Group representations 1

Wedderburn-Artin theorem in Group representations

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Representations of $\mathbb{Z}_3$ over $\mathbb{Q}$

Example

Find representations of $\mathbb{Z}_3$ over $\mathbb{Q}$.

Use the following isomorphisms of $\mathbb{Q}$-algebras:

$$\mathbb{Q}\mathbb{Z}_3 \simeq \mathbb{Q}/(x^3 - 1) \simeq \mathbb{Q} \times \mathbb{Q}[x]/(x^2 + x + 1)$$

The product on the right hand side gives us two $\mathbb{Q}\mathbb{Z}_3$-modules
$M_1 := \mathbb{Q} \times \{0\}$. The element δ_1 acts as the identity on this module.
$M_2 := \{0\} \times \mathbb{Q}[x]/(x^2 + x + 1)$ The element δ_1 acts as $\overline{x}$ on this module.
The theory says that every $\mathbb{Q}\mathbb{Z}_3$-module is isomorphic to a direct sum $M_1^{(X)} \oplus M_2^{(Y)}$
Therefore there are two 'basic' representations in $\mathrm{Rep}_{\mathbb{Q}}(G)$ such that every other representation is equivalent to a direct sum of copies of these two.

Maschke's theorem proved last time

Theorem

(Maschke) Let G be a finite group, $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ a representation of G over $\mathbb{F}$, where $\mathbb{F}$ is a field such that $\mathrm{char}\,(\mathbb{F}) \nmid |G|$*. If $U \leq V$ is a φ-invariant subspace of V, then there exists $W \leq V$ also φ-invariant such that $V = U \oplus W$. In particular, φ is equivalent to $\varphi_U \oplus \varphi_W$.*

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Corollary

Let G be a finite group, $\mathbb{F}$ be a field $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then every representation of G over $\mathbb{F}$ is equivalent to a direct sum of irreducible representations.

Proof: We prove this only for representations acting on spaces of finite dimension. Let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$ of degree $d = \dim_{\mathbb{F}}(V)$. If φ is not irreducible, there exists a φ-invariant subspace $0 < U < V$. Let W be a φ-invariant complement of U. Then φ is equivalent to $\varphi_U \oplus \varphi_W$. Apply the induction argument to φ_U and φ_W.

A short overview what are we going to do

When the theorem of Maschke applies, we need to find all irreducible representations.

Every representation is then built from irreducible ones.

The theory says that one can find all irreducible representations up to equivalence when the regular representation is written as a direct sum of irreducible representations. (but only in the case when Maschke's theorem applies)

We will use the language of rings and modules to study the regular representation.

Representations given by actions

Let G be a group, let X be a non-empty set and let $*\colon G \times X \to X$ be an action of G on X.

For a given field $\mathbb{F}$ we can construct a representation of G over $\mathbb{F}$ from this action: Let V be a vector space over $\mathbb{F}$ with basis X. We define a representation

$$\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$$

by specifying the values of $\varphi(g)$ on X for each $g \in G$.

$$\varphi(g)\colon x \mapsto g * x$$

(each $\varphi(g)$ permutes elements of basis X, so $\varphi(g) \in \mathrm{Aut}_{\mathbb{F}}(V)$ for every $g \in G$.) Note that φ is a group homomorphism, since

$$\varphi(g) \circ \varphi(h)\colon x \mapsto g * (h * x), x \in X$$

$$\varphi(gh)\colon x \mapsto (gh) * x, x \in X$$

and therefore $\varphi(g) \circ \varphi(h) = \varphi(gh)$ for every $g, h \in G$.

Regular representation

Consider the action of G on G by the left translation - that is $X = G$ and $g * x = gx$ for every $g \in G, x \in X$. If $\mathbb{F}$ is a field, the representation of G over $\mathbb{F}$ given by this action is called *the regular representation* of G over $\mathbb{F}$.

Note that essentialy same representation is the representation which is assigned to the module ${}_{\mathbb{F}G}\mathbb{F}G$ ($\mathbb{F}G$ considered as a left module over $\mathbb{F}G$).

