Group representations 1

Complex representations of symmetric groups, part 1

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Partitions

Definition

Let $n \in \mathbb{N}$. A *partition* of n is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{N}$ satisfy

1. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 1$
2. $\sum_{i=1}^{\ell} \lambda_i = n$

We write $\lambda \vdash n$.

Remark

Note there is a bijection between the set of all conjugacy classes of S_n and the set of all partitions of n. In this correspondence $\lambda \vdash n$ corresponds to the of permutations which are products of ℓ independent cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_\ell$.

For example, if $n = 3$ this bijection is

(3)	$\{(1,2,3),(1,3,2)\}$
$(2,1)$	$\{(1,2)(3),(1,3)(2),(2,3)(1)\}$
$(1,1,1)$	$\{\mathrm{id}\} = \{(1)(2)(3)\}$

An overview

For each $\lambda \vdash n$ we define $\psi^\lambda \in \mathrm{Rep}_{\mathbb{C}}(S_n)$ and show that this representation is irreducible, and whenever $\lambda \neq \mu \vdash n$ then ψ^λ and ψ^μ are not equivalent.
The general theory of representations of finite groups over $\mathbb{C}$ then implies that for every irreducible representation ψ of S_n over $\mathbb{C}$ there exists unique $\lambda \vdash n$ such that ψ and ψ^λ are equivalent.

Young diagrams (also Young frames)

Definition

Let $n \in \mathbb{N}$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $\lambda \vdash n$.
The Young diagram of λ consists of n boxes placed into ℓ rows in such a way that the i-th row contains exactly λ_i boxes $(i = 1, \ldots, \ell)$.

Dominance order

Definition

Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ we say that λ *dominates* μ (and write $\lambda \trianglerighteq \mu$ or $\mu \trianglelefteq \lambda$) if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for every $i \in \mathbb{N}$, where $\lambda_t := 0$ for every $t > \ell$ and $\mu_s := 0$ for every $s > m$.

Remark

It is easy to see that $\trianglerighteq$ is a partial order on the set of all partitions of n. Note that (n) is the greatest element and $(1, 1, \ldots, 1)$ is the least element of this poset.

Also note that the dominance order is not linear - for $n = 7$, the partitions $(3, 3, 1)$ and (4,1,1,1) are not comparable in $\trianglerighteq$.

Young tableau

Definition

Let $n \in \mathbb{N}$, $\lambda \vdash n$. A λ-tableau (a Young tableau of shape λ) is a Young diagram with numbers $\{1, 2, \ldots, n\}$ placed into the boxes (each number to one box).

Remark

For a given shape $\lambda \vdash n$ there are $n!$ different λ-tableaux.

The dominance lemma

The following result will be crucial.

Lemma

Let $n \in \mathbb{N}$*,* $\lambda, \mu \vdash n$*. Suppose there are tableaux* t^λ *and* s^μ *satisfying*

1. t^λ *is a* λ*-tableau and* s^μ *is a* μ*-tableau*
2. *every pair of different numbers located in the same row of* s^μ *is not located in the same column of* t^λ*.*

Then $\lambda \trianglerighteq \mu$*.*

The action of S_n on the set of all λ-tableaux

Let $n \in \mathbb{N}$ and $\lambda \vdash n$. Let X^λ be the set of all λ-tableaux. The group S_n has a natural action on X^λ: If $\pi \in S_n$ and $t^\lambda \in X^\lambda$, then $\pi * t^\lambda$ is the tableau obtained from t^λ by applying π to each number in t^λ

Example: $n = 4$, $\lambda = (2, 2)$, $t^\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$.

For $\pi \in S_4$ is $\pi * t^\lambda = \begin{array}{|c|c|}\hline \pi(1) & \pi(2) \\ \hline \pi(3) & \pi(4) \\ \hline \end{array}$

The column stabilizer

Definition

(Column stabilizer) Let $n \in \mathbb{N}$, $\lambda \vdash n$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $t \in X^\lambda$. The column stabilizer of t, denoted by C_t, is the set of all permutations of S_n permuting only numbers within the same column of t. More precisely,
$C_t = \{\pi \in S_n \mid \forall i \in \{1, \ldots, n\}\ i \text{ and } \pi(i)$ are placed in the same column of $t\}$.

Remark

Note that if c_i is the length of the $i-th$ column of the Young diagram of λ (in particular, $c_1 \geq c_2 \geq \cdots \geq c_{\lambda_1}$), then $|C_t| = c_1!c_2! \cdots c_{\lambda_1}!$.

