Group representations 1

Vocabulary for group representation theory

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Group algebras

Let $\mathbb{F}$ be a field and let G be a group. The group algebra $\mathbb{F}G$ is an $\mathbb{F}$-vector space with basis $\{\delta_g \mid g \in G\}$ equipped with a bilinear operation $*\colon \mathbb{F}G \times \mathbb{F}G \to \mathbb{F}G$ given on the basis by

$$\delta_g * \delta_h = \delta_{gh}, g, h \in G$$

If G is a finite group, then elements of $\mathbb{F}G$ can be written as $\sum_{g \in G} t_g \delta_g, t_g \in \mathbb{F}$. Then

$$(\sum_{g \in G} t_g \delta_g) * (\sum_{g \in G} s_g \delta_g) = \sum_{g,h \in G} t_g s_h (\delta_g * \delta_h) =$$

$$\sum_{g,h \in G} t_g s_h (\delta_{gh}) = \sum_{g \in G} (\sum_{h \in G} t_h s_{h^{-1}g}) \delta_g$$

Also note $\delta_{1_G} * f = f = f * \delta_{1_G}$ for every $f \in \mathbb{F}G$.

Proposition

$(\mathbb{F}G, +, -, 0, *, \delta_{1_G})$ *is an associative* $\mathbb{F}$*-algebra (ring having structure of an* $\mathbb{F}$*-vector which behaves well with the operations of the ring).*

Another way how to look on $\mathbb{F}G$

Let G be a finite group and let $\mathbb{F}$ be a field.
Consider $\mathbb{F}G = \{f \mid f$ is a map from G to $\mathbb{F}\}$. Note that this set can be naturally seen as a vector space over $\mathbb{F}$:

$$f_1 + f_2 \colon g \mapsto f_1(g) + f_2(g), g \in G$$

$$0 \colon g \mapsto 0, g \in G$$

$$tf \colon g \mapsto t.f(g), g \in G, t \in \mathbb{F}$$

For $f_1, f_2 \colon G \to \mathbb{F}$, define $f_1 * f_2 \colon G \to \mathbb{F}$ by

$$f_1 * f_2 \colon g \mapsto \sum_{h \in G} f_1(h) f_2(h^{-1} g)$$

Then $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$. So we defined an associative bilinear binary operation on $\mathbb{F}G$.
In fact this structure is essentially the same as the one defined on the previous slide.

Why are we interested in group algebras?

Theorem

Let G be a group and let $\mathbb{F}$ be a field. The category $\mathrm{Rep}_{\mathbb{F}}(G)$ is equivalent to the category $\mathbb{F}G\text{-}\mathrm{Mod}$.

So the theory of modules over associative algebras provides a languague we could use to study group representations.

Idea of the proof

Let us construct a functor $F\colon \mathrm{Rep}_{\mathbb{F}}(G) \to \mathbb{F}G\text{-}\mathrm{Mod}$.

Let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be an object of $\mathrm{Rep}_{\mathbb{F}}(G)$. We want to define an $\mathbb{F}G$-module $F(V)$.

The additive structure of $F(V)$ is given by the additive structure on V.

If $\sum_{g\in G} t_g\delta_g \in \mathbb{F}G$ and $v \in F(V) = V$, there is only one natural way how to define $(\sum_{g\in G} t_g\delta_g)\cdot v$:

$$(\sum_{g\in G} t_g\delta_g)\cdot v := \sum_{g\in G} t_g(\varphi_g(v))\,,$$

where $\varphi_g := \varphi(g) \in \mathrm{Aut}_{\mathbb{F}}(G)$.

It is straightforward to verify that $F(V)$ is an $\mathbb{F}G$-module.

If $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ and $\varphi'\colon G \to \mathrm{Aut}_{\mathbb{F}}(V')$ are objects of $\mathrm{Rep}_{\mathbb{F}}(G)$ and $f \in \mathrm{Rep}_{\mathbb{F}}(G)(\varphi,\varphi')$ is a morphism, then it is possible to verify that $f\colon F(V) \to F(V')$ is a homomorphism of $\mathbb{F}G$-modules.

