Problem session February 13, 2023

1. *Contragradient (dual) representations*

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. The *dual* of V (the space of linear forms on V) is defined as

$$\widetilde{V} := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$$

The $\mathbb{F}$-linear structure on $\widetilde{V}$ is defined point-wise. That is, if $f, f_1, f_2 \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and $t \in \mathbb{F}$ then we define $f_1 + f_2$ and tf by

$$f_1 + f_2 \colon v \mapsto f_1(v) + f_2(v), v \in V$$

$$tf \colon v \mapsto tf(v)$$

a) Let G be a group and let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$. We define a map $\widetilde{\varphi}\colon G \to \mathrm{Aut}_{\mathbb{F}}(\widetilde{V})$ by

$$\widetilde{\varphi}(g) \colon f \mapsto f \circ \varphi(g^{-1})$$

(so g acts on the space $\widetilde{V}$)

Show that $\widetilde{\varphi}(g)$ is an automorphism of $\widetilde{V}$ for every $g \in G$ and that the map $\widetilde{\varphi}$ is a homomorphism of groups.

The representation $\widetilde{\varphi} \in \mathrm{Rep}_{\mathbb{F}}(G)$ is called the contragradient representation of φ (also a dual of φ).

b) Assume that $\dim_{\mathbb{F}}(V)$ is finite and $B = \{b_1, \ldots, b_n\}$ is a basis of V. As explained in the lecture a representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ gives a matrix representation $\psi\colon G \to \mathrm{GL}(n, \mathbb{F})$ via

$$\psi(g) := [\varphi(g)]_B\,.$$

Let $\widetilde{\varphi}\colon G \to \mathrm{Aut}_{\mathbb{F}}(\widetilde{V})$ be the contragradient representation of φ. Consider $\widetilde{B} = \{\widetilde{b_1}, \ldots, \widetilde{b_n}\}$ the dual basis of B. This basis is described by relations

$$\widetilde{b_i}(b_j) = \delta_{i,j}, 1 \le i, j \le n\,.$$

Consider the matrix representation $\widetilde{\psi}\colon G \to \mathrm{GL}(n, \mathbb{F})$ given by $\widetilde{\psi}(g) = [\widetilde{\varphi}(g)]_{\widetilde{B}}$. Find a relation between ψ and $\widetilde{\psi}$.

c) Assume that $G = \mathbb{Z}_2$ and $\mathrm{char}\,\mathbb{F} \neq 2$. When are φ and $\widetilde{\varphi}$ equivalent?

d) (could be solved later when more basic examples are introduced in the lecture) Find an example of $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ where G is a finite group, $\mathbb{F}$ is a field and $\dim_{\mathbb{F}}(V)$ is finite and φ is not equivalent to $\widetilde{\varphi}$.

Solution

a) Obviously $\widetilde{\varphi}(1_G) = 1_{\widetilde{V}}$. If we prove $\widetilde{\varphi}(gh) = \widetilde{\varphi}(g) \circ \widetilde{\varphi}(h)$ for every $g, h \in G$, then we get $\widetilde{\varphi}(g)$ is a bijection with inverse $\widetilde{\varphi}(g^{-1})$ and that $\widetilde{\varphi}: G \to \mathrm{Aut}_{\mathbb{F}}(\widetilde{V})$ is a homomorphism of groups.

For $g \in G$ and $v \in V$ denote $g * v := [\varphi(g)](v)$. Let $f \in \widetilde{V} = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Then

$$\widetilde{\varphi}(gh)(f): v \mapsto f((gh)^{-1} * v)$$

$$\widetilde{\varphi}(h)(f): v \mapsto f(h^{-1} * v)$$

$$[\widetilde{\varphi}(g)](\widetilde{\varphi}(h)(f)): v \mapsto [\widetilde{\varphi}(h)(f)](g^{-1} * v) = f(h^{-1} * (g^{-1} * v))$$

Since $\varphi: G \to \mathrm{Aut}_{\mathbb{F}}(V)$ is a representation (i.e. a group homomorphism), $(h^{-1}g^{-1}) * v = h^{-1} * (g^{-1} * v)$ for every $g, h \in G$ and $v \in V$. Hence also $\widetilde{\varphi}(gh)$ and $\widetilde{\varphi}(g) \circ \widetilde{\varphi}(h)$ coincide as we wanted to show.

