Group representations 1

The degree theorem

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Application of Schur's lemma proved last time

Lemma

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\psi \colon G \to \mathrm{GL}(d, \mathbb{F})$ be an irreducible matrix representation and $C \subseteq G$ a conjugacy class. Then

$$\sum_{g \in C} \psi(g) = \lambda E \,,$$

where $\lambda = \frac{|C|}{d} \chi_\psi(c)$, for any $c \in C$.

Relating λ's and $h_{i,j,\ell}$

Let G be a finite group, $\mathbb{F}$ algebraically closed and $\mathrm{char}(\mathbb{F}) \nmid |G|$.
Assume again that conjugacy classes of G are labeled by $C_1, C_2, \ldots, C_k$.
For an irreducible (matrix) representation $\psi\colon G \to \mathrm{GL}(d, \mathbb{F})$ let ${\lambda_i}^{\psi} \in \mathbb{F}$ be such that

$$\sum_{g \in C_i} \psi(g) = \lambda_i^{\psi} E\,.$$

Then

$${\lambda_i}^{\psi} {\lambda_j}^{\psi} = \sum_{\ell=1}^{k} h_{i,j,\ell} {\lambda_\ell}^{\psi}$$

Of course, if we consider irreducible (linear) representation $\varphi\colon G \to \mathrm{Aut}_{\mathbb{F}}(V)$, we define ${\lambda_i}^{\varphi}$ by

$$\sum_{g \in C_i} \varphi(g) = \lambda_i^{\varphi} 1_V\,.$$

What information is encoded in the complex character table?

Theorem

The structural constants $h_{i,j,\ell}$ can be computed from the character table of G over $\mathbb{C}$.

proof: Recall $C_1, C_2, \ldots, C_k$ are the conjugacy classes of G, $\varphi_1, \varphi_2, \ldots, \varphi_k$ is a list of all distinct irreducible representations of G over $\mathbb{C}$. The character table is a matrix $A = (a_{i,j})_{1 \leq i,j, \leq k}$, where $a_{i,j}$ is the value χ_i has on C_j.
Let ${\lambda_j}^{\varphi_i} = \frac{|C_j|}{d_i} a_{i,j} = \frac{|C_j|}{\chi_i(1_G)} \chi_i(g_j)$. Note that ${\lambda_j}^{\varphi_i}$ are given by the matrix A.
Moreover for every $i, j, \ell \in \{1, 2, \ldots, k\}$

$$ {\lambda_j}^{\varphi_i} {\lambda_\ell}^{\varphi_i} = \sum_{m=1}^{k} h_{j,\ell,m} {\lambda_m}^{\varphi_i} $$

the proof, cont.

Fix $j, \ell \in \{1, 2, \ldots, k\}$.
Let Λ be a $k \times k$ complex matrix whose value at the position (m, i) is ${\lambda_m}^{\varphi_i}$. Formulae from the previous slide can be written in matrix form as

$$({\lambda_j}^{\varphi_1}\lambda_\ell^{\varphi_1}, {\lambda_j}^{\varphi_2}\lambda_\ell^{\varphi_2}, \ldots, {\lambda_j}^{\varphi_k}\lambda_\ell^{\varphi_k}) = (h_{j,\ell,1}, h_{j,\ell,2}, \ldots, h_{j,\ell,k})\Lambda$$

Note that if we prove that Λ is regular, then the values $h_{j,\ell,1}, h_{j,\ell,2}, \ldots, h_{j,\ell,k}$ can be computed from this equality and, in particular, are determined by A.
Since we can do such a computation for any $j, \ell \in \{1, 2, \ldots, k\}$, $h_{i,j,\ell}$ are determined by A.

why is Λ regular?

Note that Λ^T is a product of three regular matrices:

$$\Lambda^T = \mathrm{diag}(\frac{1}{d_1}, \frac{1}{d_2}, \dots, \frac{1}{d_k}) \cdot A \cdot \mathrm{diag}(|C_1|, |C_2|, \dots, |C_k|)$$

(in the position (u, v) of the matrix on the RHS there is $\frac{|C_v|}{d_u} a_{u,v} = \lambda_v^{\varphi_u}$). Recall, we proved that the character table A is regular.

The degree theorem

Theorem

Let G be a finite group and let φ be an irreducible representation of G over $\mathbb{C}$ of degree d. Then $d \mid |G|$.

The basic idea of the proof: We will prove that $\frac{|G|}{d}$ is an algebraic integer. Since it is also a rational number, it has to be an integer.

Some facts from algebraic number theory

Definition

A subfield of $\mathbb{C}$ is said to be a *number field* if it is a finite extension of $\mathbb{Q}$.

Definition

A complex number is called an *algebraic integer* if it is a root of a monic polynomial with integer coeficients. If K is a number field, then the set of all algebraic integers contained in K is denoted by $\mathbb{Z}_K$. That is,

$$\mathbb{Z}_K = \{\alpha \in K | \exists h \in \mathbb{Z}[x] \text{ monic } h(\alpha) = 0\}.$$

Proposition

Let K be a number field, $\mathbb{Z}_K$ the set of its algebraic integers. Then $\mathbb{Z}_K$ is a subring of $\mathbb{C}$ and $\mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$.