If we define $F(f) := f$, then F preserves units and composition of morphisms, so we indeed defined a functor.

Idea of the proof, cont.

Conversely, we want to construct a functor
$H \colon \mathbb{F}G\text{-}\mathrm{Mod} \to \mathrm{Rep}_{\mathbb{F}}(G)$.
Consider an $\mathbb{F}G$-module M. Observe that M has a natural structure of an $\mathbb{F}$-space

$$tm := (t\delta_{1_G}) \cdot m, t \in \mathbb{F}, m \in M$$

On this vector space there is a canonical action of G:

$$g * m := \delta_g \cdot m, g \in G, m \in M$$

This suggests to consider a representation $\varphi \colon G \to \mathrm{Aut}_{\mathbb{F}}(M)$

$$\varphi(g) \colon m \mapsto \delta_g \cdot m$$

and to define $H(M) := \varphi$.
If $f \in \mathrm{Hom}_{\mathbb{F}G}(M, N)$ then we check that
$f \in \mathrm{Rep}_{\mathbb{F}(G)}(H(M), H(N))$, so we complete the definition of H by putting $H(f) := f$.
From these definitions we get $HF = \mathrm{Id}_{\mathrm{Rep}_{\mathbb{F}}(G)}$ and $FH = \mathrm{Id}_{\mathbb{F}G-\mathrm{Mod}}$

Example

Let $G = \mathbb{Z}_2$, $\mathbb{F}$ an arbitrary field. Then $\mathbb{F}G \simeq \mathbb{F}[x]/(x^2 - 1)$:
Define an $\mathbb{F}$-linear map $\alpha \colon \mathbb{F}G \to \mathbb{F}[x]/(x^2 - 1)$ by

$$\alpha(\delta_0) := \overline{1}, \alpha(\delta_1) := \overline{x}$$

It can be verified that α is an isomomorphism of $\mathbb{F}$-algebras.
So the category $\mathbb{F}G$-Mod is essentially the same as the category $\mathbb{F}[x]/(x^2 - 1)$-Mod.
Now assume $\mathrm{char}(\mathbb{F}) = 2$. Then
$R := \mathbb{F}[x]/(x^2 - 1) = \mathbb{F}[x]/((x - 1)^2)$ is a principal ideal ring whose ideals are linearly ordered in inclusion and there are only finitely many of them.
A classical result from the ring theory says that every module over such a ring is isomorphic to a direct sum of cyclic modules. In our case, there are only two nonzero cyclic R-modules up to isomorphism: R, $R/(\overline{x - 1})$.
We can use this result to understand representations of G over $\mathbb{F}$ up to equivalence.

Invariant subspace

Definition

Let G be a group and let $\mathbb{F}$ be a field. Consider a representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$. A subspace $U \leq V$ is called *φ-invariant* if for every $g \in G$ and every $u \in U$ is

$$[\varphi(g)](u) \in U\,.$$

Remark

Observe that 0 *and* V *are φ-invariant subspaces.*
If we understand φ as an action of G on V, then φ-invariant subspace is a subspace of V which is a union of orbits of the action.

Representations on invariant subspaces

Remark

If U is a φ-invariant subspace then for every $g \in G$ the restriction $\varphi(g)|_U$ can be considered as a linear map from U to U. In fact, $\varphi(g)|_U \in \mathrm{Aut}_{\mathbb{F}}(U)$, since $\varphi(g^{-1})|_U$ is its inverse.
Moreover, $\varphi(gh) = \varphi(g)\varphi(h)$ implies $\varphi(gh)|_U = \varphi(g)|_U \circ \varphi(h)|_U$ for every $g, h \in G$. Therefore we get a representation of G acting on the space U.

