Group representations 1

Orthogonality relations of characters

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Some applications

Corollary

Let G be a finite group. If $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ and $\psi\colon G \to \mathrm{GL}(m, \mathbb{F})$ are irreducible matrix representations of G over $\mathbb{F}$ which are not equivalent and $X \in \mathrm{M}_{n,m}(\mathbb{F})$ is an arbitrary matrix then

$$\sum_{g \in G} \varphi(g) X \psi(g^{-1}) = 0$$

Proof.

Let $Y := \sum_{g \in G} \varphi(g) X \psi(g^{-1})$. Then for every $h \in G$ is

$$\varphi(h) Y = \sum_{g \in G} \varphi(hg) X \psi(g^{-1} h^{-1} h) = [\sum_{g \in G} \varphi(hg) X \psi((hg)^{-1})] \psi(h) .$$

In other words $\varphi(h) Y = Y \psi(h)$ for every $h \in G$. By Schur's lemma, $Y = 0$ or $m = n$ and Y is regular. If Y is regular, we obtain $\varphi(h) = Y \psi(h) Y^{-1}$ for every $h \in G$, that is, φ and ψ are equivalent.

Some applications, cont.

Corollary

Let G be a finite group. If $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ is an irreducible matrix representation and $\mathbb{F}$ is algebraically closed, then for every $X \in \mathrm{M}_n(\mathbb{F})$ there exists $\lambda \in \mathbb{F}$ such that

$$\sum_{g \in G} \varphi(g) X \varphi(g^{-1}) = \lambda E$$

(the proof is similar as the proof of the previous corollary)

Notation

When talking about matrix representation of group G we will use the following notation.
If $\psi\colon G \to \mathrm{GL}(n, \mathbb{F})$ is a matrix representation of G over $\mathbb{F}$ of degree n, then for every $1 \leq i, j \leq n$ we denote

$$\psi_{i,j}\colon G \to \mathbb{F}$$

the coordinate function of ψ. That is, for $g \in G$, $\psi_{i,j}(g)$ is the value in the (i, j)-th position of the matrix $\psi(g)$.

First important corollary of Schur's lemma

Proposition

Let G be a finite group $\varphi\colon G \to \mathrm{GL}(n,\mathbb{F})$, $\psi\colon G \to \mathrm{GL}(m,\mathbb{F})$ two irreducible matrix representations of G over $\mathbb{F}$. If φ and ψ are not equivalent then for every $1 \le i,j \le n$ and $1 \le k,\ell \le m$ we have

$$\sum_{g\in G} \varphi_{i,j}(g)\psi_{k,\ell}(g^{-1}) = 0\,.$$

Proof: Since φ and ψ are not equivalent and irreducible, we have

$$0 = \sum_{g\in G} \varphi(g)X\psi(g^{-1})$$

for every $X \in \mathrm{M}_{n,m}(\mathbb{F})$. Consider the particular choice $X = E_{j,k}$ (a $n \times m$ matrix being zero almost everywhere with the only exception in the position (j,k), where the value is one).

the proof, cont.

The general formula for the value in the (i, ℓ)-th position of the product in $\varphi(g)X\psi(g^{-1})$, i.e.

$$[\varphi(g)X\psi(g^{-1})]_{i,\ell} = \sum_{u,v} \varphi_{i,u}(g) \cdot [X]_{u,v} \cdot \psi_{v,\ell}(g^{-1})$$

then simplifies to

$$[\varphi(g)E_{j,k}\psi(g^{-1})]_{i,\ell} = \varphi_{i,j}(g)\psi_{k,\ell}(g^{-1})$$

Therefore

$$0 = [\sum_{g \in G} \varphi(g)E_{j,k}\psi(g^{-1})]_{i,\ell} = \sum_{g \in G} \varphi_{i,j}(g)\psi_{k,\ell}(g^{-1})$$

as we wanted to prove.

Orthogonality of characters - nonequivalent representations

Theorem

Let G be a finite group, $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$, $\psi\colon G \to \mathrm{GL}(m, \mathbb{F})$ irreducible representations which are not equivalent. Then

$$\sum_{g \in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = 0$$

Proof.

