Group representations 1

Character table of a finite group

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Character table of a finite group - the notation

Let G be a finite group, $C_1, C_2, \ldots, C_k$ the list of all its conjugacy classes, in particular, $G = \dot{\cup}_{i=1}^{k} C_i$.
For $i = 1, \ldots, k$ we set $n_i := |C_i|$.
In each C_i fix some $g_i \in C_i$.
Assume that $\mathbb{F}$ is algebraically closed and $\mathrm{char}(\mathbb{F}) \nmid |G|$. (In today's lecture we mostly assume $\mathbb{F} = \mathbb{C}$.)
$\varphi_1, \ldots, \varphi_k$ be a list of all irreducible representations of G over $\mathbb{F}$.
For every $1 \leq i \leq k$ we write χ_i instead of χ_{φ_i}.

Character table of G over $\mathbb{F}$

Definition

(under the introduced notation) The character table of G over $\mathbb{F}$ is a $k \times k$ matrix over $\mathbb{F}$ whose entry in the (i, j)-th position is $\chi_i(g_j)$.

Properties of G determined by the complex character table

Theorem

Let G be a finite group. The following information can be read from the character table of G over $\mathbb{C}$.

a) *Which of $C_1, C_2, \ldots, C_k$ contains 1_G*
b) *degrees of $\varphi_1, \varphi_2, \ldots, \varphi_k$*
c) *the order of G*
d) *the values $n_1, n_2, \ldots, n_k$, where $n_i = |C_i|$*
e) *$Z(G)$ as a union of some conjugacy classes*
f) *$\mathrm{Ker}\ \varphi_1, \ldots, \mathrm{Ker}\ \varphi_k$ as unions of some conjugacy classes*
g) *$[G, G]$ unions of some conjugacy classes*
h) *the lattice of normal subgroups of G*

a) How to detect C_j with $C_j = \{1_G\}$

Note that $\chi_i(1_G)$ is the degree of φ_i. So if $C_j = \{1_G\}$, then the j-th column of the character table consists of positive integers. This column has to be orthogonal with respect to a standard scalar product to all the remaining columns in the character table. In particular, there is only one column in the table containing only non-negative real entries.

b),c)How to find $|G|$ in the complex character table

Recall that if $d_i = \chi_i(1_G)$ is the degree of φ_i, then $|G| = \sum_{i=1}^{k} d_i^2$. The values $d_1, d_2, \ldots, d_k$ are in the j-th column of the character table, where $C_j = \{1_G\}$.

d) How to detect $n_i = |C_i|$

Recall the second orthogonality relations for the case $\mathbb{F} = \mathbb{C}$:

$$\sum_{i=1}^{k} \chi_i(g_j)\overline{\chi_i(g_j)} = \frac{|G|}{n_j}$$

So $n_j = \frac{|G|}{||a_j||^2}$, where a_j is the j-th column of the character table.

e) How to find $Z(G)$

Note that $g \in Z(G)$ if and only if $|\{hgh^{-1} \mid h \in G\}| = 1$. So $Z(G)$ is a union of conjugacy classes of size 1.

$$Z(G) = \cup_{1 \leq i \leq k, |C_i|=1} C_i$$

Kernels of irreducible representations

Lemma

Let G be a finite group and let $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ be a representation of degree d. Then $g \in \mathrm{Ker}\ \varphi$ if and only if $\chi_\varphi(g) = d$.

Proof: If $g \in \mathrm{Ker}\ \varphi$, then $\chi_\varphi(g)$ equals to the trace of the identity matrix, that is, $\chi_\varphi(g) = d$.
Conversely, let $A := [\varphi(g)]_B$ be the matrix of $\varphi(g)$ w.r.t. some basis B and assume $\mathrm{Tr}(A) = d$. Recall A is similar to a matrix C in the Jordan canonical form

$$C = XAX^{-1} .$$

Observe that $A^{o(g)} = E$, hence also $C^{o(g)} = E$. It is easy to verify that a regular Jordan block have finite order only if it is of size 1. Therefore C has to be diagonal.

the proof, cont.

Let $C = \mathrm{diag}(c_1, c_2, \ldots, c_d)$, then $c_i^{o(g)} = 1$ (in particular $|c_i| = 1$) for any $1 \leq i \leq d$. Observe $d = \mathrm{Tr}(A) = \mathrm{Tr}(C) = c_1 + c_2 + \cdots + c_d$. The triangle inequality gives

$$d = |c_1| + |c_2| + \cdots + |c_d| \geq |c_1 + c_2 + \cdots + c_d| = d$$

and the equality can hold only if $c_1 = c_2 = \cdots = c_d =: c$. In that case $C = cE \in Z(\mathrm{GL}(d, \mathbb{C}))$ and hence $A = X^{-1}CX = cE$. To conclude the proof observe that $d = \mathrm{Tr}(A) = cd$ implies $c = 1$ and hence $g \in \mathrm{Ker}\ \varphi$.

f) $\mathrm{Ker}\ \varphi_i$

Using the lemma it is easy to determine $\mathrm{Ker}\ \varphi_i$. Recall, we already know its degree d_i. So

$$\mathrm{Ker}\ \varphi_i = \cup_{1 \leq j \leq k, a_{i,j} = d_i} C_j \ ,$$

where $a_{i,j}$ is the value in the (i,j)-th position of the character table of G over $\mathbb{C}$.