Definition

Let G be a group and let $\mathbb{F}$ be a field, $\varphi \colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ a representation of G over $\mathbb{F}$. If $U \leq V$ is a φ-invariant subspace of V, then we denote $\varphi_U \colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ the representation of G given by the restriction to the φ-invariant subspace:

$$\varphi_U \colon G \to \mathrm{Aut}_{\mathbb{F}}(U), g \mapsto \varphi(g)|_U$$

Example

Consider the case $G = \mathbb{Z}_2$ and $\operatorname{char} \mathbb{F} \neq 2$. Last time we proved that if $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ is a representation of G over $\mathbb{F}$, then $V = V_+ \oplus V_-$, where $V_+ = \{v \in V \mid [\varphi(1)](v) = v\}$ and $V_- = \{v \in V \mid [\varphi(1)](v) = -v\}$. Note that V_+ and V_- are φ-invariant subspaces since if $v \in V_+ \cup V_-$ then $[\varphi(1)](v), [\varphi(0)](v) \in \langle v \rangle$.
Assume U is a φ-invariant subspace of V. Since $\varphi_U\colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ is a representation of G over $\mathbb{F}$, we get $U = U_+ \oplus U_-$ where $U_+ \leq V_+$ and $U_- \leq V_-$.

Invariant subspaces for matrix representation

Recall that a matrix representation $\psi\colon G \to \mathrm{GL}(n, \mathbb{F})$ induces a representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}^n)$ via

$$\varphi(g)\colon v \mapsto \psi(g) \cdot v, v \in \mathbb{F}^n, g \in G\,.$$

It is therefore natural to call a subspace $U \leq \mathbb{F}^n$ ψ-invariant if for every $g \in G$ and for every $u \in U$

$$\psi(g) \cdot u \in U$$

Assume U is a nonzero ψ-invariant subspace of $\mathbb{F}^n$. Let B_1 be a basis of U and extend this basis into a basis of $\mathbb{F}^n$. Say we have a basis $B = B_1 \cup B_2$ of $\mathbb{F}^n$. Note that for every $g \in G$ the matrix of $\varphi(g)$ w.r.t. B has a block form

$$[\varphi(g)]_B = \begin{pmatrix} [\varphi_U(g)]_{B_1} & * \\ 0 & * \end{pmatrix}$$

Invariant subspaces in $\mathbb{F}G$-Mod

Recall the correspondence $F\colon \mathrm{Rep}_{\mathbb{F}}(G) \to \mathbb{F}G$-Mod. In this correspondence the notion of a φ-invariant subspace translates to the notion of a submodule.

Irreducible representations

Definition

Let G be a group and let $\mathbb{F}$ be a field. A representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ is called *irreducible* if it has exactly two φ-invariant subspaces. Namely 0 and V.

Definition

Let G be a group and let $\mathbb{F}$ be a field. A matrix representation $\varphi\colon G \to \mathrm{GL}(n, \mathbb{F})$ is called *irreducible* if it has exactly two φ-invariant subspaces. Namely 0 and $\mathbb{F}^n$.

Remark

If $V = 0$, then φ is not irreducible. On the other hand, if $\dim_{\mathbb{F}}(V) = 1$, then φ is always irreducible.

Simple modules

In the language of $\mathbb{F}G$-Mod the notion corresponding to irreducible representation is the notion of a simple module.

Definition

Let R be a ring. An R-module M is called *simple* if it contains exactly 2 submodules, namely 0 and M.

Proposition

Let G be a group and let $\mathbb{F}$ be a field. Consider the functor $F\colon \mathrm{Rep}_{\mathbb{F}}(G) \to \mathbb{F}G\text{-}\mathrm{Mod}$ introduced earlier. Then $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is irreducible if and only if $F(\varphi)$ is a simple $\mathbb{F}G$-module.