The proof of this just a straightforward application of the previous Proposition:

$$\sum_{g \in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = \sum_{g \in G} (\sum_{i=1}^{n} \varphi_{i,i}(g))(\sum_{j=1}^{m} \psi_{j,j}(g^{-1})) =$$

$$(\sum_{i=1}^{n} \sum_{j=1}^{m} (\sum_{g \in G} \varphi_{i,i}(g)\psi_{j,j}(g^{-1})) = 0$$

Second important corollary of Schur's lemma

Proposition

Let G be a finite group, $\mathbb{F}$ algebraically closed field, $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ an irreducible matrix representation of G over $\mathbb{F}$. Then

$$n \sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,\ell}(g^{-1}) = \delta_{i,\ell}\delta_{j,k}|G|1_{\mathbb{F}}$$

Proof: Recall we know that $\sum_{g \in G} \varphi(g)X\varphi(g^{-1}) = \lambda_X E$ for some $\lambda_X \in \mathbb{F}$. Again we consider the particular choice $X = E_{j,k}$,

$$[\varphi(g)E_{j,k}\varphi(g^{-1})]_{i,\ell} = \varphi_{i,j}(g)\varphi_{k,\ell}(g^{-1})$$

Therefore $\sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,\ell}(g^{-1}) = 0$ if $i \neq \ell$.

The proof, cont.

Similarly, if $j \neq k$, then

$$\sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,\ell}(g^{-1}) = \sum_{g \in G} \varphi_{k,\ell}(g^{-1})\varphi_{i,j}(g) =$$

$$\sum_{g \in G} \varphi_{k,\ell}(g)\varphi_{i,j}(g^{-1}) = 0\,.$$

So it remains to prove that

$$n \sum_{g \in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = |G|.1_{\mathbb{F}}\,.$$

The proof, final part

Consider the formula $\sum_{g\in G} \varphi(g)X\varphi(g^{-1}) = \lambda_X E$ for the choice $X = E_{j,j}$. There exists $\lambda_j \in \mathbb{F}$ such that

$$\sum_{g\in G} \varphi(g)E_{j,j}\varphi(g^{-1}) = \lambda_j E, \sum_{g\in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = \lambda_j$$

for every $1 \le i \le n$.

Compute the trace of $\sum_{g\in G} \varphi(g)E_{j,j}\varphi(g^{-1})$ in two ways:

$$\mathrm{Tr}(\sum_{g\in G} \varphi(g)E_{j,j}\varphi(g^{-1})) = \sum_{g\in G} \mathrm{Tr}(\varphi(g)E_{j,j}\varphi(g)^{-1}) =$$

$$|G|\mathrm{Tr}(E_{j,j}) = |G|.1_{\mathbb{F}}$$

Also $\mathrm{Tr}(\sum_{g\in G} \varphi(g)E_{j,j}\varphi(g^{-1})) = \mathrm{Tr}(\lambda_j E) = n\lambda_j$.

It follows that $n\lambda_j = n\sum_{g\in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = |G|.1_{\mathbb{F}}$ as we wanted to show.

Important remark

Remark

Assume that G is a finite group, $\mathbb{F}$ algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then $\mathrm{char}(\mathbb{F})$ does not divide a degree of any irreducible matrix representation of G over $\mathbb{F}$

Proof.

Let $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ be a matrix representation of G over $\mathbb{F}$. Look at the formula

$$n \sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,\ell}(g^{-1}) = \delta_{i,\ell}\delta_{j,k}|G|1_{\mathbb{F}}$$

If $\mathrm{char}(\mathbb{F}) \mid n$, the left hand side of this formula is always zero. On the other hand, the right hand side of can be $|G|.1_{\mathbb{F}} \neq 0$ if $i = \ell$ and $j = k$ and $\mathrm{char}(\mathbb{F}) \nmid |G|$. $\square$

Theorem

Let G be a finite group, $\mathbb{F}$ algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. If $\varphi, \psi \colon G \to \mathrm{GL}(n, \mathbb{F})$ are equivalent irreducible matrix representations of G over $\mathbb{F}$, then

$$\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \chi_\psi(g^{-1}) = 1_{\mathbb{F}} \,.$$

Proof.