b) Let $A = (a_{i,j})_{1 \le i,j \le n} := \psi(g^{-1})$ and $C = (c_{i,j})_{1 \le i,j \le n} := \widetilde{\psi}(g)$. Fix $1 \le i \le n$. Then $(\widetilde{\varphi}(g))(\widetilde{b_i}) = \sum_{j=1}^n c_{j,i} \widetilde{b_j}$. Obviously $c_{j,i} = [(\widetilde{\varphi}(g))(\widetilde{b_i})](b_j) = (\widetilde{b_i})(g^{-1} * b_j)$. In our notation $g^{-1} * b_j = \sum_{k=1}^n a_{k,j} b_k$, so $(\widetilde{b_i})(g^{-1} * b_j) = (\widetilde{b_i})(\sum_{k=1}^n a_{k,j} b_k) = a_{i,j}$. We proved $c_{j,i} = a_{i,j}$ for every $1 \le i, j \le n$, in other words $C = A^T$. Therefore

$$\widetilde{\psi}(g) = \psi(g^{-1})^T .$$

c) First observe that if $\dim_{\mathbb{F}}(V)$ is not finite, then $\widetilde{V}$ has bigger dimension than V. So φ and $\widetilde{\varphi}$ cannot be equivalent in this case.

Now assume $\dim_{\mathbb{F}}(V)$ is finite. From the lecture we know that $V = V_+ \oplus V_-$, where

$$V_+ = \{v \in V \mid [\varphi(1)](v) = v\}, V_- = \{v \in V \mid [\varphi(1)](v) = -v\} .$$

Assume that $f \in \widetilde{V} = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ satisfies $f(V_-) = 0$. We claim that $[\widetilde{\varphi}(1)](f) = f$, that is $f \in \widetilde{V}_+$. Indeed, let $v \in V$. Write $v = v_+ + v_-$. Then

$$[\widetilde{\varphi}(1)(f)](v) = f(\varphi(1)(v_+ + v_-)) = f(v_+ - v_-) = f(v_+) = f(v) .$$

In other words $[\widetilde{\varphi}(1)](f) = f$.

Similarly, assume $f \in \widetilde{V}$ satisfies $f(V_+) = 0$. We claim that $[\widetilde{\varphi}(1)](f) = -f$, that is $f \in \widetilde{V}_-$. Indeed, let $v \in V$. Write $v = v_+ + v_-$. Then

$$[\widetilde{\varphi}(1)(f)](v) = f(\varphi(1)(v_+ + v_-)) = f(v_+ - v_-) = -f(v_-) = -f(v) .$$

In other words $[\widetilde{\varphi}(1)](f) = -f$.

Finally notice that if $f \in \widetilde{V}$ we define $f_+, f_- \in \widetilde{V}$ by

$$f_+(v_+ + v_-) = f(v_+), f_-(v_+ + v_-) = f(v_-), v_+ \in V_+, v_- \in V_- .$$

Then $f = f_+ + f_-$.

Overall, we proved that $\widetilde{V}_+ = \{f \in \widetilde{V} \mid f(V_-) = 0\}$ and $\widetilde{V}_- = \{f \in \widetilde{V} \mid f(V_+) = 0\}$. Notice that $\dim_{\mathbb{F}}(V_+) = \dim_{\mathbb{F}}(\widetilde{V}_+)$ and $\dim_{\mathbb{F}}(V_-) = \dim_{\mathbb{F}}(\widetilde{V}_-)$.

In particular, there exists an isomorphism $\theta \in \mathrm{Hom}_{\mathbb{F}}(V, \widetilde{V})$ such that $\theta(V_+) = \widetilde{V}_+$ and $\theta(V_-) = \widetilde{V}_-$.

We claim that this θ is a witness for the equivalence of φ and $\widetilde{\varphi}$. That is for every $g \in G = \mathbb{Z}_2$ the equality

$$\theta\varphi(g) = \widetilde{\varphi}(g)\theta$$

holds. If $g = 0$, then $\varphi(g) = 1_V$ and $\widetilde{\varphi}(g) = 1_{\widetilde{V}}$ so the equality holds.