Indeed, recall that if M is a left $\mathbb{F}G$-module the corresponding representation

$$\varphi_M \colon G \to \mathrm{Aut}_{\mathbb{F}}(M)$$

is defined by $\varphi_M(g) \colon m \mapsto \delta_g m$.

Now if $M = \mathbb{F}G$, we consider its basis $X := \{\delta_g \mid g \in G\}$. So $[\varphi_{\mathbb{F}G}(g)](\delta_x) = \delta_g * \delta_x = \delta_{gx}$. That is, $\varphi_{\mathbb{F}G}(g)$ permutes elements of X essentially using the left translation.

Definition

We denote $\mathrm{reg}_{\mathbb{F}}(G)$ the regular representation of G over $\mathbb{F}$.

$\mathrm{Rep}_{\mathbb{F}}(G)$ – $\mathbb{F}G$-Mod dictionary

representation of G over $\mathbb{F}$	$\mathbb{F}G$-module
equivalent representations	isomorphic modules
φ-invariant subspace	submodule
irreducible representation	simple module
$\mathrm{reg}_{\mathbb{F}}(G)$	${}_{\mathbb{F}G}\mathbb{F}G$

Semisimple artinian rings, why are we interested?

Definition

A ring R is said to be *semisimple artinian* if the lattice of its left ideals is complementary (i.e. for every left ideal I of R there exists a left ideal J of R such that $R = I \oplus J$).

Example

Let G be a finite group and let $\mathbb{F}$ be a field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then $\mathbb{F}G$ is semisimple artinian.

(Maschke's theorem applied to $\mathrm{reg}_{\mathbb{F}}(G)$ when translated to the language of $\mathbb{F}G$-mod exactly says that the lattice of left ideals of $\mathbb{F}G$ is complementary.)

Wedderburn-Artin theorem, summary from another lecture

Theorem

Let R be a ring. TFAE

(a) *R is a semisimple artinian ring*

(a') *the lattice of right ideals of R is complementary*

(b) *the lattice submodules of any left R-module is complementary*

(b') *the lattice of submodules of any right R-module is complementary*

(c) *There exists $k \in \mathbb{N}_0$, $n_1, n_2, \dots, n_k \in \mathbb{N}$ and division rings $D_1, D_2, \dots, D_k$ such that R is (as a ring) isomorphic to*

$$\mathrm{M}_{n_1}(D_1) \times \mathrm{M}_{n_2}(D_2) \times \cdots \times \mathrm{M}_{n_k}(D_k)$$

the role of irreducible representations

Theorem

If R is a semisimple artinian ring, then every left (right) R-module is a direct sum of simple left (right) R-modules.

Corollary

If G is a finite group and $\mathbb{F}$ is a field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$, then every representation of G over $\mathbb{F}$ is equivalent to a direct sum irreducible representations.

$\mathbb{F}$-algebras, $\mathbb{F}$ algebraically closed

Theorem

Let R be a finite dimensional $\mathbb{F}$-algebra, $\mathbb{F}$ algebraically closed. TFAE

(a) *R is a semisimple artinian ring*

(c) *There exists $k \in \mathbb{N}_0$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$*

$$R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

as $\mathbb{F}$-algebras.

Corollary

Let G be a finite group, $\mathbb{F}$ an algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Then there exists $k \in \mathbb{N}_0$, $n_1, n_2, \ldots, n_k \in \mathbb{N}$

$$\mathbb{F}G \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$$

as $\mathbb{F}$-algebras.

An example

Let $\mathbb{F}$ be a field and assume that

$$R = \mathbb{F} \times M_2(\mathbb{F}) \times M_3(\mathbb{F})$$

(so $k = 3$, $n_1 = 1, n_2 = 2, n_3 = 3$). We would like to understand simple left R-modules over R. Let

$$S_1 := \mathbb{F} \times \{0\} \times \{0\}$$

$$T_1 := \{0\} \times \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{F} \right\} \times \{0\}$$

$$T_2 := \{0\} \times \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{F} \right\} \times \{0\}$$

$$U_1 := \{0\} \times \{0\} \times \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

$$U_2 := \{0\} \times \{0\} \times \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & b & 0 \\ 0 & c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

$$U_3 := \{0\} \times \{0\} \times \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

An example, cont.

