Group representations 1

Introduction

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Groups

Definition

Group $\mathcal{G} = (G, \cdot, ^{-1}, 1)$ is a non-empty set G equipped with operations $\cdot \colon G \times G \to G$, $^{-1} \colon G \to G$ and $1 \in G$ such that

- $a(bc) = (ab)c$ for all $a, b, c \in G$
- $1 \cdot g = g = g \cdot 1$ for every $g \in G$
- $g \cdot g^{-1} = 1 = g^{-1} \cdot g$

Remark

- *For commutative groups we sometimes use the additive notation* $\mathcal{G} = (G, +, -, 0)$
- *Usually we omit the operations from the notation, i.e., instead of saying 'let* $(G, \cdot, ^{-1}, 1)$ *be a group' we say just 'let* G *be a group'.*

Group homomorphisms

Definition

Let $\mathcal{G} = (G, \cdot, ^{-1}, 1_G)$ and $\mathcal{H} = (H, *, ', 1_H)$ be groups. A map $f : G \to H$ is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ if

- $f(1_G) = 1_H$,
- $f(g^{-1}) = f(g)'$ for every $g \in G$,
- $f(g_1 \cdot g_2) = f(g_1) * f(g_2)$ for every $g_1, g_2 \in G$.

Exercise

Show that if f satisfies $f(g_1 \cdot g_2) = f(g_1) * f(g_2)$ for every $g_1, g_2 \in G$, then f is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$.

Definition

Let $\mathcal{G} = (G, \cdot, ^{-1}, 1_G)$ and $\mathcal{H} = (H, *, ', 1_H)$ be groups. We denote $\mathrm{Hom}(\mathcal{G}, \mathcal{H})$ the set of all homomorphisms from $\mathcal{G}$ to $\mathcal{H}$ (usually we use $\mathrm{Hom}(G, H)$ to denote this set).

Group actions

Definition

Let $\mathcal{G} = (G, \cdot, ^{-1}, 1)$ be a group and X a non-empty set. An action of $\mathcal{G}$ on X is a map $*\colon G \times X \to X$ such that

- $1 * x = x$ for every $x \in X$.
- $(gh) * x = g * (h * x)$ for every $x \in X, g, h \in G$

Sometimes action of $\mathcal{G}$ on X is defined to be an element of $\mathrm{Hom}(G, S(X))$ ($S(X)$ is the group of all permutations on X).

Exercise

Let $*\colon G \times X \to X$ be an action of G on X. Show that

a) *For every $g \in G$ the map $\varphi_g\colon X \to X$ defined by $\varphi_g(x) = g * x$ is a bijection*

b) *The map $\varphi\colon G \to S(X)$ defined by $\varphi(g) := \varphi_g, g \in G$ is a homomorphism of groups.*

Exercise

Let G be a group and X a nonempty set. Let $\varphi \in \mathrm{Hom}(G, S(X))$. Show that $*\colon G \times X \to X$ given by $g * x := [\varphi(g)](x)$ is an action of G on X.

Remark

These exercises gives correspondences between $\mathrm{Hom}(G, S(X))$ and the set of actions of G on X. It is easy to see that these correspondences are actually mutually inverse bijections.

Group representations as linear actions

Definition

Let G be a group and let $\mathbb{F}$ be a field. A *representation* of G over $\mathbb{F}$ is a homomorphism $\varphi \in \mathrm{Hom}(\mathrm{G}, \mathrm{Aut}_{\mathbb{F}}(\mathrm{V}))$, where V is a vector space over $\mathbb{F}$.

Remark

If V is an $\mathbb{F}$-space, the set of all its automorphisms $\mathrm{Aut}_{\mathbb{F}}(V)$ has a structure of a group $(\mathrm{Aut}_{\mathbb{F}}(V), \circ, {}^{-1}, 1_V)$, where $\alpha \circ \beta \colon v \mapsto \alpha(\beta(v))$.

Remark

A representation $\varphi \colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ of G over $\mathbb{F}$ induces an action of G on V: $g * v := [\varphi(g)](v)$. When this action is considered, the bijection of V given by action of $g \in G$, i.e., $\varphi_g \colon v \mapsto g * v$ is linear.