If $g = 1$ take an arbitrary $v \in V$ and write it as a sum $v = v_+ + v_-$. Since $\theta(v_+) \in \widetilde{V}_+$ and $\theta(v_-) \in \widetilde{V}_-$ we get

$$[\theta \circ \varphi(g)](v_+ + v_-) = \theta(v_+ - v_-) = \theta(v_+) - \theta(v_-)$$

$$[\widetilde{\varphi}(g) \circ \theta](v_+ + v_-) = \widetilde{\varphi}(g)(\theta(v_+) + \theta(v_-)) = \theta(v_+) - \theta(v_-)\,.$$

d) Observe that two representations of degree 1 (that is the group acts on a one-dimensional space) are equivalent if and only if they are equal. This observation is obvious when written in matrix form: Let $\psi_1, \psi_2 \colon G \to \mathbb{F}^* = \mathrm{GL}(1, \mathbb{F})$ be equivalent matrix representations of G over $\mathbb{F}$. That is, there exists $x \in \mathbb{F}^*$ such that

$$\psi_1(g) = x\psi_2(g)x^{-1}, g \in G$$

But $\mathbb{F}^*$ is a commutative group, so the equation above implies $\psi_1 = \psi_2$.

Let $\mathbb{F}$ be a finite field. And consider the inclusion $\iota \colon \mathbb{F}^* \to \mathbb{F}$ as a representation of $G := \mathbb{F}^*$ over $\mathbb{F}$ of degree 1. In part b) we have seen that the dual of ι is essentially the representation $\iota' \colon \mathbb{F}^* \to \mathbb{F}$, $\iota' \colon g \mapsto g^{-1}$. Note that ι and ι' are equivalent if and only if $\iota = \iota'$ if and only if $g^2 = 1$ for every $g \in \mathbb{F}^*$. So if $\mathbb{F} \neq \mathbb{F}_2$, then ι is not equivalent to its dual.

2. Representations of finite cyclic groups over $\mathbb{C}$

a) Let $G = \mathbb{Z}_n$ be a cyclic group of order n and let H be any group. Show that $f \mapsto f(1)$ gives a bijection between $\mathrm{Hom}(G, H)$ and $\{h \in H \mid h^n = 1\}$.

b) Assume $H = \mathrm{GL}(m, \mathbb{C})$, that is, $\mathrm{Hom}(G, H)$ is the set of all matrix representations of $\mathbb{Z}_n$ over $\mathbb{C}$ of degree m. Show that $\psi_1, \psi_2 \in \mathrm{Hom}(G, H)$ are equivalent if and only if the matrices $\psi_1(1), \psi_2(1)$ are similar.

c) Let B be a regular Jordan block over $\mathbb{C}$. Show that if $B^k = E$ for some $k \in \mathbb{N}$, then the size of the block is 1. Show that every matrix representation of $\mathbb{Z}_n$ is equivalent to a representation whose image consists of diagonal matrices.

d) Consider again $G = \mathbb{Z}_n$, $H = \mathrm{GL}(m, \mathbb{C})$. Show that $\psi_1, \psi_2 \in \mathrm{Hom}(G, H)$ are equivalent if and only if $\psi_1(1)$ and $\psi_2(1)$ have equal characteristic polynomials.

Solution

a) Let $G = \{0, 1, \ldots, n-1\}$. If $f \in \mathrm{Hom}(G, H)$, then $f(1)^n = f(\overbrace{1+1+\cdots+1}^{n}) = f(0) = 1_H$. So the map $f \mapsto f(1)$ is correctly defined. Recall that every homomorphism is uniquely determined by the value on a set of generators. Since 1 generates G, the map $f \mapsto f(1)$ is injective.

To prove that it is surjective take some $h \in H$ satisfying $h^n = 1$. We show that there exists $f \in \mathrm{Hom}(\mathbb{Z}_n, H)$ such that $f(1) = h$.

Consider a homomorphism $F \in \mathrm{Hom}(\mathbb{Z}, H)$ given by $F(z) := h^z, z \in \mathbb{Z}$. Since $h^{z_1+z_2} = h^{z_1}h^{z_2}$, F is indeed a homomorphism of groups. Note that $h^n = 1$ implies $\mathbb{Z}n \subseteq \mathrm{Ker}\ F$. The homomorphism theorem shows that there exists $f' \in \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, H)$ such that $f'\pi = F$, where $\pi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the canonical projection. The composition of f' and the obvious isomorphism of $\mathbb{Z}_n \simeq \mathbb{Z}/n\mathbb{Z}$ gives $f \in \mathrm{Hom}(\mathbb{Z}_n, H)$ such that $f(1) = h$.

b) Just a straightforward verification: If ψ_1, ψ_2 are equivalent, there exists $X \in H$ such that $\psi_1(g) = X\psi_2(g)X^{-1}$ for every $g \in G$. In particular, $\psi_1(1)$ and $\psi_2(1)$ are similar.