The following statements are easy to prove

- $R = S_1 \oplus T_1 \oplus T_2 \oplus U_1 \oplus U_2 \oplus U_3$
- $S_1, T_1, T_2, U_1, U_2, U_3$ are simple left R-modules
- Every simple left R-module is isomorphic to one of these modules
- $T_1 \simeq T_2$ and $U_1 \simeq U_2 \simeq U_3$
- There are up to isomorphism 3 different simple R-modules. That is S_1 is not isomorphic neither to T_1 nor to U_1 and T_1 is not isomorphic to U_1.
- Note that $k = 3$ is the number distinct simple left R-modules
- Note that for each simple left R-module X the number of copies of X appearing in the expression of R as a direct sum of simple R-modules is exactly $\dim_{\mathbb{F}}(X)$.

A straightforward generalization

Let $R = \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F})$. Then the left module ${}_R R$ is decomposed as $S_1^{n_1} \oplus S_2^{n_2} \oplus \cdots \oplus S_k^{n_k}$, where $S_1, S_2, \dots, S_k$ are pair-wise non-isomorphic simple R-modules and every simple R-module is isomorphic to (exactly 1) of these modules.
Moreover, $\dim_{\mathbb{F}}(S_i) = n_i$ for every $i = 1, \dots, k$.
There may be more ways how to write ${}_R R$ as a direct sum of simple modules. But if $R \simeq S_1^{m_1} \oplus S_2^{m_2} \oplus \cdots \oplus S_k^{m_k}$ for some $m_1, \dots, m_k \in \mathbb{N}_0$ then $m_1 = n_1, \dots, m_k = n_k$.

Centers of matrix rings

If R is a ring, the center of R is the set

$$Z(R) = \{r \in R \mid \forall s \in R \;\; rs = sr\}$$

Let $R = \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times M_{n_k}(\mathbb{F})$. Then $Z(R)$ can be determined using these two easy facts (proofs of these are subjects of a forthcoming problem session)

Exercise

1. *If R_1 and R_2 are rings then $Z(R_1 \times R_2) = Z(R_1) \times Z(R_2)$*
2. $Z(\mathrm{M}_n(\mathbb{F})) = \{\lambda E \mid \lambda \in \mathbb{F}\}$

In our example

$$Z(R) = \left\{ \left((a), \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right) \mid a, b, c \in \mathbb{F} \right\}$$

Back to Homework # 1

Definition

We say that G has k distinct irreducible representations over $\mathbb{F}$ up to equivalence if there are $\varphi_1, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible such that every irreducible $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is equivalent to exactly one of these representations (in particular if φ_i and φ_j are equivalent, then $i = j$).

Theorem

Let G be a finite group and let $\mathbb{F}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let k be the number of conjugacy classes in G. Then G has k distinct irreducible representations over $\mathbb{F}$ up to equivalence.

Proof of the theorem

In the language of $\mathbb{F}G\text{-}\mathrm{Mod}$ we have to prove that there exist simple left $\mathbb{F}G$-modules $S_1, \ldots, S_k$ such that every simple left $\mathbb{F}G$-module is isomorphic to exactly one of these modules.
The theorem of Maschke implies that $\mathbb{F}G$ is semisimple artinian.
The Wedderburn-Artin theorem gives an isomorphism of $\mathbb{F}$-algebras

$$\mathbb{F}G \simeq R \simeq \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_\ell}(\mathbb{F}) \, .$$

The Homework # 1 gives $\dim_{\mathbb{F}}(Z(\mathbb{F}G)) = k$ while the matrix structure of R gives $\dim_{\mathbb{F}}(Z(R)) = \ell$, where ℓ can be also interpreted as the number of simple left R-modules up to isomorphism.
Since $Z(\mathbb{F}G)$ and $Z(R)$ are isomorphic vector spaces, we conclude $k = \ell$.

That's all for today

Thank you for your attention.