Homework # 2 - An alternative approach to the decomposition of the regular representation

Let G be a group, $\mathbb{F}$ a field and $\varphi\colon G \to \mathrm{GL}(d,\mathbb{F})$ a matrix representation of G over $\mathbb{F}$. Recall that for each $1 \leq i,j \leq d$ there is a function $\varphi_{i,j}\colon G \to \mathbb{F}$ such that for any $g \in G$ is $\varphi(g) = (\varphi_{i,j}(g))_{1\leq i,j\leq d}$ (these functions are called coordinate functions of φ).

Let $V = \{f \mid f\colon G \to \mathbb{F}\}$ be the vector space of all $\mathbb{F}$-valued functions on G, with point-wise $\mathbb{F}$-linear structure:

$$f_1 + f_2\colon g \mapsto f_1(g) + f_2(g), (tf)\colon g \mapsto tf(g), g \in G, t \in \mathbb{F}, f_1, f_2, f \in V\,.$$

Given a matrix representation φ we define the space $M(T) \subseteq V$ as the subspace of V generated by the coordinate functions of G, i.e.,

$$M(\varphi) := \langle \{\varphi_{i,j} \mid 1 \leq i,j \leq d\}\rangle\,.$$

a) Show that if φ and φ' are equivalent representations of G over $\mathbb{F}$, then $M(\varphi) = M(\varphi')\,.$

b) For a given a matrix representation $\varphi\colon G \to \mathrm{GL}(d,\mathbb{F})$, consider a map $\mu\colon \mathrm{M}_d(\mathbb{F}) \to V$ given by $\mu(C)\colon g \mapsto \mathrm{Tr}(C\varphi(g)), C \in \mathrm{M}_d(\mathbb{F})$. Show that μ is an $\mathbb{F}$-linear map with image $M(\varphi)$.

Let us define two representations of $G \times G$ over $\mathbb{F}$:

- $R\colon G\times G \to \mathrm{Aut}_{\mathbb{F}}(V)$ given by $[R(g_1,g_2)](f) = (g_1,g_2)*f$, where $(g_1,g_2) \in G \times G$, $f \in V$ and $[(g_1,g_2)*f](g) = f(g_2^{-1}gg_1)$.
- $L\colon G \times G \to \mathrm{Aut}_{\mathbb{F}}(\mathrm{M}_d(\mathbb{F}))$ given by

$$L(g_1,g_2)\colon X \mapsto \varphi(g_1)X\varphi(g_2^{-1}), X \in \mathrm{M}_d(\mathbb{F})$$

Note that the representation L of $G\times G$ is constructed from a representation φ of G.

c) Verify that R and L are representations of $G \times G$ over $\mathbb{F}$ and that $M(\varphi)$ is an R-invariant subspace.

d) Let $U = \mathbb{F}^d$ and $\psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ a linear representation given by φ, that is, $\psi(g)(u) := \varphi(g)\cdot u$ for $g \in G$ and $u \in U$. Let $U^* = \mathrm{Hom}_{\mathbb{F}}(U,\mathbb{F})$ be the dual space of U and $\psi^*\colon G \to \mathrm{Aut}_{\mathbb{F}}(U^*)$ the contragradient representation of ψ (see the first problem session). Show that L is equivalent to $\psi\otimes\psi^*$ where $\psi\otimes\psi^*\colon G\times G \to \mathrm{Aut}_{\mathbb{F}}(V\otimes V^*)$ is a representation of $G\times G$ acting on $V\otimes V^*$ via $\psi\otimes\psi^*(g_1,g_2) = \psi(g_1)\otimes\psi^*(g_2)$.

e) Show that μ is a homomorphism from L to R, that is $\mu([L(g_1,g_2)](X)) = R(g_1,g_2)(\mu(X))$ for every $g_1,g_2 \in G$ and $X \in \mathrm{M}_d(\mathbb{F})$.

f) Now assume that G is finite, $\mathbb{F}$ algebraically closed and $\text{char}(\mathbb{F})$ does not divide $|G|$. Show that if φ is irreducible, then L is irreducible and $\dim_{\mathbb{F}}(M(\varphi)) = d^2$.

g) (do it only if you want) Assume G finite and $\mathbb{F}$ is algebraically closed of characteristic 0. Show that if $\varphi_1, \ldots, \varphi_k$ are pair-wise non-equivalent representations of G over $\mathbb{F}$ then $M(\varphi_1), \ldots, M(\varphi_k)$ are independent subspaces of V.