Group representations 1

Wedderburn-Artin theorem in Group representations, part 2

March 29, 2021

Small recapitulation

Let G be a finite group, let $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$.

- Then every $\mathbb{F}G$-module is a direct sum of simple modules.
- $\mathbb{F}G \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$ as $\mathbb{F}$-algebras.
- The basic goal is to find all simple left $\mathbb{F}G$-modules up to isomorphism (or all irreducible representations of G over $\mathbb{F}$ up to equivalence).
- This is roughly the same problem as to find explicit expression of the isomorphism above.

Centers of matrix rings

If R is a ring, the center of R is the set

$$Z(R) = \{r \in R \mid \forall s \in R \;\; rs = sr\}$$

Let $R = \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times M_{n_k}(\mathbb{F})$. Then $Z(R)$ can be determined using these two easy facts (proofs of these are subjects of the last problem session).

Exercise

1. *If R_1 and R_2 are rings then $Z(R_1 \times R_2) = Z(R_1) \times Z(R_2)$*
2. $Z(\mathrm{M}_n(\mathbb{F})) = \{\lambda E \mid \lambda \in \mathbb{F}\}$

For example, if $R = \mathbb{F} \times \mathrm{M}_2(\mathbb{F}) \times \mathrm{M}_3(\mathbb{F})$, then

$$Z(R) = \left\{ \left((a), \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right) \mid a, b, c \in \mathbb{F} \right\}$$

Back to Homework # 1

Definition

We say that G has k distinct irreducible representations over $\mathbb{F}$ up to equivalence if there are $\varphi_1, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible such that every irreducible $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is equivalent to exactly one of these representations (in particular if φ_i and φ_j are equivalent, then $i = j$).

Theorem

Let G be a finite group and let $\mathbb{F}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let k be the number of conjugacy classes in G. Then G has k distinct irreducible representations over $\mathbb{F}$ up to equivalence.

Proof of the theorem

In the language of $\mathbb{F}G\text{-}\mathrm{Mod}$ we have to prove that there exists simple left $\mathbb{F}G$-modules $S_1, \ldots, S_k$ such that every simple left $\mathbb{F}G$-module is isomorphic to exactly one of these modules.
The theorem of Maschke implies that $\mathbb{F}G$ is semisimple artinian.
The Wedderburn-Artin theorem gives an isomorphism of $\mathbb{F}$-algebras

$$\mathbb{F}G \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_\ell}(\mathbb{F}) =: R\,.$$

The Homework # 1 gives $\dim_{\mathbb{F}}(Z(\mathbb{F}G)) = k$ while the matrix structure of R gives $\dim_{\mathbb{F}}(Z(R)) = \ell$, where ℓ can be also interpreted as the number of simple left R-modules up to isomorphism.
Since $Z(\mathbb{F}G)$ and $Z(R)$ are isomorphic vector spaces, we conclude $k = \ell$.

Endomorphism ring of a module

Let $\mathbb{F}$ be a field, R an $\mathbb{F}$-algebra. For every pair of (right) R-modules M, N the set $\mathrm{Hom}_R(M, N)$ has a natural structure of a vector space over $\mathbb{F}$.

If $f, g \in \mathrm{Hom}_R(M, N)$ and $t \in \mathbb{F}$, we define

$$f + g\colon m \mapsto f(m) + g(m), m \in M$$

$$tf\colon m \mapsto t.f(m), m \in M$$

If $M = N$, the set $\mathrm{Hom}_R(M, M) = \mathrm{End}_R(M)$ has a canonical structure of an $\mathbb{F}$-algebra:

$$f \cdot g\colon m \mapsto f(g(m)), m \in M$$

$$1_{\mathrm{End}_R(M)} := 1_M\colon m \mapsto m, m \in M$$

Definition

If R is an $\mathbb{F}$-algebra and M is an R-module, then the $\mathbb{F}$-algebra $\mathrm{End}_R(M)$ is called *the endomorphism algebra* of M.

Schur's lemma for modules

Lemma

(Schur) Let R be a ring and M, N simple (left) R-modules. Then

a) *If M, N are not isomorphic then $\mathrm{Hom}_R(M, N) = 0$.*
b) *If $M = N$ then every nonzero element of $\mathrm{End}_R(M)$ has an inverse (such rings are called division rings)*
c) *If R is a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$, then the $\mathbb{F}$-algebra $\mathrm{End}_R(M)$ is isomorphic to $\mathbb{F}$.*

Proof.

a)+b) If $f \in \mathrm{Hom}_R(M, N)$, then $\mathrm{Ker}\ f \leq M$ and $\mathrm{Im}\ f \leq N$. Since M, N are simple modules only two cases may happen:

1. $\mathrm{Ker}\ f = M$ and hence also $\mathrm{Im}\ f = 0$, that is, $f = 0$.
2. $\mathrm{Ker}\ f = 0$ and hence $\mathrm{Im}\ f = N$. In this case f is an isomorphism.