What is important for us

Remark

Note that if $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ has finite degree then the values of χ_φ are contained in $\mathbb{Z}_K$, where $K := \mathbb{Q}[e^{\frac{2\pi \mathrm{i}}{|G|}}]$.
Recall if $g \in G$ then the eigenvalues of $\varphi(g)$ are of the form $e^{\frac{2\pi \mathrm{i} k}{|o(g)|}}, k \in \mathbb{Z}$. Since $o(g) \mid |G|$, these eigenvalues are also in K.
But in fact, the eigenvalues are roots of polynomial $x^{|o(g)|} - 1$, hence they are in $\mathbb{Z}_K$. Since $\mathbb{Z}_K$ is closed under addition, $\chi_\varphi(g) \in \mathbb{Z}_K$.

The Key Lemma:

Lemma

Let G be a finite group, $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ irreducible of degree d. Let C be a conjugacy class of G and $c \in C$. Then

$$\lambda_C = \frac{|C|}{d}\chi_\varphi(c) \in \mathbb{Z}_K\,,$$

where $K = \mathbb{Q}[e^{\frac{2\pi \mathrm{i}}{|G|}}]$

Proof.

Let $C_1, C_2, \ldots, C_k$ be a list of all conjugacy classes of G.
Recall $\lambda_{C_i}\lambda_{C_j} = \sum_{\ell=1}^{k} h_{i,j,\ell}\lambda_{C_\ell}$.
Let H_i be the $k \times k$-matrix whose (j,ℓ) entry is $h_{i,j,\ell}$ and let $\vec{\lambda} = (\lambda_{C_1}, \lambda_{C_2}, \ldots, \lambda_{C_k})^T \in \mathbb{C}^k$.
From $H_i\vec{\lambda} = \lambda_{C_i}\vec{\lambda}$, we see that λ_{C_i} has to be a root of the characteristic polynomial of H_i. This is a polynomial from $\mathbb{Z}[x]$ with leading coefficient ± 1. Hence $\lambda_{C_i} \in \mathbb{Z}_K$. $\square$

A slide added after the lecture

In the presented proof of the key lemma two details are omitted.
First every λ_C is in K, since $\chi_\varphi(c) \in K$ for every $c \in C$.
Of course, it is also necessary to check $\vec{\lambda} \neq 0$. But this is easy: If $C = \{1_G\}$, then

$$\lambda_C = \frac{1}{d}\chi_\varphi(1_G) = 1\,,$$

so at least one coordinate of $\vec{\lambda} \neq 0$.

The proof of the degree theorem

The starting point is equation

$$|G| = \sum_{g \in G} \chi_\varphi(g)\overline{\chi_\varphi(g)} = \sum_{i=1}^{k} |C_i|\chi_\varphi(g_i)\overline{\chi_\varphi(g_i)}\,,$$

where $C_1, C_2, \ldots, C_k$ are the conjugacy classes of G and $g_i \in C_i$ for every $1 \leq i \leq k$. If d is the degree of φ, we get

$$\frac{|G|}{d} = \sum_{i=1}^{k} \frac{|C_i|\chi_\varphi(g_i)}{d}\overline{\chi_\varphi}(g_i)\,.$$

Of course, $\frac{|G|}{d} \in \mathbb{Q}$, $\overline{\chi_\varphi}(g_i) = \chi_\varphi(g_i^{-1}) \in \mathbb{Z}_K$ and, by the key lemma, $\frac{|C_i|\chi_\varphi(g_i)}{d} = \lambda_{C_i} \in \mathbb{Z}_K$.
Since $\mathbb{Z}_K$ is a subring of $\mathbb{C}$, we conclude $\frac{|G|}{d} \in \mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$. This proves the degree theorem.

Alternative proof of the degree theorem

There exists another proof of the degree theorem which is based on an elementary result about abelian groups and formulae for so called primitive central idempotents of $\mathbb{C}G$.

A lemma about abelian groups

Lemma

Assume that V is a vector space over $\mathbb{Q}$ and G is a nonzero finitely generated subgroup of $(V, +)$. If some $q \in \mathbb{Q}$ satisifies $qG \subseteq G$, i.e.,

$$\forall g \; g \in G \;\Rightarrow qg \in G$$

then $q \in \mathbb{Z}$.

Proof.

Since G is a finitely generated torsion free abelian group, it is a free abelian group. Let $\{b_1, b_2, \ldots, b_n\}$ be a free basis of G. Since $qb_1 \in G$, there are integers $z_1, z_2, \ldots, z_n$ such that $qb_1 = \sum_{i=1}^{n} z_i b_i$. Now if $d \in \mathbb{N}$ is such that $dq \in \mathbb{Z}$, we obtain

$$(dq)b_1 = \sum_{i=1}^{n} dz_i b_i \,.$$

Since $\{b_1, \ldots, b_n\}$ is a free basis of G, we get $dq = dz_1$, so $q \in \mathbb{Z}$.

