Problem session February 27, 2023

Averaging principle

1. Assume G finite group and $\mathbb{F}$ is a field such that $\mathrm{char}(\mathbb{F})$ does not divide $|G|$. Let $\varphi: G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$ and let U be a φ-invariant subspace of V. Let $\psi: G \to \mathrm{Aut}_{\mathbb{F}}(W)$ be another representation of G over $\mathbb{F}$ and let $\theta_0 \in \mathrm{Hom}_{\mathbb{F}}(U, W)$ be a morphism in $\mathrm{Mor}(\varphi_U, \psi)$ (that is $[\theta_0\varphi(g)](u) = [\psi(g)\theta_0](u)$ holds for every $g \in G$ and $u \in U$). Pretending no knowledge of Maschke's theorem show there exists $\theta \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ such that

(i) θ extends θ_0, i.e., $\theta(u) = \theta_0(u)$ for every $u \in U$

(ii) $\theta \in \mathrm{Mor}(\varphi, \psi)$, i.e., $[\theta\varphi(g)](v) = [\psi(g)\theta](v)$ for every $g \in G$ and $v \in V$.

In the module-theoretic language we have that every $\mathbb{F}G$-module is injective and hence $\mathbb{F}G$ is semisimple artinian.

Solution: First we extend θ_0 somehow, for sure there exists $\theta' \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ such that $\theta'(u) = \theta_0(u)$ for every $u \in U$. We improve θ' in the same way as we improved π in the proof of Maschke's theorem:

$$\theta := \frac{1}{|G|} \sum_{g \in G} \psi(g)\theta'\varphi(g^{-1})\,.$$

It remains to verify that θ has required properties:

(i) if $u \in U$, then $[\varphi(g^{-1})](u) \in U$ for every $g \in G$ (U is φ-invariant). To make the formulas understandable, write φ_g for $\varphi(g)$ and ψ_g for $\psi(g)$.Then

$$\theta(u) = \frac{1}{|G|} \sum_{g \in G} \psi_g(\theta(\varphi_{g^{-1}}(u))) = \frac{1}{|G|} \sum_{g \in G} \psi_g(\theta_0(\varphi_{g^{-1}}(u)))\,.$$

Using that $\theta_0 \in \mathrm{Mor}(\varphi_U, \psi)$ we get $\theta(u) = \frac{1}{|G|} \sum_{g \in G} \psi_g(\psi_{g^{-1}}(\theta_0(u)))\,.$ Since ψ_g and $\psi_{g^{-1}}$ are mutually inverse, this formula simplifies to

$$\theta(u) = \frac{1}{|G|} \sum_{g \in G} \theta_0(u) = \theta_0(u)\,.$$

(ii) Let $h \in G$ be arbitrary. Then

$$\psi(h)\theta = \frac{1}{|G|} \sum_{g \in G} \psi(hg)\theta'\varphi(g^{-1}h^{-1}h) = (\frac{1}{|G|} \sum_{x \in G} \psi(x)\theta'\varphi(x^{-1}))\varphi(h) = \theta\varphi(h)\,.$$

Therefore $\theta \in \mathrm{Mor}(\varphi, \psi)$.

2. Now assume G has a normal subgroup H such that G/H is a finite group and $\mathbb{F}$ is a field whose characteristic does not divide $|G/H|$. Let $\varphi: G \to \mathrm{Aut}_{\mathbb{F}}(V)$,

$\psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(W)$ be representations of G over $\mathbb{F}$, $U \leq V$ a φ-invariant subspace and $\theta_0\colon V \to W$ an $\mathbb{F}$-linear map such that $[\theta_0\varphi(h)](v) = [\psi(h)\theta_0](v)$ for every $v \in V$ and for every $h \in H$. Assume further that $[\theta_0\varphi(g)](u) = [\psi(g)\theta_0](u)$ holds for every $g \in G$ and for every $u \in U$. Show there exists $\theta\colon V \to W$ $\mathbb{F}$-linear such that

(i) for every $u \in U$ is $\theta(u) = \theta_0(u)$

(ii) $[\theta\varphi(g)](v) = [\psi(g)\theta](v)$ for every $v \in V$ and every $g \in G$.

