Group representations 1

Consequeces of orthogonality relations, character tables

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Some consequences of orthogonality relations - a reminder

Let G be a finite group, let $\mathbb{F}$ be a field. Consider the space

$$V := \{f \mid f\colon G \to \mathbb{F}\}$$

of all $\mathbb{F}$-valued functions on G with point-wise linear structure:

$$f_1 + f_2\colon g \mapsto f_1(g) + f_2(g), g \in G$$

$$tf\colon g \mapsto tf(g), g \in G$$

for $f_1, f_2, f \in V$ and $t \in \mathbb{F}$.
Let U be the subspace of V consisting of all functions which are constant on conjugacy classes of G, i.e.

$$U := \{f \in V \mid f(x) = f(y) \text{ whenever } \exists z \in G \ y = zxz^{-1}\}.$$

Note that if $\mathbb{F}G$ is understand as the set of all $\mathbb{F}$-valued functions on G, then U is exactly the center of $\mathbb{F}G$.

A bilinear form on V - a reminder

Assume that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then we define a bilinear form $b\colon V \times V \to \mathbb{F}$ by the rule

$$b(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}), f_1, f_2 \in V$$

Recall that

- $\chi_\varphi \in U$ for every (matrix) representation φ of G over $\mathbb{F}$
- $b(\chi_\varphi, \chi_\psi) = 0$ if φ, ψ are non-equivalent irreducible (matrix) representations of G over $\mathbb{F}$
- $b(\chi_\varphi, \chi_\varphi) = 1_{\mathbb{F}}$ if $\mathbb{F}$ is algebraically closed and φ is an irreducible (matrix) representation of G over $\mathbb{F}$.

Multiplicities

Let G be a finite group, $\mathbb{F}$ a field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be the list of all irreducible representations of G over $\mathbb{F}$ up to equivalence.
For every representation $\varphi \colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ of finite degreee there are $n_1, n_2, \ldots, n_k \in \mathbb{N}_0$ such that φ is equivalent to

$$\overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{n_1} \oplus \overbrace{\varphi_2 \oplus \cdots \oplus \varphi_2}^{n_2} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{n_k}$$

Numbers $n_1, \ldots, n_k$ are uniquely determined by φ and n_i is called the multiplicity of φ_i in φ.
The uniqueness of $n_1, n_2, \ldots, n_k$ is a consequence of the Krull-Schmidt theorem.
For algebraically closed fields of characteristic 0 multiplicities can be defined via characters.

Characters and multiplicity

Let us keep the previous notation but assume also that $\mathbb{F}$ is algebraically closed and $\mathrm{char}(\mathbb{F}) = 0$.
Maschke's theorem gives $n_1, n_2, \ldots, n_k \in \mathbb{N}_0$ such that φ is equivalent to

$$\overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{n_1} \oplus \overbrace{\varphi_2 \oplus \cdots \oplus \varphi_2}^{n_2} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{n_k}$$

Then $\chi_\varphi = \sum_{i=1}^{k} n_i \chi_{\varphi_i}$ and, consequently,

$$b(\chi_\varphi, \chi_{\varphi_i}) = n_i 1_{\mathbb{F}}$$

The multiplicity of φ_i in φ can be therefore computed as $b(\chi_\varphi, \chi_{\varphi_i})$.

$b(\chi_\varphi, \chi_\psi)$

Assume now G finite, $\mathbb{F}$ algebraically closed and $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi, \psi \in \mathrm{Rep}_{\mathbb{F}}(G)$ of finite degree. Let n_i be the multiplicity of φ_i in φ and let m_i be the multiplicity of φ_i in ψ. That is,

$$\varphi \simeq \overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{n_1} \oplus \overbrace{\varphi_2 \oplus \cdots \oplus \varphi_2}^{n_2} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{n_k}$$

$$\psi \simeq \overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{m_1} \oplus \overbrace{\varphi_2 \oplus \cdots \oplus \varphi_2}^{m_2} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{m_k}$$