Some remarks on the column stabilizer

Remark

Consider the relation $\wr$ *on* X^λ *given by* $s, t \in X^\lambda$ *satisfy* $s \wr t$ *if and only if for each* $1 \leq i \leq \ell$ *the* i*-th column of* s *contains the same set of numbers as the* i*-th column of* t*. Then* $\wr$ *is obviously an equivalence on* X^λ *and the action of* S_n *on* X^λ *can be transfered to the action of* S_n *on* $X^\lambda/\wr := \{[t]_\wr \mid t \in X^\lambda\}$ *by* $\pi * [t]_\wr = [\pi * t]_\wr$*. Then* C_t *is actually the stabilizer of* $[t]_\wr$ *in this action, that is*

$$C_t = \{\pi \in S_n \mid \pi * [t]_\wr = [t]_\wr\}$$

Tabloid

Definition

Let $n \in \mathbb{N}$, $\lambda \vdash n$ and let $s, t \in X^\lambda$. We say that s and t are equivalent λ-tableaux ($s \sim t$), if for each $1 \leq i \leq \ell$ the set of numbers contained in the i-th row of s is the same as the set of numbers in the i-th row of t.

Note that $\sim$ is an equivalence on X^λ.

Definition

An equivalence class of a λ-tableau is called a λ-tabloid. If $t \in X^\lambda$, the equivalence class

$$[t] := \{s \in X^\lambda \mid s \sim t\}$$

is called the *tabloid* of t. The set of all λ-tabloids is denoted by T^λ, i. e., $T^\lambda := X^\lambda / \sim$.

The action of S_n on T^λ

Note that if $t_1, t_2 \in X^\lambda$ and $\pi \in S_n$, then

$$t_1 \sim t_2 \Leftrightarrow \pi * t_1 \sim \pi * t_2$$

It follows that $\pi * [t] := [\pi * t], \pi \in S_n, [t] \in T^\lambda$ is a correctly defined map from $S_n \times T^\lambda \to T^\lambda$. Obviously

$$\mathrm{id} * [t] = [t], \pi_1 * (\pi_2 * [t]) = (\pi_1 \pi_2) * [t] (= [\pi_1 * (\pi_2 * t)])$$

for every $[t] \in T^\lambda$ nad $\pi_1, \pi_2 \in S_n$. Thus we get natural action of S_n on the set of all λ-tabloids.

Representation φ^λ

Since S_n acts on T^λ we can consider the representation of S_n induced by this action.
Let us briefly recall its construction: Let M^λ be a vector space over $\mathbb{C}$ with basis T^λ, $\varphi^\lambda \colon S_n \to \mathrm{Aut}_{\mathbb{C}}(M^\lambda)$ be the representation of S_n over $\mathbb{C}$ given by $\varphi^\lambda(\pi) := \varphi^\lambda_\pi$, where

$$\varphi^\lambda_\pi([t]) := \pi * [t] = [\pi * t] \,.$$

Note that if $\lambda = (n)$, then $|T^\lambda| = 1$ and the φ^λ is the trivial representation of S_n over $\mathbb{C}$.
For other partitions of n the space M^λ has dimension ≥ 2 and contains a one dimensional φ^λ-invariant subspace $\langle \sum_{[t]\in T^\lambda} [t] \rangle$.
The irreducible representations we are searching for are given by some φ^λ-invariant subspaces of M^λ.

Polytabloid

Definition

Let $n \in \mathbb{N}$, $\lambda \vdash n$ and $t \in X^\lambda$. The element

$$e_t = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)[\pi * t] \in M^\lambda$$

is called the *polytabloid associated to t*.

Remark

Important: Note that the polytabloid is a linear combination of tabloids. But the parametr t is a tableau. So we can have tableaux $s, t \in X^\lambda$ such that $s \sim t$ and $e_s \neq e_t$.

Action of φ^λ on polytabloids

Proposition

Let $n \in \mathbb{N}, \lambda \vdash n, t \in X^\lambda$. For every $\sigma \in S_n$ is

$$\varphi^\lambda_\sigma(e_t) = e_{\sigma * t}$$

Proof.

Recall that if G is a group acting on the set X, the stabilizer of x is defined as $C_x := \{g \in G \mid g * x = x\}$. Then for every $x \in X$ and $g \in G$ we have $C_{g*x} = gC_xg^{-1}$.
Apply this rule on the action of S_n on $X^\lambda/\wr$ - the column stabilizer $C_{\sigma * t}$ is the stabilizer of $\sigma * [t]_\wr$ in this action. Therefore

$$C_{\sigma * t} = \sigma C_t \sigma^{-1}\,.$$

The rest of the proof is just a standard computation □

some standard computation

$$\varphi^{\lambda}_{\sigma}(e_t) = \varphi^{\lambda}_{\sigma}(\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi^{\lambda}_{\pi}([t])) = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi^{\lambda}_{\sigma\pi}([t]) =$$

$$\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi^{\lambda}_{\sigma\pi\sigma^{-1}}[\sigma * t] = \sum_{\pi \in C_t} \mathrm{sgn}(\sigma\pi\sigma^{-1})\varphi^{\lambda}_{\sigma\pi\sigma^{-1}}[\sigma * t] =$$

$$\sum_{\pi' \in \sigma C_t \sigma^{-1}} \mathrm{sgn}(\pi')\varphi^{\lambda}_{\pi'}([\sigma * t]) = e_{\sigma * t} \,.$$