Commutant as an intersection of kernels of representations

Lemma

Let G be a finite group. Then

$$[G, G] = \cap_{\varphi \in \mathrm{Hom}(G,\mathbb{C}^*)} \mathrm{Ker}\ \varphi\,.$$

Proof.

Recall $[G, G]$ is a subgroup of G generated by elements $xyx^{-1}y^{-1}$. Since $\mathbb{C}^*$ is a commutative group, $\varphi(xyx^{-1}y^{-1}) = 1$ for every $x, y \in G$ and $\varphi \in \mathrm{Hom}(G, \mathbb{C}^*)$. Therefore

$$[G, G] \subseteq \cap_{\varphi \in \mathrm{Hom}(G,\mathbb{C}^*)} \mathrm{Ker}\ \varphi\,.$$

Conversely, let $g \in G \setminus [G, G]$. Note that $G/[G, G]$ is a finite abelian group and hence a direct sum of finite cyclic groups. Recall that every finite cyclic group is a subgroup of $\mathbb{C}^*$. It follows that for every $g \in G \setminus [G, G]$ there exists $f \in \mathrm{Hom}(G/[G, G], \mathbb{C}^*)$ such that $f(g[G, G]) \neq 1$. Let $\varphi := f\pi$, where $\pi\colon G \to G/[G, G]$ is the canonical projection. Then $\varphi \in \mathrm{Hom}(G, \mathbb{C}^*)$ and $\varphi(g) \neq 1$.

f) $[G, G]$ from the complex character table

Let $I \subseteq \{1, \ldots, k\}$ be the set indexing degree one representations of G over $\mathbb{C}$. That is,

$$i \in I \Leftrightarrow d_i = 1\,.$$

Further let $J \subseteq \{1, \ldots, k\}$ be given by

$$j \in J \Leftrightarrow a_{i,j} = 1, \forall i \in I$$

It means that elements of C_j are contained in the kernel of every degree one representation of G over $\mathbb{C}$.
Then the lemma implies

$$[G, G] = \cup_{j \in J} C_j\,.$$

The lattice of normal subgroups

The idea is that if G is a finite group and N is its normal subgroup, then

$$N = \cap_{i \in I} \mathrm{Ker}\ \varphi_i,$$

where $I \subseteq \{1, 2, \ldots, k\}$ satisfies

$$i \in I \Leftrightarrow N \subseteq \mathrm{Ker}\ \varphi_i$$

The structural constants of $Z(\mathbb{F}G)$ w.r.t. a homework basis

Fix $i, j, \ell \in \{1, 2, \ldots, k\}$ and consider conjugacy classes C_i, C_j, C_ℓ. For a fixed element $g \in C_\ell$ consider the number of ways how g can be written as a product of an element from C_i and an element from C_j. That is,

$$|\{(x, y) \mid x \in C_i, y \in C_j, g = xy\}|$$

Lemma

This quantity depends only on i, j, ℓ and not on the choice of g

proof of the lemma

Assume that $g, g' \in C_\ell$ and consider sets

$$X_g = \{(x, y) \mid x \in C_i, y \in C_j, g = xy\}$$

$$X_{g'} = \{(x, y) \mid x \in C_i, y \in C_j, g' = xy\}$$

Since g, g' are in the same conjugacy class, there exists $h \in G$ such that $g' = hgh^{-1}$.
Then it is easy to see that

$$(x, y) \in X_g \mapsto (hxh^{-1}, hyh^{-1}) \in X_{g'}$$

$$(x, y) \in X_{g'} \mapsto (h^{-1}xh, h^{-1}yh) \in X_g$$

are mutually inverse bijections between X_g and $X_{g'}$

The structural constants of $Z(\mathbb{F}G)$ w.r.t. a homework basis 2

Definition

For $i, j, \ell \in \{1, 2, \ldots, k\}$ we define

$$h_{i,j,\ell} = |\{(x, y) \mid x \in C_i, y \in C_j, xy = g\}|$$

where g is a fixed element of C_ℓ.

