mathrm{reg}(g)\colon v \mapsto \delta_g * v$ for every $v \in \mathbb{F}G$. Therefore $(\sum_{g \in G} t_g \delta_g) * v = 0$ for every $v \in \mathbb{F}G$. Substitute $v = \delta_{1_G}$ and conclude $\sum_{g \in G} t_g \delta_g = 0$, therefore φ is indeed mono.