Group representations 1

Complex representations of symmetric groups, part 1

June 1, 2021

How the Dominance lemma is applied

Let μ, λ be partitions of $n \in \mathbb{N}$, $t \in X^\lambda$.
We define an operator $A_t^\mu \in \mathrm{End}_{\mathbb{C}}(M^\mu)$ by
$A_t^\mu := \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi_\pi^\mu$.
If $\lambda = \mu$, then $A_t^\mu([t]) = e_t$, the polytabloid associated to t.

Lemma

Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$, $t^\lambda \in X^\lambda$, $s^\mu \in X^\mu$. Assume $A_{t^\lambda}^\mu([s^\mu]) \neq 0$.
Then $\lambda \trianglerighteq \mu$.
Moreover, if $\lambda = \mu$ and $A_{t^\lambda}^\mu([s^\mu]) \neq 0$, then $A_{t^\lambda}^\mu([s^\mu]) = \pm e_{t^\lambda}$.

proof of the lemma part 1:

We show that if $A^{\mu}_{t^{\lambda}}([s^{\mu}]) \neq 0$, then the assumption of the dominance lemma is satisfied for t^{λ} and s^{μ}.
If it is not the case there are $i \neq j \in \{1, \ldots, n\}$ such that i, j are located in the same row of s^{μ} and also in the same column of t^{λ}.
The second condition says that $H := \{\mathrm{id}, (i,j)\} \subseteq C_{t^{\lambda}}$. Let $\sigma_1, \ldots, \sigma_k \in C_{t^{\lambda}}$ be a transversal of left cosets of H in $C_{t^{\lambda}}$, that is

$$C_{t^{\lambda}} = \dot{\cup}_{r=1}^{k} \sigma_r H \, .$$

Then $A^{\mu}_{t^{\lambda}}([s^{\mu}]) = \sum_{\pi \in C_{t^{\lambda}}} \mathrm{sgn}(\pi) \varphi^{\mu}_{\pi}([s^{\mu}]) =$
$\sum_{r=1}^{k} \mathrm{sgn}(\sigma_r)[\sigma_r * s^{\mu}] + \mathrm{sgn}(\sigma_r(i,j))[\sigma_r * ((i,j) * s^{\mu})]$
Since i, j are located in the same row of s^{μ}, we get $(i,j) * [s^{\mu}] = [s^{\mu}]$. Therefore $A^{\mu}_{t^{\lambda}}([s^{\mu}]) = 0$.
The dominance lemma implies that $\lambda \trianglerighteq \mu$.

proof of the lemma, part 2

The proof of the dominance lemma in the case $\lambda = \mu$ shows that there exists a λ-tableau u^λ such that

- There exists $\sigma \in C_{t^\lambda}$ such that $\sigma * [t^\lambda] = [u^\lambda]$
- $s^\lambda \sim u^\lambda$ (recall the assumption $\lambda = \mu$)

Now compute
$A^\mu_{t^\lambda}[s^\mu] = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([s^\mu]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([u^\lambda]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([\sigma * t^\lambda]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi\sigma)\varphi^\lambda_{\pi\sigma}([t^\lambda]) = \mathrm{sgn}(\sigma) \sum_{\pi' \in C_{t^\lambda}} \mathrm{sgn}(\pi')\varphi^\lambda_{\pi'}([t^\lambda]) = \pm e_{t^\lambda}$.

A corollary

Corollary

Let $n \in \mathbb{N}$ *and let* $t \in X^\lambda$*. Let* $A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\varphi_\pi^\lambda \in \operatorname{End}_{\mathbb{C}}(M^\lambda)$*. Then* $\operatorname{Im} A_t = \mathbb{C}e_t$

Proof.

Since M^λ has basis T^λ, $\operatorname{Im} A_t$ is the subspace of M^λ generated by $\{A_t([s]) \mid [s] \in T^\lambda\}$. If $A_t([s]) \neq 0$, then $A_t([s]) = \pm e_t$.
On the other hand, $A_t([t]) = e_t$. □

Scalar product on M^λ

On M^λ we consider a standard scalar product, $\langle -, - \rangle$ such that T^λ is an orthogonal basis w.r.t. $\langle -, - \rangle$. That is, for $[s], [t] \in T^\lambda$

$$\langle [s], [t] \rangle := \delta_{[s],[t]} \,.$$

Note that the product is invariant w.r.t. φ^λ, since for every $\sigma \in S_n$

$$\langle \varphi^\lambda_\sigma([s]), \varphi^\lambda_\sigma([t]) \rangle = \langle \sigma * [s], \sigma * [t] \rangle = \delta_{[\sigma * s],[\sigma * t]} = \delta_{[s],[t]} = \langle [s], [t] \rangle$$

A_t is self adjoint

Proposition

Let $n \in \mathbb{N}$, $\lambda \vdash n$, $t \in X^\lambda$ and $A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\varphi_\pi^\lambda$. For every $u, v \in M^\lambda$ $\langle A_t(u), v\rangle = \langle u, A_t(v)\rangle$

Proof.

