Homework # 4 - Normal basis in characteristic zero

Let $\mathbb{F} \subseteq \mathbb{E}$ be a finite Galois extension of fields. The aim of this homework is to provide a representation theoretical proof of the fact that such an extension has a normal basis, i.e., there exists an element $e \in \mathbb{E}$ such that $\{\varphi(e) \mid \varphi \in \mathrm{Gal}(\mathbb{E} \mid \mathbb{F})\}$ is a basis of the space ${}_{\mathbb{F}}\mathbb{E}$.

Recall that $\Gamma := \mathrm{Gal}(\mathbb{E} \mid \mathbb{F})$ is the group of all automorphisms of the field $\mathbb{E}$ which fix any element of the field $\mathbb{F}$, so every $\varphi \in \Gamma$ is $\mathbb{F}$-linear. Therefore the inclusion $\iota\colon \Gamma \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{E})$ is a representation of Γ over $\mathbb{F}$.

In this homework you can use the following result:

Theorem: Two (matrix) representations of a finite group over a field of characteristic zero are equivalent if and only if they have equal characters.

Note that we proved this result in the lecture under the assumption that the field is also algebraically closed. In fact, using the ideas introduced in the problem session devoted to extension of scalars it is possible to remove this additional assumption easilly.

In the following exercises we keep the notation introduced above.

1. Prove the equivalence of the following statements.

   a) There exists a normal basis of the extension $\mathbb{F} \subseteq \mathbb{E}$.

   b) The representation ι is equivalent to the regular representation of Γ over $\mathbb{F}$.

2. Let $\mathbb{F}_0$ be a subfield of $\mathbb{F}$ such that $[\mathbb{F}\colon \mathbb{F}_0] < \infty$. Let $\varphi \in \mathrm{End}_{\mathbb{F}}(\mathbb{E})$. Note that $\varphi \in \mathrm{End}_{\mathbb{F}_0}(\mathbb{E})$. Find a relation between $\mathrm{Tr}_{\mathbb{F}}(\varphi)$ and $\mathrm{Tr}_{\mathbb{F}_0}(\varphi)$, where $\mathrm{Tr}_X(\varphi)$ means the trace of φ when considered as an operator of the space ${}_X\mathbb{E}$.

3. Assume that $\mathrm{char}(\mathbb{F}) = 0$. Prove that ι is equivalent to $\mathrm{reg}_{\mathbb{F}}(\Gamma)$ if and only if $\chi_\iota(\varphi) = 0$ for every $\varphi \in \Gamma \setminus \{\mathrm{id}_{\mathbb{E}}\}$.

4. Use the main theorem of the Galois theory to prove the equivalence of the following statements:

   a) Every finite Galois extension of fields of characteristic zero has a normal basis.

   b) Every finite Galois extension of fields of characteristic zero with cyclic Galois group has a normal basis.

5. (do it only if you don't want to miss the point of this homework) Assume that $\mathrm{char}(\mathbb{F}) = 0$ and that Γ is a cyclic group of order $n > 1$, and let φ the generator of Γ. Prove that $(-1)^n(x^n - 1)$ is the characteristic polynomial of φ and hence $\mathrm{Tr}_{\mathbb{F}}(\varphi) = 0$. (Hint: It is possible to use Dedekind's lemma but it would be nice to find a direct argument.)