Solution: Note that the first exercise is basically a special case of this one for $H = 1$. This time θ_0 is already defined on the whole V, so we just have to improve it. Let $g_1, \ldots, g_k$ be a transversal of cosets of H in G, i.e., $G = \dot\bigcup_{i=1}^{k} g_i H$. Define $\theta := \frac{1}{k}\sum_{i=1}^{k} \psi(g_i)\theta_0\varphi(g_i^{-1})$. Then

(i) If $u \in U$, $[\varphi(g_i^{-1})](u) \in U$ and $\theta_0([\varphi(g_i)^{-1}](u)) = \psi(g_i^{-1})\theta_0(u)$. The same computation as in the first exercise shows that $\theta(u) = \theta_0(u)$ for every $u \in U$.

(ii) Let $g \in G$. Since H is a normal subgroup of G there are $h_1, \ldots, h_k \in H$ and a permutation $\sigma \in S_k$ such that $gg_i = g_{\sigma(i)}h_i$ for every $1 \leq i \leq k$. Then

$$\psi(g)\theta = \frac{1}{k}\sum_{i=1}^{k} \psi(gg_i)\theta_0\varphi(g_i^{-1}) = \frac{1}{k}\sum_{i=1}^{k} \psi(g_{\sigma(i)})\psi(h_i)\theta_0\varphi(g_i^{-1})\,.$$

Now use that θ_0 commutes with the action of H.

$$\psi(g)\theta = \frac{1}{k}\sum_{i=1}^{k} \psi(g_{\sigma(i)})\theta_0\varphi(h_i g_i^{-1}) = \frac{1}{k}\sum_{i=1}^{k} \psi(g_{\sigma(i)})\theta_0\varphi(g_{\sigma(i)}^{-1}g) = \theta\varphi(g)$$

3. Now assume G is a finite group, V is a finite dimensional real vector space with scalar product $\langle -,- \rangle$ and $\varphi\colon G \to \mathrm{Aut}_{\mathbb{R}}(V)$ is a representation of G over $\mathbb{R}$. Show that φ is equivalent to a representation $\varphi'\colon G \to \mathrm{Aut}(V)$ which respect the scalar product, that is, for every $g \in G, u, v \in V$ the equality $\langle [\varphi'(g)](u), [\varphi'(g)](v)\rangle = \langle u, v\rangle$ is true.
Solution: First, we define another scalar product on V, $\langle -,- \rangle_G\colon V \times V \to \mathbb{R}$ which is φ-invariant. The basis change transfering $\langle -,- \rangle$ to $\langle -,- \rangle_G$ then defines φ' of required properties. No one should be surprised by the definition of $\langle -,- \rangle_G$:

$$\langle v_1, v_2\rangle_G := \frac{1}{|G|}\sum_{g\in G} \langle [\varphi(g)](v_1), [\varphi(g)](v_2)\rangle$$

This is obviously a positive definite form on V. Moreover, if $h \in G$

$$\langle [\varphi(h)](v_1), [\varphi(h)](v_2)\rangle_G = \frac{1}{|G|}\sum_{g\in G} \langle [\varphi(gh)](v_1), [\varphi(gh)](v_2)\rangle = \langle v_1, v_2\rangle_G\,.$$

Consider $b_1, \dots, b_n$ an orthonormal basis of $\langle -, - \rangle$ and $b'_1, \dots, b'_n$ an orthonormal basis of $\langle -, - \rangle_G$. That is, $\langle b_i, b_j \rangle = \delta_{i,j} = \langle b'_i, b'_j \rangle_G$. If $\psi \in \mathrm{End}_{\mathbb{F}}(V)$ satisfies $\psi(b_i) = b'_i$ for every $i = 1, \dots, n$, then $\langle v_1, v_2 \rangle = \langle \psi(v_1), \psi(v_2) \rangle_G$ for every $v_1, v_2 \in V$. Finally for every $g \in G, v_1, v_2 \in V$ we have

$$\langle \psi^{-1}(\varphi(g)(\psi(v_1))), \psi^{-1}(\varphi(g)(\psi(v_2))) \rangle = \langle \varphi(g)(\psi(v_1)), \varphi(g)(\psi(v_2)) \rangle_G =$$

$$\langle \psi(v_1), \psi(v_2) \rangle_G = \langle v_1, v_2 \rangle .$$

So $\varphi' \colon g \mapsto \psi^{-1}\varphi(g)\psi$ is a representation equivalent to φ whose action respects scalar product $\langle -, - \rangle$.

Remark: We can use this result to give a quick proof of Maschke's theorem for representations of finite groups over $\mathbb{R}$:

Note that the lattice of φ-invariant subspaces is complementary if and only if the lattice of φ'-invariant subspaces is complementary.