□

proof of part c)

We want to prove: If R is a finite-dimensional algebra over an algebraically closed field $\mathbb{F}$, then the $\mathbb{F}$-algebra $\mathrm{End}_R(M)$ is isomorphic to $\mathbb{F}$.
We prove that $\mathrm{End}_R(M) = \{\lambda 1_M \mid \lambda \in \mathbb{F}\}$. Then the map

$$\lambda \mapsto \lambda \cdot 1_M, \lambda \in \mathbb{F}$$

gives the isomorphism between $\mathbb{F}$ and $\mathrm{End}_R(M)$.
Recall that $M \simeq R/I$ for some maximal right ideal of R. In particular, $\dim_{\mathbb{F}}(M) < \infty$.
Let $f \in \mathrm{End}_R(M)$. Then f can be considered as an element of $\mathrm{End}_{\mathbb{F}}(M)$:

$$f(tm) = f((t1_R) \cdot m) = (t1_R) \cdot f(m) = t \cdot f(m), t \in \mathbb{F}, m \in M$$

Let $\lambda \in \mathbb{F}$ be an eigenvalue of f with eigenvector $0 \neq m \in M$.
Then $f - \lambda \cdot 1_M \in \mathrm{End}_R(M)$ has nontrivial kernel, since $f(m) = \lambda \cdot m$.
Since M is a simple R-module, $M = \mathrm{Ker}\ (f - \lambda \cdot 1_M)$, i.e., $f = \lambda \cdot 1_M$.

Comments on the proof of the Wedderburn-Artin theorem

Assume that R is semisimple artinian ring which is also a finite-dimensional $\mathbb{F}$-algebra, where $\mathbb{F}$ is algebraically closed. Then

$$R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

as $\mathbb{F}$-algebras for some $n_1, n_2, \ldots, n_k \in \mathbb{N}$.
Idea of the proof: Decompose R_R into a direct sum of simple right modules

$$R_R = T_1 \oplus T_2 \oplus \cdots \oplus T_m$$

Group together isomorphic simple modules. If $S_1, S_2, \ldots, S_k$ are pairwise non-isomorphic simple right modules such that each T_i is isomorphic to exactly one of these modules, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$R_R \simeq S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}$$

Some isomorphisms of $\mathbb{F}$-algebras

The proof is based on the following isomorphisms of $\mathbb{F}$-algebras:

$$R \simeq \mathrm{End}_R(R_R), r \mapsto \lambda_r,$$

where $\lambda_r(s) = rs$ for every $s \in R$.
Endomorphism algebras of isomorphic modules are isomorphic. In our case

$$\mathrm{End}_R(R_R) \simeq \mathrm{End}_R(\oplus_{i=1}^k S_i^{n_i}) .$$

Apply Schur's lemma part a)

$$\mathrm{End}_R(\oplus_{i=1}^k S_i^{n_i}) \simeq \prod_{i=1}^k \mathrm{End}_R(S_i^{n_i})$$

The principle from linear algebra - linear maps can be displayed as matrices can be generalized

$$\mathrm{End}_R(S^m) \simeq \mathrm{M}_m(\mathrm{End}_R(S))$$

for every right R-module S and every $m \in \mathbb{N}$.

Some isomorphisms of $\mathbb{F}$-algebras, cont.

Finally use Schur's lemma part c) to see $\mathrm{End}_R(S_i) \simeq \mathbb{F}$ and hence also

$$\mathrm{End}_R(S_i^{n_i}) \simeq \mathrm{M}_{n_i}(\mathbb{F})$$

for every $1 \leq i \leq n$. Composing all these isomorphisms we get the desired one:

$$R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

The multiplicity

Let us state without a proof a particular case of the Krull-Schmidt theorem:

Theorem

Let R be a ring and let $S_1, \dots, S_m$, $T_1 \dots T_n$ be simple left R-modules. Then $S_1 \oplus \cdots \oplus S_m \simeq T_1 \oplus \cdots \oplus T_n$ if and only if $n = m$ and there exists a permutation $\sigma \in S(\{1, 2, \dots, n\})$ such that $S_i \simeq T_{\sigma(i)}$ for every $i \in \{1, 2, \dots, n\}$

Recall that if R is a semisimple artinian ring, then every left R-module is a direct sum of simple left modules.