Specht's module

Definition

Let $n \in \mathbb{N}, \lambda \vdash n$. Let S^λ be the subspace of M^λ given by all polytabloids related to tableaux of shape λ. That is,

$$S^\lambda := \langle e_t \mid t \in X^\lambda \rangle$$

The proposition implies that S^λ is a φ^λ-invariant subspace of M^λ, therefore there exists a representation $\psi^\lambda \colon S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ given by $\psi^\lambda := \varphi^\lambda_{S^\lambda}$.

Definition

The representation $\psi^\lambda \colon S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ is called the Specht's representation of S_n associated to λ.

So finally we defined representations ψ^λ. It remains to show that every Specht's representation is irreducible and $\psi^\lambda \not\simeq \psi^\mu$ if λ and μ are different partitions of n.

How the Dominance lemma is applied

Let μ, λ be partitions of $n \in \mathbb{N}$, $t \in X^\lambda$. We define an operator $A_t \in \mathrm{End}_{\mathbb{C}}(M^\mu)$ by $A_t := \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi^\mu_\pi$.
Note that if $\lambda = \mu$, then $A_t([t]) = e_t$, the polytabloid associated to t.

Lemma

Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$, $t^\lambda \in X^\lambda$, $s^\mu \in X^\mu$. Assume $A_{t^\lambda}([s^\mu]) \neq 0$. Then $\lambda \trianglerighteq \mu$.
Moreover, if $\lambda = \mu$ and $A_{t^\lambda}([s^\mu]) \neq 0$, then $A_{t^\lambda}([s^\mu]) = \pm e_{t^\lambda}$.

proof of the lemma part 1:

We show that if $A_{t^\lambda}([s^\mu]) \neq 0$, then the assumption of the dominance lemma is satisfied for t^λ and s^μ.
If it not the case there are $i \neq j \in \{1, \ldots, n\}$ such that i, j are located in the same row of s^μ and also in the same column of t^λ.
The second condition says that $H := \{\mathrm{id}, (i,j)\} \subseteq C_{t^\lambda}$. Let $\sigma_1, \ldots, \sigma_k \in C_{t^\lambda}$ be a transversal of left cosets of H in C_{t^λ}, that is

$$C_{t^\lambda} = \dot\cup_{r=1}^{k} \sigma_r H \,.$$

Then $A_{t^\lambda}([s^\mu]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi) \varphi_\pi^\mu([s^\mu]) =$
$\sum_{r=1}^{k} \mathrm{sgn}(\sigma_r)[\sigma_r * s^\mu] + \mathrm{sgn}(\sigma_r(i,j))[\sigma_t * ((i,j) * s^\mu)]$
Since i, j are located in the same row of s^μ, we get $(i,j) * [s^\mu] = [s^\mu]$. Therefore $A_{t^\lambda}([s^\mu]) = 0$.
The dominance lemma implies that $\lambda \trianglerighteq \mu$.

proof of the lemma, part 2

The proof of the dominance lemma in the case $\lambda = \mu$ shows that there exists a λ-tableau u^λ such that

- There exists $\sigma \in C_{t^\lambda}$ such that $\sigma * [t^\lambda] = [u^\lambda]$
- $s^\lambda \sim u^\lambda$ (recall the assumption $\lambda = \mu$)

Now compute

$A_{t^\lambda}[s^\mu] = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([s^\mu]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([u^\lambda]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\pi)\varphi^\lambda_\pi([\sigma * t^\lambda]) = \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma\pi)\varphi^\lambda_{\pi\sigma}([t^\lambda]) = \mathrm{sgn}(\sigma) \sum_{\pi' \in C_{t^\lambda}} \mathrm{sgn}(\pi')\varphi^\lambda_{\pi'}([t^\lambda]) = \pm e_t$.

A corollary

Corollary

Let $n \in \mathbb{N}$ *and let* $t \in X^\lambda$. *Let* $A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\varphi_\pi^\lambda \in \operatorname{End}_{\mathbb{C}}(M^\lambda)$. *Then* $\operatorname{Im} A_t = \mathbb{C}e_t$

Proof.

Since M^λ has basis T^λ, $\operatorname{Im} A_t$ is the subspace of M^λ generated by $\{A_t([s]) \mid [s] \in T^\lambda\}$. If $A_t([s]) \neq 0$, then $A_t([s]) = \pm e_t$.
On the other hand, $A_t([t]) = e_t$. □

End of part 1

Thank you for your attention.