If $U \leq V$ is a φ'-invariant subspace, then $U^\perp = \{v \in V \mid \langle U, v \rangle = 0\}$ obviously satisfies $U \oplus U^\perp = V$. Note that $U^\perp$ is φ'-invariant, since for every $u \in U, v \in U^\perp$ and $g \in G$

$$\langle u, \varphi'(g)(v) \rangle = \langle [\varphi'(g^{-1})](u), v \rangle = 0$$

that is $[\varphi'(g)](v) \in U^\perp$.

Assumptions in Maschke's theorem

Recall the theorem of Maschke: If G is a finite group, $\mathbb{F}$ a field and $\mathrm{char}(\mathbb{F}) \nmid |G|$ and $\varphi \colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ a representation of G over $\mathbb{F}$, then the lattice of φ-invariant subspaces of V is complementary (or in the module-theoretical form $\mathbb{F}G$ is semisimple artinian).

1. Assume $G = (\mathbb{R}, +)$, $V = \mathbb{R}^2$ and $\varphi \colon a \mapsto \varphi_a$, where

$$\varphi_a \colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + ay \\ y \end{pmatrix}$$

(i.e., φ_a acts as a multiplication by a matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$)

a) Find a φ-invariant subspace of V which has no φ-invariant complement.

b) Find all φ-invariant subspaces of V.

Solution: Consider $U := \left\{ \begin{pmatrix} r \\ 0 \end{pmatrix} \mid r \in \mathbb{R} \right\}$. Obviously, this is a φ-invariant subspace of V (and φ_U is a trivial representation). If $w \in V \setminus U$ then $\varphi_a(w) - \varphi_b(w)$ is a nonzero element of U whenever $a \neq b$. Therefore if W is a φ-invariant subspace not containing V, then $W \cap U \neq 0$, therefore U has no φ-invariant complement.

Note that since V is 2-dimensional, every nontrivial φ-invariant subspace of V is of the form $\langle u\rangle$, where u is a common eigenvector of the set $\{\varphi_a \mid a \in \mathbb{R}\}$. It is easy to see that every such u has to be a scalar multiple of $(1,0)^T$.

2. Let G be a finite group. Consider $\varepsilon\colon \mathbb{F}G \to \mathbb{F}$ a map given by

$$\varepsilon\colon \sum_{g\in G} t_g\delta_g \mapsto \sum_{g\in G} t_g$$

Obviously, this is an $\mathbb{F}$-linear map and, in fact, it is not hard to check that this map is actually a homomorphism of $\mathbb{F}$-algebras. Therefore Ker ε is an ideal in $\mathbb{F}G$. This ideal is often called the *augmentation ideal* of $\mathbb{F}G$. Show that if char $\mathbb{F}$ divides $|G|$ then there is no left ideal I of $\mathbb{F}G$ such that $I \oplus \operatorname{Ker}\varepsilon = \mathbb{F}G$. So in this case the theorem of Maschke does not hold.

Solution: Consider $e = \sum_{g\in G}\delta_g$. Since char($\mathbb{F}$) divides $|G|$, $e \in \operatorname{Ker}\varepsilon$. Consider an arbitrary element $f = \sum_{g\in G} t_g\delta_g, t_g \in \mathbb{F}$. Then

$$e * f = \sum_{g,h\in G} t_g\delta_{hg} = \sum_{x\in G}(\sum_{h\in G} t_{h^{-1}x})\delta_x$$[1]

So if $f \notin \operatorname{Ker}\varepsilon$, then $t := \sum_{g\in G} t_g \neq 0$ and $e * f = te$ is a nonzero element of Ker ε. In particular, every left ideal of $\mathbb{F}G$ not contained in Ker ε has nonzero intersection with Ker ε.

3. (only if you know what a Jacobson radical is) Assume that G is a finite p-group and $\mathbb{F}$ is a field of characteristic p. Show that the augmentation ideal of $\mathbb{F}G$ is exactly the Jacobson radical of $\mathbb{F}G$.

Solution: First of all, the augmentation ideal I is a maximal ideal of $\mathbb{F}G$, since $\mathbb{F}G/I \simeq \mathbb{F}$ (cf. the homomorphism ε from Exercise 2). Therefore it is enough to show, that I is nil, that is, for every $f \in I$ there exists $n \in \mathbb{N}$ such that $f^n = 0$. In this case $1 - f$ has inverse $(1-f)^{-1} = 1 + f + f^2 + \cdots f^{n-1}$. Therefore if K is a maximal left ideal different from I, then $I + K = \mathbb{F}G$ and $\delta_1 = 1 = f + k$ for some $f \in I$ and $k \in K$. But then $k = 1 - f$ is invertible and $K = \mathbb{F}G$ which is not possible. Hence if we prove that I is nil, then I is the only maximal left ideal of $\mathbb{F}G$ and hence also the Jacobson radical of the group algebra.