Using the bilinearity and $b(\chi_{\varphi_i}, \chi_{\varphi_j}) = \delta_{i,j} 1_{\mathbb{F}}$ we have

$$b(\chi_\varphi, \chi_\psi) = b(\sum_{i=1}^{k} n_i \chi_{\varphi_i}, \sum_{i=1}^{k} m_i \chi_{\varphi_i}) = \sum_{i=1}^{k} n_i m_i . 1_{\mathbb{F}}$$

Intertwining number

Assume $\varphi, \psi \in \mathrm{Rep}_{\mathbb{F}}(G)$ of finite degree. The set of morphisms $\mathrm{Rep}_{\mathbb{F}}(\varphi, \psi)$ has a canonical structure of an $\mathbb{F}$-space.
Its dimension is called the intertwining number of φ and ψ

Proposition

Assume G is a finite group, $\mathbb{F}$ algebraically closed such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi_1, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be a list of all distinct irreducible representations of G over $\mathbb{F}$ up to equivalence and $\varphi, \psi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be representations of finite degree. Then the intertwining number of φ and ψ is $\sum_{i=1}^{k} n_i m_i$, where n_i is the multiplicity of φ_i in φ and m_i is the multiplicity of φ_i in ψ.

Note that if we assume also $\mathrm{char}(\mathbb{F}) = 0$ then $b(\chi_\varphi, \chi_\psi)$ is essentially the intertwining number of φ and ψ.

The summary

Theorem

Let G be a finite group, $\mathbb{F}$ algebraically closed field of characteristic 0. Then

a) *If φ and ψ are representations of G over $\mathbb{F}$ of finite degree, then φ and ψ are equivalent if and only if $\chi_\varphi = \chi_\psi$.*

b) *If $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ has finite degree then φ is irreducible if and only if*

$$\frac{1}{|G|}\sum_{g\in G}\chi_\varphi(g)\chi_\varphi(g^{-1}) = 1$$

Proof.
a) The direct implication is a general property of characters, the converse is a consequence of the fact that if $\chi_\varphi = \chi_\psi$ then for every $\theta \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible, then $b(\chi_\theta, \chi_\varphi) = b(\chi_\theta, \chi_\psi)$. Therefore the multiplicity of θ in φ is the same as the multiplicity of θ in ψ. In particular, φ and ψ are equivalent.
b) The direct implication follows from the Schur's lemma (and does not need $\mathrm{char}(\mathbb{F}) = 0$). The converse follows from the fact that the intertwining number of a representation which is not irreducible is always at least 2. □

Character table of a finite group - the notation

Let G be a finite group, $C_1, C_2, \ldots, C_k$ the list of all its conjugacy classes, in particular, $G = \dot{\bigcup}_{i=1}^{k} C_i$.
For $i = 1, \ldots, k$ we set $n_i := |C_i|$.
In each C_i fix some $g_i \in C_i$.
Assume that $\mathbb{F}$ is algebraically closed and $\mathrm{char}(\mathbb{F}) \nmid |G|$.
$\varphi_1, \ldots, \varphi_k$ be a list of all irreducible representations of G over $\mathbb{F}$.
For every $1 \leq i \leq k$ we write χ_i instead of χ_{φ_i}.

Character table of G over $\mathbb{F}$

Definition

(under the introduced notation) The character table of G over $\mathbb{F}$ is a $k \times k$ matrix over $\mathbb{F}$ whose entry in the (i, j)-th position is $\chi_i(g_j)$.

Remark

Note that the table keeps information about the values of characters of every irreducible representation of G over $\mathbb{F}$. Indeed, every such a representation is equivalent to exactly one representation from the list $\varphi_1, \ldots, \varphi_k$ and every character is constant on conjugacy classes of G.

The character table is regular

Proposition

(keeping the introduced notation) Let A be a character table of G over $\mathbb{F}$. Then A is regular, in fact, $A^{-1} = B = (b_{i,j})_{1 \le i,j \le k}$, where $b_{i,j} = \frac{n_i}{|G|}\chi_j(g_i^{-1})$.

Proof.