Category $\mathrm{Rep}_{\mathbb{F}}(G)$

Definition

Let G be a group, $\mathbb{F}$ be a field. Let $\varphi\colon G \to \mathrm{Aut}_F(V)$ and $\psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ be representations of G over $\mathbb{F}$. A map $f \in \mathrm{Hom}_{\mathbb{F}}(V, U)$ is said to be a *homomorphism* between these representations (a homomorphism from φ to ψ) if for every $g \in G$

$$f \circ \varphi(g) = \psi(g) \circ f\,.$$

Exercise

Let G be a group and let $\mathbb{F}$ be a field. Show that

1. *Show that 1_V is a homomorphism from φ to φ for every representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$.*
2. *Let $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(U), \psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V), \theta\colon G \to \mathrm{Aut}_{\mathbb{F}}(W)$ be representations of G over $\mathbb{F}$. If $f \in \mathrm{Hom}_{\mathbb{F}}(U, V)$ is a homomorphism from φ to ψ and $g \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ is a homomorphism from ψ to θ, then gf is a homomorphism from φ to θ*

Definition

Let G be a group and let $\mathbb{F}$ be a field. The category of representations of G over $\mathbb{F}$ denoted by $\mathrm{Rep}_{\mathbb{F}}(G)$ consists of

- objects of $\mathrm{Rep}_{\mathbb{F}}(G)$ are all representations of G over $\mathbb{F}$
- morphisms: if φ, ψ are two representations of G over $\mathbb{F}$ then $\mathrm{Rep}_{\mathbb{F}}(G)(\varphi, \psi)$ is the set of all homomorphisms from φ to ψ
- units are given by identities and morphisms are composed as maps

Equivalent representations

Definition

Two representations of a group G over a field $\mathbb{F}$ are *equivalent* if they are isomorphic objects in the category $\mathrm{Rep}_{\mathbb{F}}(G)$.
That is, representations $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(U)$ and $\psi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ of G over $\mathbb{F}$ are equivalent if there exists an isomorphism $f \in \mathrm{Hom}_{\mathbb{F}}(U, V)$ such that for every $g \in G$

$$f \circ \varphi(g) = \psi(g) \circ f$$

A basic goal for the theory of group representations: For a given group G and a field $\mathbb{F}$ describe all representations of G over $\mathbb{F}$ up to equivalence.

General linear group

Let $n \in \mathbb{N}$, and let $\mathbb{F}$ be a field.
$\mathcal{GL}(n, \mathbb{F}) = (\mathrm{GL}(n, \mathbb{F}), \cdot, ^{-1}, E)$ is the group of all regular $n \times n$ matrices over $\mathbb{F}$.
Such a group is called a general linear group (of degree n over $\mathbb{F}$).
If V is an $\mathbb{F}$-vector space of dimension n, then
$\mathrm{Aut}_{\mathbb{F}}(V) \simeq \mathrm{GL}(n, \mathbb{F})$ via

$$\alpha \mapsto [\alpha]_B, \alpha \in \mathrm{Aut}_{\mathbb{F}}(V)\,,$$

where B is a fixed basis of V

Working in coordinates - matrix representations

Definition

Let G be a group and let $\mathbb{F}$ be a field. A *matrix representation* of G over $\mathbb{F}$ is a homomorphism $\psi \in \mathrm{Hom}(G, \mathrm{GL}(n, \mathbb{F}))$, where $n \in \mathbb{N}$ is called the *degree* of representation ψ.

Let V is an $\mathbb{F}$-space of dimension n, B a basis of V and $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$.
Then $\psi\colon g \mapsto [\varphi(g)]_B$ is a matrix representation of degree n.

Conversely assume $\psi\colon G \to \mathrm{GL}(n, \mathbb{F})$ is a matrix representation of G over $\mathbb{F}$. Let $V := \mathbb{F}^n$ and define $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ by

$$\varphi(g)\colon v \mapsto \psi(g) \times v$$

It is easy to verify that

- $\varphi(g) \in \mathrm{Aut}_{\mathbb{F}}(V)$ for every $g \in G$
- $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}_{\mathbb{F}}(V))$
- if B is the canonical basis of $V = \mathbb{F}^n$ then $[\varphi(g)]_B = \psi(g)$ for every $g \in G$.

Equivalence of matrix representations

Definition

Let G be a group and let $\mathbb{F}$ be a field. Two matrix representations $\psi_1 \colon G \to \mathrm{GL}(n, \mathbb{F})$ and $\psi_2 \colon G \to \mathrm{GL}(m, \mathbb{F})$ are equivalent if $n = m$ and there exists an $X \in \mathrm{GL}(n, \mathbb{F})$ such that

$$\psi_1(g) = X\psi_2(g)X^{-1}$$

for every $g \in G$.