By induction on $|G|$ we prove that $I^{|G|} = 0$, which implies that I is nil.

First let us consider $|G| = p = \operatorname{char}(\mathbb{F})$, let g be a generator of G. Observe that I is a left ideal generated by $1 - \delta_g$. Indeed, $\mathbb{F}G(1-\delta_g)$ contains $\delta_{g^i} - \delta_{g^{i+1}}$ for every $0 \le i < p-1$, hence also $1 - \delta_{g^i}$ for every $1 \le i \le p-1$ and it is a basis of I.

Since G is a commutative group, $\mathbb{F}G$ is a commutative ring. Because of the characteristic $(f_1 + f_2)^p = f_1^p + f_2^p$ for every $f_1, f_2 \in \mathbb{F}G$ (write a binomial formula and note that most of the binomial coefficients are divisible by p). Then

$$I^p = [\mathbb{F}G(1-\delta_g)]^p = \mathbb{F}G(\delta_{1_G} - \delta_g)^p = \mathbb{F}G(\delta_{1_G} - \delta_{g^p}) = 0\,.$$

[1] $e * f$ denotes the product in $\mathbb{F}G$

So we are done if $|G| = p$.

The induction step uses the fact that every finite p-group has non-trivial center. Let $h \in Z(G)$ be an element of order p. Then $H := \langle h \rangle$ is a normal subgroup of G and we may consider a map

$$\varepsilon_H : \mathbb{F}G \to \mathbb{F}G/H$$

which is an $\mathbb{F}$-linear map defined by $\varepsilon_H(\delta_g) := \delta_{gH}$. Since $\varepsilon_H(\delta_{g_1} * \delta_{g_2}) = \varepsilon_H(\delta_{g_1}) * \varepsilon_H(\delta_{g_2})$, and $\varepsilon_H(\delta_{1_G}) = \delta_{1_{G/H}}$ this map is in fact a homomorphism of $\mathbb{F}$-algebras.

Note that the augmentation ideal I of $\mathbb{F}G$ is mapped onto the augmentation ideal of $\mathbb{F}G/H$. The induction hypothesis for G/H says that $I^{|G/H|} \subseteq \operatorname{Ker} \varepsilon_H =: J$.

We claim that $J^p = 0$. Let $O_1, O_2, \ldots, O_m$ be the list of the cosets of H in G. So every of these cosets is of the form $O_i = \{x_i, hx_i, h^2x_i, \ldots, h^{p-1}x_i\}$. Then $\sum_{g \in G} t_g \delta_g \in J$ if and only if for each $1 \le i \le m$ $\sum_{g \in O_i} t_g = 0$. Note that $\sum_{g \in O_i} t_g \delta_g = (\sum_{j=0}^{p-1} t_{h^j x_i} \delta_{h^j}) \delta_{x_i}$. Let $z := 1 - \delta_h$. Note that $z \in Z(\mathbb{F}G)$ and by the first induction step every element of J is a (left) multiple of z. That is $J = \mathbb{F}Gz$. Finally $J^p = \mathbb{F}Gz^p$. From the first part of the induction we already know $z^p = 0$.

Overall: $I^{|G|} = (I^{|G/H|})^p \subseteq J^p = 0$.

Centers

If R is a ring, then $Z(R) := \{s \in R \mid \forall r \in R rs = sr\}$ is a subring of R called the *center of* R.
Show that

a) If R is a ring and $n \in \mathbb{N}$, then

$$Z(\mathrm{M}_n(R)) = \{\operatorname{diag}(z, \ldots, z) \mid z \in Z(R)\}.$$

b) If R, S are rings, then $Z(R \times S) = Z(R) \times Z(S)$.

!c) Let $\mathbb{F}$ be a field. Show that $Z(\mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F}))$ is an $\mathbb{F}$-vector space of dimension k.

d) Find $Z(\mathrm{GL}(n, \mathbb{F}))$, where $\mathbb{F}$ is a field (in this exercise you can apply previous ones to find center of the general linear group).