Recall $\frac{1}{|G|}\sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \delta_{i,j}$ for every $1 \le i, j \le k$.
Note that if $g_i^{-1} \in C_j$ then $x \mapsto x^{-1}$ is a bijection between C_i and C_j. In particular, $n_i = |C_i| = |C_j|$. Thus $\sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) = \sum_{t=1}^{k}\sum_{g \in C_t} \chi_i(g)\chi_j(g^{-1}) = \sum_{t=1}^{k} n_t\chi_i(g_t)\chi_j(g_t^{-1})$.
The first orthogonality relations can be rewritten as $\frac{1}{|G|}\sum_{t=1}^{k} n_t\chi_i(g_t)\chi_j(g_t^{-1}) = \delta_{i,j}$. This equation can be read as 'the product of the i-th row of A and the j-th column of B is $\delta_{i,j}$' which is $AB = E$. In particular, A is regular and $A^{-1} = B$. □

Second orthogonality relations

Theorem

(keeping the introduced notation) Let $x, y \in G$. Then

a) *If x, y are not conjugated, then $\sum_{i=1}^{k} \chi_i(x)\chi_i(y^{-1}) = 0$*

b) *If $x, y \in C_j$, then $\sum_{i=1}^{k} \chi_i(x)\chi_i(y^{-1}) = \frac{|G|}{|C_j|} 1_{\mathbb{F}}$*

Proof.

In the proof of the previous Proposition we established equality $AB = E$. Of course, this implies $BA = E$. Multiply the i-th row of B and j-th column of A gives

$$\sum_{t=1}^{k} \frac{n_i}{|G|} \chi_t(g_i^{-1})\chi_t(g_j) = \delta_{i,j}$$

If $x \in C_j$ and $y \in C_i$ and $i \neq j$ we obtain a).
If $x, y \in C_j$ the equality (with $i = j$) gives b). □

Second orthogonality relations over $\mathbb{C}$

Theorem

(keeping the introduced notation) Assume that $\mathbb{F} = \mathbb{C}$. Let $x, y \in G$. Then

a) *If x, y are not conjugated, then $\sum_{i=1}^{k} \chi_i(x)\overline{\chi_i(y)} = 0$*

b) *If $x, y \in C_j$, then $\sum_{i=1}^{k} \chi_i(x)\overline{\chi_i(y)} = \frac{|G|}{|C_j|}$.*

Properties of G determined by the complex character table

Theorem

Let G be a finite group. The following information can be read from the character table of G over $\mathbb{C}$.

a) *Which of* $C_1, C_2, \ldots, C_k$ *contains* 1_G
b) *degrees of* $\varphi_1, \varphi_2, \ldots, \varphi_k$
c) *the order of* G
d) *the values* $n_1, n_2, \ldots, n_k$*, where* $n_i = |C_i|$
e) $Z(G)$ *as a union of some conjugacy classes*
f) $\mathrm{Ker}\ \varphi_1, \ldots, \mathrm{Ker}\ \varphi_k$ *as unions of some conjugacy classes*
g) $[G, G]$ *unions of some conjugacy classes*
h) *the lattice of normal subgroups of* G

a) How to detect C_j with $C_j = \{1_G\}$

Note that $\chi_i(1_G)$ is the degree of φ_i. So if $C_j = \{1_G\}$, then the j-th column of the character table consists of positive integers. This column has to be orthogonal with respect to a standard scalar product to all the remaining columns in the character table. In particular, there is only one column in the table containing only non-negative real entries.

b),c)How to find $|G|$ in the complex character table

Recall that if $d_i = \chi_i(1_G)$ is the degree of φ_i, then $|G| = \sum_{i=1}^{k} d_i^2$. The values $d_1, d_2, \ldots, d_k$ are in the j-th column of the character table, where $C_j = \{1_G\}$.

d) How to detect $n_i = |C_i|$

Recall the second orthogonality relations for the case $\mathbb{F} = \mathbb{C}$:

$$\sum_{i=1}^{k} \chi_i(g_j)\overline{\chi_i(g_j)} = \frac{|G|}{n_j}$$

So $n_j = \frac{|G|}{||a_j||^2}$, where a_j is the j-th column of the character table.

e) How to find $Z(G)$

Note that $g \in Z(G)$ if and only if $|\{hgh^{-1} \mid h \in G\}| = 1$. So $Z(G)$ is a union of conjugacy classes of size 1.

$$Z(G) = \cup_{1 \leq i \leq k, |C_i|=1} C_i$$

That's all for today

Thank you for your attention