Remark

If H is a group and $x \in H$, the automorphism $\Omega_x \colon H \to H$ given by

$$\Omega_x(h) = xhx^{-1}$$

is called the inner automorphism of H (induced by x). Note that ψ_1 is a composition of ψ_2 and the inner automorphism of $\mathrm{GL}(n, \mathbb{F})$ induced by X.

Another way how to look at equivalent matrix reps

Assume $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$ is a representation of G over $\mathbb{F}$, $n = \dim_{\mathbb{F}}(V)$. Let B_1,B_2 two bases of V. Let $\psi_1(g) := [\varphi(g)]_{B_1}$ and $\psi_2(g) := [\varphi(g)]_{B_2}$ for $g \in G$.
Then $\psi_1, \psi_2 \colon G \to \mathrm{GL}(n, \mathbb{F})$ are equivalent matrix representations of G over $\mathbb{F}$.
Indeed, if $X = [1_V]_{B_2}^{B_1}$, then $\psi_1(g) = X^{-1}\psi_2(g)X$.
Another goal for the theory of group representations: Given two (matrix) representations of G over $\mathbb{F}$. Decide whether they are equivalent or not.

Group algebras

Let $\mathbb{F}$ be a field and let G be a group. The group algebra $\mathbb{F}G$ is an $\mathbb{F}$-vector space with basis $\{\delta_g \mid g \in G\}$ equipped with a bilinear operation $*\colon \mathbb{F}G \times \mathbb{F}G \to \mathbb{F}G$ given on the basis by

$$\delta_g * \delta_h = \delta_{gh}, g, h \in G$$

If G is a finite group, then elements of $\mathbb{F}G$ can be written as $\sum_{g\in G} t_g\delta_g, t_g \in \mathbb{F}$. Then

$$(\sum_{g\in G} t_g\delta_g) * (\sum_{g\in G} s_g\delta_g) = \sum_{g,h\in G} t_g s_h(\delta_g * \delta_h) =$$

$$\sum_{g,h\in G} t_g s_h(\delta_{gh}) = \sum_{g\in G}(\sum_{h\in G} t_h s_{h^{-1}g})\delta_g$$

Also note $\delta_{1_G} * f = f = f * \delta_{1_G}$ for every $f \in \mathbb{F}G$.

Proposition

$(\mathbb{F}G, +, -, 0, *, \delta_{1_G})$ *is an associative* $\mathbb{F}$*-algebra (ring having structure of an* $\mathbb{F}$*-vector which behaves well with the operations of the ring).*

Another way how to look on $\mathbb{F}G$

Let G be a finite group and let $\mathbb{F}$ be a field.
Consider $\mathbb{F}G = \{f \mid f \text{ is a map from } G \text{ to } \mathbb{F}\}$. Note that this set can be naturally seen as a vector space over $\mathbb{F}$:

$$f_1 + f_2 \colon g \mapsto f_1(g) + f_2(g), g \in G$$

$$0 \colon g \mapsto 0, g \in G$$

$$tf \colon g \mapsto t.f(g), g \in G, t \in \mathbb{F}$$

For $f_1, f_2 \colon G \to \mathbb{F}$, define $f_1 * f_2 \colon G \to \mathbb{F}$ by

$$f_1 * f_2 \colon g \mapsto \sum_{h \in G} f_1(h) f_2(h^{-1}g)$$

Then $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$. So we defined an associative bilinear binary operation on $\mathbb{F}G$.
In fact this structure is essentially the same as the one defined on the previous slide.

Why are we interested in group algebras?

Theorem

Let G be a group and let $\mathbb{F}$ be a field. The category $\mathrm{Rep}_{\mathbb{F}}(G)$ is equivalent to the category $\mathbb{F}G\text{-}\mathrm{Mod}$.

So the theory of modules over associative algebras provides a languague we could use to study group representations.

Homework #1

Let G be a finite group and let $\mathbb{F}$ be a field. Find a basis of the center of the group algebra, i.e., a basis of the space $Z(\mathbb{F}G) = \{f \in \mathbb{F}G \mid \forall h \in \mathbb{F}G \ f * h = h * f\}$.

That's all for today

http://artax.karlin.mff.cuni.cz/~ppri7485/group_rep1/