Solution: a) Assume $A \in Z(\mathrm{M}_n(R))$, for $1 \le i, j \le n$ let $E_{i,j}$ be the matrix having the only non-zero entry in the position (i, j), where is placed 1. Notice $E_{i,j}A$ has nonzero entries only in the i-th row which is $(a_{j,1}, \ldots, a_{j,n})$ and $AE_{i,j}$ has nonzero elements only in the j-th column which is $(a_{1,i}, \ldots, a_{n,i})^T$. Therefore $E_{i,j}A = AE_{i,j}$ can hold only if $a_{j,k} = 0$ for every $k \ne j$, $a_{k,i}$ for every $k \ne i$ and $a_{j,j} = a_{i,i}$. Since i, j can be chosen arbitrary, we see that every matrix in $Z(\mathrm{M}_n(R))$ are of the form $\mathrm{diag}(r,\dots,r)$, where $r \in R$. Since the center of $\mathrm{M}_n(R)$ has to commute with all diagonal matrices we see that

$$Z(\mathrm{M}_n(R)) \subseteq \{\mathrm{diag}(r,\dots,r) \mid r \in Z(R)\}$$

The opposite inclusion is checked by a direct computation.

b) This is just a straightforward verification $(r,s) \in Z(R\times S) \Leftrightarrow (r,s).(r',s') = (r',s')(r,s)$ for every $(r',s') \in R\times S \Leftrightarrow (rr',ss') = (r'r,s's)$ for every $r' \in R, s' \in S \Leftrightarrow r \in Z(R), s \in Z(S)$.

c) Combine a) and b). If $E_i \in$ is the identity matrix of $\mathrm{M}_{n_i}(\mathbb{F})$ then the center of $\mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times M_{n_k}(\mathbb{F})$ is $\{(\lambda_1 E_1,\dots,\lambda_k E_k) \mid \lambda_1,\dots,\lambda_k \in \mathbb{F}\}$. This is obviously an $\mathbb{F}$-vector space of dimension k.

d) If $n = 1$, then $\mathrm{GL}(1,\mathbb{F}) \simeq \mathbb{F}^*$ is commutative, so assume $n \geq 2$. If $\mathbb{F}$ is algebraically closed we can use Schur's lemma: The identity $i\colon \mathrm{GL}(n,\mathbb{F}) \to \mathrm{GL}(n,\mathbb{F})$ is an irreducible representation of $\mathrm{GL}(n,\mathbb{F})$ over $\mathbb{F}$. If $A \in Z(\mathrm{GL}(n,\mathbb{F}))$, then $AX = XA$ for every $X \in \mathrm{GL}(n,\mathbb{F})$. (Multiplication by A is considered as a morphism in $\mathrm{Mor}(i,i)$.) Schur's lemma imply that A is a scalar multiple of identity. Therefore

$$Z(\mathrm{GL}(n,\mathbb{F})) = \{\lambda E \mid \lambda \in \mathbb{F}^*\}$$

If $\mathbb{F}$ is not algebraically closed we can prove the following claim: Every element of $\mathrm{M}_n(\mathbb{F})$ is a sum of (finitely many) regular matrices. Then every element of $Z(\mathrm{GL}(n,\mathbb{F}))$ is also in the center of $\mathrm{M}_n(\mathbb{F})$ and, by a), we get the same result.

Consider a matrix $A \in \mathrm{M}_n(\mathbb{F})$. Our goal is to write A as a sum of finitely many regular matrices. Write $A = L + \Delta + U$, where L is a strictly lower triangular matrix, U is a strictly upper triangular matrix and Δ is a diagonal matrix. Then $A = (L+E) + \Delta + (U-E)$, where $L+E, U-E$ are regular. So we have to write a diagonal matrix Δ as a sum of regular matrices. Observe it is easy if $\mathbb{F} \neq \mathbb{F}_2$. The following method should work also for $\mathbb{F}_2$. Let $X = (x_{i,j})$ satisfy $x_{i,n-i+1} = 1$ for $1 \leq i \leq n$ and $x_{i,j} = 0$ if $i+j \neq n+1$. Consider $Y = E_{1,1} + X$ (so we change the top left corner of X to 1). Note that $E_{1,1} = Y - X$ is a sum of 2 regular matrices, and so is every nonzero scalar multiple of $E_{1,1}$. Note that for every $1 \leq i \leq n$ matrices $E_{1,1}$ and $E_{i,i}$ are similar, so also every nonzero element of $E_{i,i}$ is a sum of regular matrices. It follows that every diagonal matrix is a sum of regular matrices.