Conversely, if $\psi_1(1)$ and $\psi_2(1)$ are similar, there exists $X \in H$ such that $\psi_1(1) = X\psi_2(1)X^{-1}$. For every $g \in G = \{0, 1, \ldots, n-1\}$ is then $\psi_1(g) = \psi_1(1)^g = X\psi_2(1)^g X^{-1} = X\psi_2(g)X^{-1}$.

c) This is just a reminder from the linear algebra course. Take for example a block of size 3. Using a binomial formula we obtain

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^k = \left(\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right)^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}^k$$

for $k \geq 2$. Since characteristic of the field is zero, we cannot obtain identity matrix for any k. The argument works in the same way for any block of size at least 2.

d) This follows from uniqueness of Jordan normal form. More pedestrian argument could look like this: If ψ_1, ψ_2 are equivalent, then $\psi_1(1)$ and $\psi_2(1)$ are similar and hence have equal characteristic polynomials. Conversely, assume that $\psi_1(1), \psi_(2)$ have equal characteristic polynomials. Then $\psi_1(1)$ is similar to $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $\psi_2(1)$ is similar to $\mathrm{diag}(\lambda'_1, \ldots, \lambda'_m)$. Equality of characteristic polynomials means that $(\lambda_1, \ldots, \lambda_m)$ is just a permutation of $(\lambda'_1, \ldots, \lambda'_m)$. Recall that a conjugation by a permutation matrix permutes elements on the diagonal. So $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ and $\mathrm{diag}(\lambda'_1, \ldots, \lambda'_m)$ are similar and hence $\psi_1(1)$ and $\psi_2(1)$ are similar too.

3. Faithful representations

Let G be a finite group, let $\mathbb{F}$ be a field and let $\psi\colon G \to \mathrm{GL}(m, \mathbb{F})$ be a matrix representation of G over $\mathbb{F}$. We say that ψ is *faithful* if $\psi(g), g \in G$ are linearly vectors of space $\mathrm{M}_m(\mathbb{F})$.

a) Let ψ_1, ψ_2 be equivalent representations of G over $\mathbb{F}$. Show that ψ_1 is faithful if and only if ψ_2 is faithful.

b) Let $n \geq 3$, consider representation $\psi\colon S_n \to \mathrm{GL}(n, \mathbb{C})$ which maps a permutation to the corresponding permutation matrix. Show that ψ is not faithful.

c) Find a faithful representation of S_3 of degree 4.

d) Describe faithful representations of $\mathbb{Z}_n$ over $\mathbb{C}$.

e) Show that G has a faithful representation over $\mathbb{F}$.

f) (maybe better to do it later with some theory in hands) Show that there is no faithful representation of S_3 over $\mathbb{C}$ of degree 3.

Solution

a) Let $X \in \mathrm{GL}(m, \mathbb{F})$ be such that $X\psi_1(g)X^{-1} = \psi_2(g)$ for every $g \in G$. If $G_0 \subseteq G$ is a nonempty finite set and $t_g, g \in G_0$ are nonzero elements of $\mathbb{F}$ such that $\sum_{g\in G_0} t_g\psi_1(g) = 0$, then $\sum_{g\in G} t_g\psi_2(g) = \sum_{g\in G_0} t_g X^{-1}\psi_1(g)X = X^{-1}(\sum_{g\in G_0} t_g\psi_1(g))X = 0$. So if ψ_1 is not faithful then ψ_2 is not faithful. The other implication is proved by symmetric arguments.

b) Let us first consider the case when $n = 3$. Look at the representation

$$\psi(\mathrm{id}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \psi((1,2,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \psi((1,3,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\psi((1,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \psi((1,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \psi((2,3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note that the sum of all matrices in the first row is the same as the the sum of the matrix in the second row. This shows that ψ is not faithful and suggests how to proceed in the general case.

Let $\psi\colon S_n \to \mathrm{GL}(n, \mathbb{C})$ be the representation of S_n given by permutation matrices. We claim that $\sum_{\pi\in A_n} \psi(\pi) = \sum_{\pi\in S_n\setminus A_n} \psi(\pi) = \frac{(n-1)!}{2}U$, where U is the matrix having all its entries equal to 1.