Standard arguments using linearity of A_t and properties of scalar products enable reduction to the case $u, v \in T^\lambda$.
Note that if $u \notin \{\sigma * [t] \mid \sigma \in C_t\}$, then $A_t(u) = 0$. Since $A_t(u)$ is a multiple of e_t, $\langle u, v\rangle$ can be nonzero only if $v \in \{\sigma * [t] \mid \sigma \in C_t\}$.
Similar arguments show that $\langle u, A_t(v)\rangle \neq 0$ only for $u, v \in \{\sigma * [t] \mid \sigma \in C_t\}$.
Now assume $u = \sigma_u * [t], v = \sigma_v * [t]$ for some $\sigma_u, \sigma_v \in C_t$. Then
$\langle A_t(u), v\rangle = \operatorname{sgn}(\sigma_u)\langle e_t, \sigma_v * [t]\rangle = \operatorname{sgn}(\sigma_u)\operatorname{sgn}(\sigma_v)$.
$\langle u, A_t(v)\rangle = \operatorname{sgn}(\sigma_v)\langle \sigma_u * [t], e_t\rangle = \operatorname{sgn}(\sigma_v)\operatorname{sgn}(\sigma_u)$. □

Subrepresentation theorem

Theorem

Let $n \in \mathbb{N}$ and let $\lambda \vdash n$. If $V \subseteq M^\lambda$ is a φ^λ-invariant subspace of M^λ, then either $S^\lambda \subseteq V$ or $V \subseteq (S^\lambda)^\perp$.

Remark

Recall S^λ is the subspace of M^λ spanned by $\{e_t \mid t \in X^\lambda\}$. We already know that S^λ is φ^λ-invariant.

The proof of subrepresentation theorem

Proof.

We distinguish two cases: a) Assume there exists $t \in X^\lambda$ and $v \in V$ such that $A_t(v) \neq 0$, i.e.,

$$\sum_{\pi \in C_t} \mathrm{sgn}(\pi) \varphi^\lambda_\pi(v) \neq 0$$

Since V is φ^λ-invariant, $A_t(v) \in V$.
By the Corollary, $\mathrm{Im}\, A_t = \mathbb{C} e_t$, so $e_t \in V$. We also proved $\varphi^\lambda_\sigma(e_t) = e_{\sigma * t}$. Since V is φ_λ-invariant, $e_{\sigma * t} \in V$ for every $\sigma \in S_n$.
Hence also $S^\lambda \subseteq V$.
b) For every $t \in X^\lambda$ and every $v \in V$ is $A_t(v) = 0$. Then

$$\langle e_t, v \rangle = \langle A_t([t]), v \rangle = \langle [t], A_t(v) \rangle = 0$$

for every $t \in X^\lambda$ and for every $v \in V$. Since S^λ is spanned by $\{e_t \mid t \in t^\lambda\}$, we get $V \subseteq (S^\lambda)^\perp$. □

Specht's representations are irreducible

Corollary

Let $n \in \mathbb{N}$ and let $\lambda \vdash n$, $\psi^\lambda \colon S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ the Specht's representation associated to λ. Then ψ^λ is irreducible.

Proof.

The important observation is that every ψ^λ-invariant subspace of S^λ is also a φ^λ-invariant subspace of M^λ.
If $0 \subsetneq V \subsetneq S^\lambda$ is a ψ^λ-invariant subspace, then it is φ^λ-invariant subspace and we may apply the subrepresentation theorem.
Since V cannot contain S^λ it has to be contained in $(S^\lambda)^\perp$. But then $V \subseteq S^\lambda \cap (S^\lambda)^\perp = 0$. This is not possible, so S^λ contains no ψ^λ-invariant subspaces other than 0 and S^λ.

□

Equivalence of S^λ and S^μ

Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$ and assume that ψ^λ is equivalent to ψ^μ. Maschke's theorem implies there exits a φ^λ-invariant subspace $C^\lambda \leq M^\lambda$ and a φ^μ-invariant subspace $C^\mu \leq M^\mu$ such that

$$M^\lambda = S^\lambda \oplus C^\lambda, M^\mu = S^\mu \oplus C^\mu .$$

Let $T \in \mathrm{Hom}_\mathbb{C}(S^\lambda, S^\mu)$ be a witness of the equivalence of ψ^μ and ψ^λ. That is, T is an isomorphism satisfying

$$\psi^\mu_\pi T = T \psi^\lambda_\pi$$

for every $\pi \in S_n$. Extend T to $\overline{T} = T \oplus 0 \in \mathrm{Hom}_\mathbb{C}(M^\lambda, M^\mu)$. It is easy to check that $\overline{T} \in \mathrm{Rep}_\mathbb{C}(S_n)(\varphi^\lambda, \varphi^\mu)$, in other words

$$\varphi^\mu_\pi \overline{T} = \overline{T} \varphi^\lambda_\pi$$

Some computations

Let $t \in X^\lambda$. Then

$$T(e_t) = \overline{T}(e_t) = \overline{T}(\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi_\pi^\lambda([t])) = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi_\pi^\mu(\overline{T}([t])) =$$

$A_t^\mu(\overline{T}[t])$, where $A_t^\mu = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi_\pi^\mu \in \mathrm{End}_{\mathbb{C}}(M^\mu)$.
Since $T(e_t) \neq 0$ for some $t \in X^\lambda$ the value of $A_t^\mu([s])$ has to be nonzero for some $[s] \in T^\mu$.
The dominance lemma implies $\lambda \trianglerighteq \mu$. But using the symmetric arguments we can obtain also $\mu \trianglerighteq \lambda$.
Therefore if ψ^λ and ψ^μ are equivalent, we have $\lambda = \mu$.

The conclusion

Theorem

Let $n \in \mathbb{N}$. Then

a) *For each $\lambda \vdash n$ the representation $\psi^\lambda \colon S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ is irreducible.*

b) *For every irreducible representation of S_n over $\mathbb{C}$ there exists exactly one $\lambda \vdash n$ such that ψ is equivalent to ψ^λ.*

c) *Every representation of S_n over $\mathbb{C}$ is equivalent to a direct sum of Specht's representations.*

The End.

Thank you for following the course. To schedule the date and the form of the exam, please write me an email (preferably about a week in advance).