Definition

Let R be a semisimple artinian ring and let M be a finitely generated R-module. Then there are simple modules $S_1, S_2, \dots, S_m$ such that $M \simeq \oplus_{i=1}^{m} S_i$. If S is a simple left R-module, the *multiplicity of S* in M is defined as

$$|\{i \in \{1, \dots, m\} \mid S_i \simeq S\}|$$

An example from the last lecture

Let $\mathbb{F}$ be a field and assume that

$$R = \mathbb{F} \times M_2(\mathbb{F}) \times M_3(\mathbb{F})$$

Recall there are up to isomorphism 3 different simple R-modules:

$$S_1 := \mathbb{F} \times \{0\} \times \{0\}$$

$$T_1 := \{0\} \times \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{F} \right\} \times \{0\}$$

$$U_1 := \{0\} \times \{0\} \times \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

And ${}_RR \simeq S_1 \oplus T_1^2 \oplus U_1^3$. So the multiplicity of S_1 in ${}_RR$ is 1, the multiplicity of T_1 in ${}_RR$ is 2 and the multiplicity of U_1 in ${}_RR$ is 3. Note that the multiplicity of a simple module in ${}_RR$ coincides with its dimension.

multiplicity = dimension

Proposition

Assume that R be a semisimple artinian ring which is a finite dimensional $\mathbb{F}$-algebra, $\mathbb{F}$ algebraically closed. For any simple left R-module the multiplicity of S in ${}_RR$ coincides with $\dim_{\mathbb{F}}(S)$.

Proof.

Use the Wedderburn-Artin theorem, there are $k \in \mathbb{N}_0$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

as $\mathbb{F}$-algebras. So we may assume

$$R = \mathrm{M}_{n_1}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

and in this case we know that there are simple left modules $S_1, S_2, \ldots, S_k$ pair-wise non-isomorphic such that ${}_RR \simeq S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}$. So the multiplicity of S_i in R is n_i which coincides with $\dim_{\mathbb{F}}(S_i)$.

The regular representation

Let G be a finite group, $\mathbb{F}$ an algebraically closed field with $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $S_1, S_2, \ldots, S_k$ be simple left $\mathbb{F}G$ modules such that every simple left $\mathbb{F}G$ module is isomorphic to exactly 1 of these modules. We already know that k is the number of conjugacy classes in G. For $1 \leq i \leq k$ let $d_i := \dim_{\mathbb{F}}(S_i)$.
By the previous proposition the multiplicity of S_i in ${}_{\mathbb{F}G}\mathbb{F}G$ is exactly d_i, so $\mathbb{F}G \simeq \oplus_{i=1}^{k} S_i^{d_i}$ as left $\mathbb{F}G$-modules.
Now look at dimensions of these modules: $\dim_{\mathbb{F}}(\mathbb{F}G) = |G|$, $\dim_{\mathbb{F}}(\oplus_{i=1}^{k} S_i^{d_i}) = \sum_{i=1}^{k} d_i^2$.
Thus we conclude

$$|G| = \sum_{i=1}^{k} d_i^2 \, .$$

Summary in the language of $\mathrm{Rep}_{\mathbb{F}}(G)$

Theorem

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$.

1. *Every representation of G over $\mathbb{F}$ is equivalent to a direct sum of irreducible representations.*
2. *If k is the number of conjugacy classes of G, then there are $\varphi_1, \varphi_2, \cdots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible such that every irreducible representation of G over $\mathbb{F}$ is equivalent to exactly one of these representations.*
3. *If d_i is the degree of φ_i then the multiplicity of φ_i in $\mathrm{reg}_{\mathbb{F}}(G)$ is d_i*

$$\mathrm{reg}_{\mathbb{F}}(G) \simeq \overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{d_1} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{d_k}$$

4. $|G| = \sum_{i=1}^{k} d_i^2$

The Krull-Schmidt theorem in $\mathrm{Rep}_{\mathbb{F}}(G)$

Theorem

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi_1, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be a list of all distinct irreducible representations in $\mathrm{Rep}_{\mathbb{F}}(G)$ up to equivalence. Then the representations $\overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{m_1} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{m_k}$ and $\overbrace{\varphi_1 \oplus \cdots \oplus \varphi_1}^{\ell_1} \oplus \cdots \oplus \overbrace{\varphi_k \oplus \cdots \oplus \varphi_k}^{\ell_k}$ are equivalent if and only if $m_i = \ell_i$ for every $1 \leq i \leq k$.

Remark

This theorem holds whithout assumptions on $\mathbb{F}$ and can be generalized also to representations of infinite groups - the only problem is that there can be infinitely many distinct irreducible representations of G over $\mathbb{F}$.

Thank you for your attention

There is no lecture next Monday.
But I will try to prepare something for Tuesday, April 6.