Fix $(i,j) \in \{1, \ldots, n\}$. To evaluate the value in the position (i,j) in $\sum_{\pi\in A_n} \psi(\pi)$ we have to calculate how many permutations of A_n sends j to i. Note that $\{\pi \in S_n \mid \pi(j) = i\} = (j,i)\{\pi \in S_n \mid \pi(j) = j\}$, since if $\pi(j) = i$, then $\pi = (i,j)((i,j)\pi)$ and $(i,j)\pi$ stabilizes j. So we have to calculate the number of odd permuations in the set $\{\pi \in S_n \mid \pi(j) = j\}$ which is obviously $(n-1)!/2$.

Similarly we count the number of odd permutations in the set $\{\pi \in S_n \mid \pi(j) = i\}$.

c) Looking at 3×3 matrices introduced in part b) we note that they generate a subspace of $\mathrm{M}_3(\mathbb{C})$ of dimension 5. An elementary argument from linear algebra shows that if $\sum_{\pi\in S_3} t_\pi \psi(\pi) = 0$, then $t_{\mathrm{id}} = t_{(1,2,3)} = t_{(1,3,2)} = -t_{(1,2)} = -t_{(1,3)} = -t_{(2,3)}$. Now consider a map $\psi'\colon S_3 \to \mathrm{GL}(4,\mathbb{C})$ given by

$$\psi'(\mathrm{id}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \psi'((1,2,3)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\psi'((1,3,2)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \psi'((1,2)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\psi'((1,3)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \psi'((2,3)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is easy to check that ψ' is a homomorphism of groups and if $0 \neq t \in \mathbb{C}$ then $\sum_{\pi\in A_3} t\psi'(\pi) - \sum_{\pi\in S_3\setminus A_3} t\psi'(\pi)$ is nonzero. As noticed above, all remaining nontrivial linear combinations have non-zero top-left block of size 3. Therefore ψ' is faithful.

d) We already know that every matrix representation of $\mathbb{Z}_n$ over $\mathbb{C}$ is equivalent to a representation given by diagonal matrices. Consider such a representation

$$\psi\colon t \mapsto \mathrm{diag}(\lambda_1^t, \lambda_2^t, \dots, \lambda_m^t), t \in \{0,\dots,n-1\},$$

where $\lambda_1,\dots,\lambda_m$ are complex roots of a polynomial $x^n - 1$.

For $t \in \{0,\dots,n-1\}$ let $v_t := (\lambda_1^t, \lambda_2^t, \dots, \lambda_m^t)^T \in \mathbb{C}^m$. We wonder when $v_0,\dots,v_{n-1}$ are linearly independent vectors. So consider a matrix A with columns $v_0,\dots,v_{n-1}$ and are interested when its rank equals n. Note that if $\lambda_i = \lambda_j$, then the the i-th and the j-th rows of A are equal. So the rank of A can be n only if the set $\{\lambda_1,\dots,\lambda_m\}$ contains all roots of $x^n - 1$. Conversely, if every root of $x^n - 1$ is contained in $\{\lambda_1,\dots,\lambda_m\}$, then there are n different values of in this set. Assume for example that $\lambda_1,\dots,\lambda_n$ are pair-wise different. When deleting last $m-n$ rows in A, we obtain Vandermonde matrix

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

This matrix is regular, hence A has rank n.

To summarize our results: ψ is faithful if and only if A has rank n if and only if $x^n - 1 \mid \prod_{j=1}^m (x - \lambda_j)$. That is ψ is faithful if and only if $x^n - 1$ divides the characteristic polynomial of $\psi(1)$.

e) We can linearize the action on G an G by left translation to obtain a regular representation of G over $\mathbb{F}$.

Let V be a vector space with basis G, i.e., every element of v is uniquely expressed as $\sum_{g \in G} t_g g$, $t_g \in \mathbb{F}$. For every $g \in G$ define an $\mathbb{F}$-linear map $\varphi_g \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ by

$$\varphi_g : h \mapsto gh, h \in G$$

(so φ_g is given by its values on basis G of V).

Let $g_1, g_2 \in G$ and $h \in G$. Then $[\varphi_{g_1} \circ \varphi_{g_2}](h) = g_1 g_2 h = \varphi_{g_1 g_2}(h)$. It follows $\varphi(g_1) \circ \varphi(g_2) = \varphi(g_1 g_2)$. Therefore we have a representation $\varphi : G \to \mathrm{Aut}_{\mathbb{F}}(V)$. Now look at the matrix form of this representation $\psi : G \to \mathrm{GL}(|G|, \mathbb{F})$, $\psi(g) := [\varphi(g)]_G$. Matrices $\psi(g), g \in G$ are linearly independent, since if we look at their first columns, we obtain a canonical basis of $\mathbb{F}^{|G|}$.