Back to homework # 1: If G is a finite group, $Z(\mathbb{F}G)$ has a basis $z_1, z_2, \ldots, z_k$, where $z_i = \sum_{g \in C_i} \delta_g$. The structural constants of $Z(\mathbb{F}G)$ w.r.t. this basis says how $z_i * z_j$ can be expressed as a linear combination of $z_1, z_2, \ldots, z_k$.
Assume $z_i * z_j = \sum_{g \in G} c_g \delta_g$, Using the distributivity of $*$ it is easy to compute that

$$c_g = |\{(x, y) \mid x \in C_i, y \in C_j, xy = g\}|.1_{\mathbb{F}}$$

Therefore

$$z_i * z_j = \sum_{\ell=1}^{k} h_{i,j,\ell} z_\ell$$

Yet another application of Schur's lemma

Lemma

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\psi\colon G \to \mathrm{GL}(d, \mathbb{F})$ be an irreducible matrix representation and $C \subseteq G$ a conjugacy class. Then

$$\sum_{g \in C} \psi(g) = \lambda E\,,$$

where $\lambda = \frac{|C|}{d}\chi_\psi(c)$, for any $c \in C$.

Proof of the lemma

Proof.

Let $X := \sum_{g \in G} \psi(g)$. Then it is easy to verify $\psi(h)X = X\psi(h)$ for every $h \in G$. Part b) of Schur's lemma for irreducible matrix representations now gives $X = \lambda E$ for some $\lambda \in \mathbb{F}$.

Apply trace on both sides of the equality $\sum_{g \in C} \psi(g) = \lambda E$ to obtain

$$|C|\chi_\psi(c) = d\lambda$$

Recall that if $\mathrm{char}(\mathbb{F}) \nmid |G|$, the degree of any irreducible representation of G over $\mathbb{F}$ does not divide $|G|$. Therefore $\lambda = \frac{|C|}{d}\chi_\psi(c)$ for any $c \in C$. □

Relating λ's and $h_{i,j,\ell}$

Let G be a finite group, $\mathbb{F}$ algebraically closed and $\mathrm{char}(\mathbb{F}) \nmid |G|$. Assume again that conjugacy classes of G are labeled by $C_1, C_2, \ldots, C_k$.

For an irreducible (matrix) representation $\psi\colon G \to \mathrm{GL}(d, \mathbb{F})$ let ${\lambda_i}^{\psi} \in \mathbb{F}$ be such that

$$\sum_{g \in C_i} \psi(g) = \lambda_i^{\psi} E\,.$$

Then

$${\lambda_i}^{\psi} {\lambda_j}^{\psi} = \sum_{\ell=1}^{k} h_{i,j,\ell} {\lambda_\ell}^{\psi}$$

Of course, if we consider irreducible (linear) representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$, we define ${\lambda_i}^{\varphi}$ by

$$\sum_{g \in C_i} \varphi(g) = \lambda_i^{\varphi} 1_V\,.$$

What infromation is encoded in the complex character table?

Theorem

The structural constants $h_{i,j,\ell}$ can be computed from the character table of G over $\mathbb{C}$.

proof Recall $C_1, C_2, \ldots, C_k$ are the conjugacy classes of G, $\varphi_1, \varphi_2, \ldots, \varphi_k$ is a list of all distinct irreducible representations of G over $\mathbb{C}$. The character table is a matrix $A = (a_{i,j})_{1 \leq i,j, \leq k}$, where $a_{i,j}$ is the value χ_i has on C_j.
Let $\lambda_j{}^{\varphi_i} = \frac{|C_j|}{d_i} a_{i,j} = \frac{|C_j|}{\chi_i(1_G)} \chi_i(g_j)$. Note that $\lambda_j{}^{\varphi_i}$ are given by the matrix A.
Moreover for every $i, j, \ell \in \{1, 2, \ldots, k\}$

$$\lambda_j{}^{\varphi_i} \lambda_\ell{}^{\varphi_i} = \sum_{m=1}^{k} h_{j,\ell,m} \lambda_m{}^{\varphi_i}$$

the proof, cont.

Fix $j, \ell \in \{1, 2, \dots, k\}$.
Let Λ be a $k \times k$ complex matrix whose value at the position (m, i) is ${\lambda_m}^{\varphi_i}$. Formulae from the previous slide can be written in matrix form as

$$({\lambda_j}^{\varphi_1}\lambda_\ell^{\varphi_1}, {\lambda_j}^{\varphi_2}\lambda_\ell^{\varphi_2}, \dots, {\lambda_j}^{\varphi_k}\lambda_\ell^{\varphi_k}) = (h_{j,\ell,1}, h_{j,\ell,2}, \dots, h_{j,\ell,k})\Lambda$$

Note that if we prove that Λ is regular, then the values $h_{j,\ell,1}, h_{j,\ell,2}, \dots, h_{j,\ell,k}$ can be computed from this equality and, in particular, are determined by the matrix A.
Since we can do such a computation for any $j, \ell \in \{1, 2, \dots, k\}$,

why is Λ regular?

Note that Λ^T is a product of three regular matrices:

$$\Lambda^T = \mathrm{diag}(\frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_k}) \cdot A \cdot \mathrm{diag}(|C_1|, |C_2|, \dots, |C_k|)$$

(in the position (u, v) of the matrix on the RHS there is $\frac{|C_v|}{d_u} a_{u,v} = \lambda_v^{\varphi_u}$)

That's all for today

Thank you for your attention