# Characters – basic properties

Characters can be seen as numerical invariants identifying representations (at least under some assumptions on group and field). We consider only representations of finite degree, i.e., linear actions on finite dimensional spaces

Def: Let $G$ be a group, $\mathbb{F}$ a field, $\varphi : G \to \mathrm{Aut}_{\mathbb{F}}(V)$ a rep of $G$ over $\mathbb{F}$, $\dim_{\mathbb{F}}(V) < \infty$

The character of $\varphi$ is a function $\chi_\varphi : G \to \mathbb{F}$ defined by $\boxed{\chi_\varphi(g) := \mathrm{Tr}\, \varphi(g)}$, $g \in G$

Similarly, if $\psi : G \to \cancel{\mathrm{Aut}}\ GL(d, \mathbb{F})$ is a matrix rep. of $G$, the character of $\psi$ is defined by

$$\boxed{\chi_\psi(g) := \mathrm{Tr}\, \psi(g)}$$

Remark $\mathrm{Tr}\, \psi(g)$ is the trace of a matrix $\psi(g)$, ~~so~~ i.e. the sum of the diagonal elements of $\psi(g)$.

If $B$ is a basis of $V$, then $\mathrm{Tr}\, \varphi(g)$ is $\mathrm{Tr}\, [\varphi(g)]_B$; this is correctly defined ~~is~~ since the value $\mathrm{Tr}\, [\varphi(g)]_B$ does not depend on the choice of $B$

Next proposition summarizes basic properties of characters. We formulate them only for the case of matrix representations

<u>Proposition</u> Let $G$ be a group and let $\mathbb{F}$ be a field

(a) equivalent ~~charact~~ representations have equal characters

(b) characters are constant on conjugacy classes of $G$

(c) character of direct sum is a sum of characters

(d) if $g \in G$, $n = o(g) \in \mathbb{N}_0$, $\mathbb{E}$ is an extension of $\mathbb{F}$ containing all roots of $x^n - 1$, $\Psi: G \to GL(d, \mathbb{F})$ a matrix rep. of $G$ over $\mathbb{F}$. Then $\chi_\Psi(g) = \lambda_1 + \lambda_2 + \dots + \lambda_d$ where $\lambda_1, \dots, \lambda_d \in \mathbb{E}$ are roots of $x^n - 1$

(e) if $\mathbb{F} = \mathbb{C}$, $g \in G$ an <u>element of finite order</u> then $\chi_\Psi(g^{-1}) = \overline{\chi_\Psi(g)}$ for every $\Psi: G \to GL(d, \mathbb{C})$ matrix rep. of $G$ over $\mathbb{C}$.

Proof: (a) Consider equivalent reps $\Psi, \Psi' : G \to GL(d, \mathbb{F})$

There exists $X \in GL(d, \mathbb{F})$ such that $\forall g \in G$

$\Psi'(g) = X \Psi(g) X^{-1}$! Then $\chi_{\Psi'}(g) = Tr\, X \Psi(g) X^{-1}$

$Tr\,AB = Tr\,BA$ --- $= Tr\, \Psi(g) X^{-1} X = Tr\, \Psi(g) = \chi_\Psi(g)$

So $\Psi = \Psi'$!

(b) The same principle ~~in~~ as in (a)

Let $g_1, g_2 \in G$ such that $g_1 = h g_2 h^{-1}$ for some $h \in G$

$\Psi: G \to GL(d, \mathbb{F})$ a hom. of groups. Then

$\chi_\Psi(g_1) = Tr\, \Psi(h) \Psi(g_2) \Psi(h^{-1}) = Tr\, \Psi(g_2) = \chi_\Psi(g_2)$

(c) For $i=1,\dots,k$ let $\psi_i : G \to GL(d_i, \mathbb{F})$ be matrix reps of $G$ over $\mathbb{F}$, $d = d_1 + \dots + d_k$. Then $\psi := \psi_1 \oplus \dots \oplus \psi_k : G \to GL(d, \mathbb{F})$ is defined by

$$\psi : g \mapsto \begin{pmatrix} \psi_1(g) & 0 & & \\ 0 & \psi_2(g) & & \\ & & \ddots & \\ & & & \psi_k(g) \end{pmatrix}$$

Obviously $\chi_\psi(g) = \chi_{\psi_1}(g) + \dots + \chi_{\psi_k}(g)$, hence $\chi_\psi = \chi_{\psi_1} + \dots + \chi_{\psi_k}$

(d) Note that $\psi(g)^n = \psi(1) = E$ ... identity matrix

so if $\lambda \in \mathbb{E}$ is an eigenvalue of $\psi(g)$, then $\lambda^n = 1$. The matrix $\psi(g)$ is similar to an upper triangular matrix, that is $\exists X \in GL(d, \mathbb{E})$ such that $X\psi(g)X^{-1} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$

$\lambda_1, \dots, \lambda_d$ are eigenvalues of $\psi(g)$, therefore $\lambda_i^n = 1$ for $i = 1, \dots, d$

Then $\chi_\psi(g) = \mathrm{Tr}\, X\psi(g)X^{-1} = \lambda_1 + \lambda_2 + \dots + \lambda_d$ where $\lambda_1, \dots, \lambda_d \in \mathbb{E}$ are roots of $x^n - 1$

(e) Consider the proof of (d) for $\mathbb{E} = \mathbb{F} = \mathbb{C}$

In this notation we get $X\psi(g)X^{-1} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$

$X\psi(g)^{-1}X^{-1} = \begin{pmatrix} \lambda_1^{-1} & & * \\ & \lambda_2^{-1} & \\ 0 & & \ddots \;\; \lambda_d^{-1} \end{pmatrix}$ Since $\lambda_t^n = 1$, $|\lambda_t| = 1$ and $\lambda_t^{-1} = \overline{\lambda_t}$

Therefore $\chi_\psi(g^{-1}) = \cancel{\chi}\, \mathrm{Tr}\, X\psi(g)^{-1}X^{-1} = \overline{\lambda_1} + \dots + \overline{\lambda_d} = \overline{\chi_\psi(g)}$.