Factor representations

Assume that G is a group and that $\mathbb{F}$ is a field, $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$. Let $U \leq V$ be a φ-ivariant subspace. Then we can define a representation of G on V/U

$$\overline{\varphi_U}\colon G \to \mathrm{Aut}_{\mathbb{F}}(V/U)$$

in an obvious way:

$$\overline{\varphi_U}(g)\colon v + U \mapsto [\varphi(g)](v) + U$$

The important point is that the map $\overline{\varphi_U}(g)$ is defined correctly. That is if $v + U = v' + U$, then $[\varphi(g)](v) + U = [\varphi(g)](v') + U$. Indeed, if $v \in V$ and $v' = v + u$ for some $u \in U$, then

$$[\varphi(g)](v) - [\varphi(g)](v') = [\varphi(g)](v - v') \in U$$

and therefore $[\varphi(g)](v) + U = [\varphi(g)](v') + U$.

Matrix form of factor representations

Consider a representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$, where $\dim_{\mathbb{F}}(V) < \infty$. Let U be a φ-invariant subspace of V. Let $B = B_1 \cup B_2$ be a basis of V such that B_1 is a basis of U. The corresponding matrix representation of φ w.r.t. B has a block form

$$[\varphi(g)]_B = \begin{pmatrix} [\varphi_U(g)]_{B_1} & * \\ 0 & [\overline{\varphi_U}(g)]_{B_2} \end{pmatrix}$$

A bit out of time

There should be one more obvious frame that representations on V/U corresponds to factor modules.

Direct sums

Definition

Let $\varphi_1 \colon G \to \mathrm{Aut}_{\mathbb{F}}(V_1)$ and $\varphi_2 \colon G \to \mathrm{Aut}_{\mathbb{F}}(V_2)$ be representations of a group G over $\mathbb{F}$. The direct sum of these representations is a representation

$$\varphi_1 \oplus \varphi_2 \colon G \to \mathrm{Aut}_{\mathbb{F}}(V_1 \oplus V_2)$$

defined by

$$\varphi_1 \oplus \varphi_2 \colon g \mapsto \varphi_1(g) \oplus \varphi_2(g)$$

Exercise

Verify that $\varphi_1 \oplus \varphi_2$ is indeed a representation of G over $\mathbb{F}$.

Remark

If U, W are spaces over $\mathbb{F}$, $\alpha \in \mathrm{Hom}_{\mathbb{F}}(U, U)$ and $\beta \in \mathrm{Hom}_{\mathbb{F}}(W, W)$, $\alpha \oplus \beta \in \mathrm{Hom}_{\mathbb{F}}(U \oplus W, U \oplus W)$ is defined in the obvious way: $\alpha \oplus \beta \colon (u, w) \mapsto (\alpha(u), \beta(w)), u \in U, w \in W$.

Direct sums for matrix representations

Definition

Let G be a group, $\mathbb{F}$ a field, and $\psi_1 \colon G \to \mathrm{GL}(n, \mathbb{F})$ $\psi_2 \colon G \to \mathrm{GL}(m, \mathbb{F})$ be two matrix representations of G over $\mathbb{F}$. Their direct sum is a representation

$$\psi_1 \oplus \psi_2 \colon G \to \mathrm{GL}(n + m, \mathbb{F})$$

defined by

$$\psi_1 \oplus \psi_2 \colon g \mapsto \begin{pmatrix} \psi_1(g) & 0 \\ 0 & \psi_2(g) \end{pmatrix}, g \in G$$

Remark

Similarly we can define a direct sum of finitely many matrix representations of G over $\mathbb{F}$.
Note that $\psi_1 \oplus \psi_2$ and $\psi_2 \oplus \psi_1$ are equivalent.

Homework #1

Let G be a finite group and let $\mathbb{F}$ be a field. Find a basis of the center of the group algebra, i.e., a basis of the space $Z(\mathbb{F}G) = \{f \in \mathbb{F}G \mid \forall h \in \mathbb{F}G \; f * h = h * f\}$.

Thanks for your attention

New set of exercises will be available at
http://artax.karlin.mff.cuni.cz/~ppri7485/group_rep1/