Since equivalent representations have equal characters, we may assume $\varphi = \psi$. Therefore

$$\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \chi_\psi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} (\sum_{i=1}^{n} \varphi_{i,i}(g))(\sum_{j=1}^{n} \varphi_{j,j}(g^{-1})) =$$

$$\frac{1}{|G|} \sum_{i,j=1}^{n} \sum_{g \in G} \varphi_{i,i}(g) \varphi_{j,j}(g^{-1}) = \frac{1}{|G|} \sum_{i=1}^{n} \sum_{g \in G} \varphi_{i,i}(g) \varphi_{i,i}(g^{-1}) = \frac{n|G|}{|G|n} = 1_{\mathbb{F}}$$

The summary of orthogonality relations

Theorem

Let G be a finite group, $\mathbb{F}$ an algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Consider two irreducible matrix representations φ, ψ of G over $\mathbb{F}$. Then

a) *If φ and ψ are equivalent, then $\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = 1_\mathbb{F}$*

b) *If φ and ψ are not equivalent, then $\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = 0$*

Remark

Of course, the same theorem holds for linear representations of finite degree.

Definition

We call the relations from this theorem as 'the first orthogonality relations of characters'.

A particular case of $\mathbb{F} = \mathbb{C}$

Recall that if φ is a matrix representation of a finite group G over $\mathbb{C}$ then $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$ for every $g \in G$. The previous theorem then says that that characters of non-equivalent irreducible representations are orthogonal with respect to the standard scalar product on the space of complex valued functions of G.

Theorem

Let G be a finite group and φ, ψ irreducible complex (matrix) representations of G. Then

a) *If φ and ψ are not equivalent, then*

$$\sum_{g \in G} \chi_\varphi(g)\overline{\chi_\psi(g)} = 0$$

b) *If φ and ψ are equivalent, then*

$$\frac{1}{|G|}\sum_{g \in G} \chi_\varphi(g)\overline{\chi_\psi(g)} = 1$$

Some consequences of orthogonality relations

Let G be a finite group, let $\mathbb{F}$ be a field. Consider the space

$$V := \{f \mid f\colon G \to \mathbb{F}\}$$

of all $\mathbb{F}$-valued functions on G with point-wise linear structure:

$$f_1 + f_2\colon g \mapsto f_1(g) + f_2(g), g \in G$$

$$tf\colon g \mapsto tf(g), g \in G$$

for $f_1, f_2, f \in V$ and $t \in \mathbb{F}$.
Let U be the subspace of V consisting of all functions which are constant on conjugacy classes of G, i.e.

$$U := \{f \in V \mid f(x) = f(y) \text{ whenever } \exists z \in G \;\, y = zxz^{-1}\}.$$

Note that if $\mathbb{F}G$ is understand as the set of all $\mathbb{F}$-valued functions on G, then U is exactly the center of $\mathbb{F}G$.

A bilinear form on V

Assume that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then we define a bilinear form $b\colon V \times V \to \mathbb{F}$ by the rule

$$b(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}), f_1, f_2 \in V$$

Recall that

- $\chi_\varphi \in U$ for every (matrix) representation φ of G over $\mathbb{F}$
- $b(\chi_\varphi, \chi_\psi) = 0$ if φ, ψ are non-equivalent irreducible (matrix) representations of G over $\mathbb{F}$
- $b(\chi_\varphi, \chi_\varphi) = 1_\mathbb{F}$ if $\mathbb{F}$ is algebraically closed and φ is an irreducible (matrix) representation of G over $\mathbb{F}$.

A basis of U

Proposition

Let G be a finite group, $\mathbb{F}$ and algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be a list of all different irreducible representations of G over $\mathbb{F}$ up to equivalence (i.e., every irreducible representation of $\mathrm{Rep}_{\mathbb{F}}(G)$ is equivalent to exactly one represenation on this list). Then $\chi_{\varphi_1}, \chi_{\varphi_2}, \ldots, \chi_{\varphi_k}$ is a basis of U.

Proof.

Recall $k = \dim_{\mathbb{F}}(U) = \#$ of conjugacy classes in G. So it is sufficient to verify linear independence of $\chi_{\varphi_1}, \chi_{\varphi_2}, \ldots, \chi_{\varphi_k}$. Assume $t_1, t_2, \ldots, t_k \in \mathbb{F}$ are such that $\sum_{i=1}^{k} t_i \chi_{\varphi_i} = 0$. Then

$$0 = b\Big(\sum_{i=1}^{k} t_i \chi_{\varphi_i}, \chi_{\varphi_j}\Big) = t_j$$

for every $1 \leq j \leq k$. Therefore the characters of the representations on the list are linearly independent elements of U.

End

Thank you for your attention.