Homework # 5 - Representations of symmetric groups

Let $n \in \mathbb{N}$, $\lambda \vdash n$.

(1) Recall the definition of polytabloid associated to a tableau $t \in X^\lambda$.

$$e_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)[\pi * t] .$$

Show that all the tabloids in the sum are mutually different, i.e., if $\pi \neq \sigma \in C_t$ then $[\pi * t] \neq [\sigma * t]$.

(2) Let M^λ be a $\mathbb{C}$-space with basis T^λ, where T^λ is the set of all tabloids of shape λ. Recall the representation φ^λ of S_n acting of M^λ via

$$\varphi^\lambda(\pi) = \varphi^\lambda_\pi \colon [t] \mapsto [\pi * t], [t] \in T^\lambda .$$

Further let $\langle -, - \rangle$ be the scalar product on M^λ such that $\langle [t], [s] \rangle = \delta_{[t],[s]}$ for $[s], [t] \in T^\lambda$. For a λ-tableau $t \in X^\lambda$ consider the operator

$$A_t := \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \varphi^\lambda_\pi \in \operatorname{End}_{\mathbb{C}}(M^\lambda) .$$

Show that for every $u, v \in M^\lambda$ is $\langle A_t(u), v \rangle = \langle u, A_t(v) \rangle$.

(3) Let $\mu \vdash n$ and let $\psi^\mu \colon S_n \to \operatorname{Aut}_{\mathbb{C}}(S^\lambda)$ be the Specht's representation of S_n associated to μ. Show that if the multiplicity of ψ^μ in φ^λ is nonzero, then $\mu \trianglerighteq \lambda$.

For practical computations with Specht's representations the following result can be useful:

Definition: A λ-tableau is called *standard* if the sequence of numbers in any row and in any column of t is increasing. For example, if $n = 4$ and $\lambda = (2, 2)$ there are only two standard tableaux of shape λ, namely

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \, , \; \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

Theorem: Let $n \in \mathbb{N}$, $\lambda \vdash n$. Let $Y^\lambda \subseteq X^\lambda$ be the set of all standard tableaux of shape λ. The space S^λ, i.e. the subspace of M^λ generated by all polytabloids associated to elements of X^λ, has basis $\{e_t \mid t \in Y^\lambda\}$, in particular $\dim_{\mathbb{C}}(S^\lambda) = |Y^\lambda|$.

(4) Let $n = 5$, $\lambda = (3, 2)$. Compute the character of ψ^λ. For every $\mu \vdash n$ determine the multiplicity of ψ^μ in φ^λ (it can be easier to find the multiplicities first).