Central idempotents in group algebras

Exercise

Let $\mathbb{F}$ be a field and consider a semisimple artinian ring

$$R = \mathrm{M}_{n_1}(\mathbb{F}) \times \mathrm{M}_{n_2}(\mathbb{F}) \times \cdots \times \mathrm{M}_{n_k}(\mathbb{F}) \,.$$

Find central idempotents of R, that is $\{e \in Z(R) \mid e^2 = e\}$.

The result of the exercise is that there are idempotents $e_1 = (E, 0, \ldots, 0), e_2 = (0, E, 0, \ldots, 0), \ldots, e_k = (0, \ldots, 0, E)$ with the following properties:

- $e_i . e_j = \delta_{i,j} e_i$ (orthogonality)
- $\sum_{i=1}^{k} e_i = 1_R$ (completeness)
- $e_i = e_i + 0$ is essentially the only way how to express e_i as a sum of two orthogonal central idempotents (primitivity)
- For every central idempotent $e \in R$ there exists unique set $I \subseteq \{1, \ldots, k\}$ such that $e = \sum_{i \in I} e_i$

Primitive central idempotents in $\mathbb{F}G$

Theorem

Let G be a finite group, $\mathbb{F}$ an algebraically closed field, $\mathrm{char}(\mathbb{F}) \nmid |G|$. Let $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ be a list of all distinct irreducible represenations of G over $\mathbb{F}$ up to equivalence. For each $t \in \{1, \ldots, k\}$ set

$$e_t = \frac{d_t}{|G|} \sum_{g \in G} \chi_{\varphi_t}(g^{-1}) \delta_g \,,$$

where d_t is the degree of φ_t. Then $e_1, e_2, \ldots, e_k$ is a complete set of orthogonal primitive central idempotents of $\mathbb{F}G$.

This year we omit the proof of this theorem.

How to apply this for the proof of the degree theorem?

Consider $V = \mathbb{C}G$ as a vector space over $\mathbb{Q}$.
Fix $i \in \{1, 2, \dots, k\}$, and let $d_i := \chi_{\varphi_i}(1_G)$ be the degree of φ_i.
All we have to do is to find a finitely generated nonzero group $G_i \subseteq (V, +)$ such that $\frac{|G|}{d_i} G_i \subseteq G_i$.
Let G_i be the subgroup of V generated by the set
$S_i := \{e_i * e^{\frac{2\pi \mathrm{i} \ell}{|G|}} \delta_g \mid g \in G, \ell \in \{0, 1, \dots, |G| - 1\}\}$.
Fix $\ell \in \{0, \dots, |G| - 1\}, g \in G$ and compute

$$\frac{|G|}{d_i} e_i * e^{\frac{2\pi \mathrm{i} \ell}{|G|}} \delta_g = \left(\frac{|G|}{d_i} e_i\right) * e_i * e^{\frac{2\pi \mathrm{i} \ell}{|G|}} \delta_g$$

$$\frac{|G|}{d_i} e_i = \sum_{h \in G} \chi_{\varphi_i}(h^{-1}) \delta_h$$

$$\frac{|G|}{d_i} e_i * e^{\frac{2\pi \mathrm{i} \ell}{|G|}} \delta_g = \sum_{h \in G} \chi_{\varphi_i}(h^{-1}) e^{\frac{2\pi \mathrm{i} \ell}{|G|}} e_i * \delta_{hg} \in G_i$$

since $\chi_{\varphi_i}(h^{-1}) e^{\frac{2\pi \mathrm{i} \ell}{|G|}}$ is a sum of roots of the polynomial $x^{|G|} - 1$.

Some details added after the lecture

$\chi_{\varphi_i}(h^{-1}) = \lambda_{h,1} + \lambda_{h,2} + \cdots + \lambda_{h,d_i}$, the sum on the RHS is the sum of all eigenvalues of $\varphi_i(h^{-1})$. In particular, $\lambda_{h,j} = e^{\frac{2\pi \mathrm{i} m_{h,j}}{|G|}}$ for some $m_{h,j} \in \{0, 1, \ldots, |G| - 1\}$. Then

$$\chi_{\varphi_i}(h^{-1}) e^{\frac{2\pi \mathrm{i}\ell}{|G|}} = \sum_{t=1}^{d_i} e^{\frac{2\pi \mathrm{i}((m_{h,t}+\ell) \mod |G|)}{|G|}}$$

$$\sum_{h \in G} \chi_{\varphi_i}(h^{-1}) e^{\frac{2\pi \mathrm{i}\ell}{|G|}} e_i * \delta_{hg} = \sum_{h \in G} \sum_{t=1}^{d_i} e^{\frac{2\pi \mathrm{i}((m_{h,t}+\ell) \mod |G|)}{|G|}} e_i * \delta_{hg}$$

Note that the summands in the last sum are elements of from S_i. This proves that $\frac{|G|}{d_i} s \in G_i$ for every $s \in S_i$ and hence also $\frac{|G|}{d_i} G_i \subseteq G_i$.
Note that in the proof we used $e_i \in Z(\mathbb{C}G)$ and also $e_i * e_i = e_i$. (orthogonality and primitivity of $e_1, \ldots, e_k$ is not needed in this proof)

End

Thank you for your attention.