4. Actions and representations

Let G be a group. Consider category $\mathcal{A}$ whose objects are actions of G on sets and morphisms are maps compatible with the actions. To be more precise, consider nonempty sets X, Y and $* : G \times X \to X$, $\otimes : G \times Y \to Y$ actions of G on X and Y. Then $*$ and $\otimes$ are objects of $\mathcal{A}$ and the set of morphisms from $*$ to $\otimes$ is

$$\mathcal{A}(*, \otimes) := \{h : X \to Y \mid \forall x \in X, g \in G \ h(g * x) = g \otimes h(x)\}$$

Morphisms in $\mathcal{A}$ are composed as maps between sets.

If $\mathbb{F}$ is a field find a 'natural' pair of adjoint functors between $\mathcal{A}$ and $\mathrm{Rep}_{\mathbb{F}}(G)$.

Solution: The functor $F : \mathcal{A} \to \mathrm{Rep}_{\mathbb{F}}(G)$ will be a linearization of the action: Consider an action of G on X, $* : G \times X \to X$. Let V be an $\mathbb{F}$-vector space with basis X. For $g \in G$ define $F(*)_g \in \mathrm{End}_{\mathbb{F}}(V)$ by its values on the basis X: $F(*)_g(x) := g * x$. Note that every $F(*)_g$ permutes basis, and hence is an automorphism of V. Further $g * (h * x) = (gh) * x, g, h \in G, x \in X$ implies $F(*)_g \circ F(*)_h = F(*)_{gh}, g, h \in G$. Therefore $F(*) : G \to \mathrm{Aut}_{\mathbb{F}}(V)$which maps g to $F(*)_g$ is a homomorphism of groups. So F assigns a representation of G over $\mathbb{F}$ to every object of $\mathcal{A}$.

Now consider $*, \otimes$ objects of $\mathcal{A}$ ($*$ acts on X and $\otimes$ acts on Y) and a morphism $h \in \mathcal{A}(*, \otimes)$. If V is a space with basis X and W is a space with basis Y and $F(*) : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $F(\otimes) : G \to \mathrm{Aut}_{\mathbb{F}}(W)$, define $F(h) \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ by $[F(h)](x) := h(x)$. It is easy to see that $F(h)$ is a morphism in $\mathrm{Rep}_{\mathbb{F}}(G)$. Indeed, $[F(h)](F(*)_g(x)) = [F(h)](g * x) = h(g * x) = g \otimes [h(x)] = F(\otimes)_g([F(h)](x))$ for every $x \in X$. Since $F(h), F(*)_g$ and $F(\otimes)_g$ are $\mathbb{F}$-linear, the equality also holds for any linear combination of elements from X.

In the opposite direction we consider the forgetful functor $H : \mathrm{Rep}_{\mathbb{F}}(G) \to \mathcal{A}$ which is based on the fact that $\mathrm{Rep}_{\mathbb{F}}(G)$ is a subcategory of $\mathcal{A}$.

If $* : G \times X \to X$ is an object of $\mathcal{A}$ we define $\eta_* : * \to HF(*)$ to be the inclusion of $X \to V$, where V is the space on which $F(*)$ acts (recall that V is a space with basis X). By the definition $\eta_* \in \mathcal{A}(*, HF(*))$. It remains to show that η is the unit of the adjunction. Consider a representation $\psi : G \to \mathrm{Aut}_{\mathbb{F}}(W)$ and a morphism $h \in \mathcal{A}(*, H(\psi))$. We have to show that there exists unique $t \in \mathrm{Rep}_{\mathbb{F}}(G)(F(*), \psi)$ such that $h = H(t)\eta_*$. Such a t has to be a linear map satisfying $t(x) = h(x)$ for every $x \in X$, hence there can be at most one such h. On the other hand, if $t \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ satisfies $t(x) = h(x)$ for every $x \in X$ then $[\psi(g)](t(x)) = h(g * x) = t(F(*)_g(x))$ for every $x \in X$ and $g \in G$. Using linearity of $\psi(g), h$ and $F(*)_g$ we get that $[\psi(g)](t(v)) = t(F(*)_g(v))$ for every $v \in V$, i.e., t is a morphism in $\mathrm{Rep}_{\mathbb{F}}(G)$.