Homework # 3 - Complex and real representations

1. Let G be a finite group, $g \in G$. Show that $\chi_\varphi(g) \in \mathbb{R}$ for every $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ of finite degree if and only if g and g^{-1} are conjugated.

2. A conjugacy class C is called self-inversive if $C = C^{-1}$. Show that the number of self-inversive conjugacy classes of G equals to the number of real valued rows in the character table of G over $\mathbb{C}$. (Hint: Let N be a 'normalized' character table of G over $\mathbb{C}$ - columns of the character table introduced in the lecture are scaled to have norm 1 in the standard Euclidean norm. Compute the trace of NN^T and also of N^TN.)

3. Let G be a finite group of odd order. Show that every irreducible representation of G over $\mathbb{C}$ with real valued character is trivial (so the trivial representation is the only irreducible complex representation of G defined over $\mathbb{R}$).

4. (Burnside) Let G be a group of odd order, let k be a number of conjugacy classes in G. Show that $|G| \equiv k \pmod{16}$ (Hint: consider degrees of irreducible representations modulo 8.)

Remark: The character theory can be used to decide whether an irreducible complex representation of a finite group G is defined over $\mathbb{R}$. If $\psi\colon G \to \mathrm{GL}(d, \mathbb{C})$ is an irreducible matrix representation of G over $\mathbb{C}$, its Frobenius-Schur indicator is defined as $\mathrm{fs}(\psi) := \frac{1}{|G|}\sum_{g\in G} \chi_\psi(g^2)$. Its value is always $0, 1$ or -1. If χ_ψ is not real-valued, then $\mathrm{fs}(\psi) = 0$ and,of course, ψ cannot be defined over $\mathbb{R}$. If χ_ψ is real-valued, then $\mathrm{fs}(\psi) \in \{1, -1\}$ and $\mathrm{fs}(\psi) = 1$ if and only if ψ is defined over $\mathbb{R}$. The proof of this result is more involved than the homework problems listed on this page.