Consequences of orthogonality relations

1.

Let G be a finite group, $\mathbb{F}$ a field

Consider $V := \{$all maps from G to $\mathbb{F}\}$ with an $\mathbb{F}$-space structure given pointwise: $f, g \in V$, $t \in \mathbb{F}$

$$f+g : x \mapsto f(x) + g(x), \quad x \in G$$

$$tg : x \mapsto tg(x), \quad x \in G$$

$U := \{ f \in V \mid f(x) = f(y)$ whenever $x, y \in G$ are conjugated$\}$

That is U is a subspace of V consisting of all maps which are constant on conjugacy classes of G

__Remark:__ As a vector space V is canonically isomorphic to $\mathbb{F}G$ and U is isomorphic to $Z(\mathbb{F}G)$.
But we do not care about the multiplication in $\mathbb{F}G$ resp. $Z(\mathbb{F}G)$ right now, so the notation is different

If $\operatorname{char} \mathbb{F} \nmid |G|$, we define a bilinear form ~~of~~ on V

$$b(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1}), \quad f_1, f_2 \in V$$

Note that if $\varphi: G \to \operatorname{Aut}_{\mathbb{F}}(V)$ is a rep. of G over $\mathbb{F}$ $\dim_{\mathbb{F}}(V) < \infty$ then $\chi_\varphi \in U$.

If $\varphi_1, \varphi_2 \in \operatorname{Rep}_{\mathbb{F}}(G)$ are irreducible and not equivalent, then $b(\chi_{\varphi_1}, \chi_{\varphi_2}) = 0$.

If $\mathbb{F}$ is alg. closed and char $\mathbb{F} \nmid |G|$,
$\varphi_1, \varphi_2 \in \mathrm{Rep}_{\mathbb{F}}(G)$ are equivalent irreducible reps,
then $b(\chi_{\varphi_1}, \chi_{\varphi_2}) = 1$

Corollary If $\varphi_1, \varphi_2, \dots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ is the list of all the different irreducible reps. of G over $\mathbb{F}$ up to equivalence, then $\chi_{\varphi_1}, \dots, \chi_{\varphi_k}$ is a basis of U.

Proof: Recall $k = \#$ conjugacy classes of $G = \dim_{\mathbb{F}}(U)$
So it is sufficient to prove that $\chi_{\varphi_1}, \dots, \chi_{\varphi_k}$ are linearly independent
Assume $t_1, \dots, t_k \in \mathbb{F}$ such that $t_1 \chi_{\varphi_1} + \dots + t_k \chi_{\varphi_k} = 0$
Then $0 = b(t_1 \chi_{\varphi_1} + \dots + t_k \chi_{\varphi_k}, \chi_{\varphi_j}) = t_j b(\chi_{\varphi_j}, \chi_{\varphi_j})$
$= t_j$
for every $1 \leq j \leq k$ $\square$

Assume again $\mathbb{F}$ alg. closed, char $\mathbb{F} \nmid |G|$,
$\varphi_1, \varphi_2, \dots, \varphi_k \in \mathrm{Rep}_{\mathbb{F}}(G)$ the list of all different irred. reps. of G over $\mathbb{F}$ up to equivalence

If $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ is a representation of finite degree
then $\varphi \simeq \underbrace{\varphi_1 \oplus \dots \oplus \varphi_1}_{m_1} \oplus \underbrace{\varphi_2 \oplus \dots \oplus \varphi_2}_{m_2} \oplus \dots \oplus \underbrace{\varphi_k \oplus \dots \oplus \varphi_k}_{m_k}$

where $m_1, \dots, m_k \in \mathbb{N}_0$ are unique (Krull-Schmidt theorem)
Recall m_i is called the multiplicity of φ_i in φ

When $\operatorname{char}\mathbb{F} = 0$, multiplicities can be computed from characters: Recall that equivalent reps have equal characters and character is "additive" with respect to direct sums. Therefore $\chi_\varphi = \sum_{j=1}^{k} n_j \chi_{\varphi_j}$

Then $b(\chi_\varphi, \chi_{\varphi_i}) = n_i \cdot 1_{\mathbb{F}}$

(recall we assume $\mathbb{F} = \overline{\mathbb{F}}$, $\operatorname{char}\mathbb{F} \nmid |G|$; so if $\operatorname{char}\mathbb{F} > 0$ the multiplicity is determined by $b(\chi_\varphi, \chi_{\varphi_i})$ only modulo $\operatorname{char}\mathbb{F}$)

This formula can be generalized. Assume $\mathbb{F}, \varphi_1, \dots, \varphi_k, G$ are as above and consider $\varphi, \psi \in \operatorname{Rep}_{\mathbb{F}}(G)$ of finite degree. Write φ, ψ as a direct sum of irreducible reps

$$\varphi \simeq \underbrace{\varphi_1 \oplus \dots \oplus \varphi_1}_{n_1} \oplus \underbrace{\varphi_2 \oplus \dots \oplus \varphi_2}_{n_2} \oplus \dots \oplus \underbrace{\varphi_k \oplus \dots \oplus \varphi_k}_{n_k}$$

$$\psi \simeq \underbrace{\varphi_1 \oplus \dots \oplus \varphi_1}_{m_1} \oplus \underbrace{\varphi_2 \oplus \dots \oplus \varphi_2}_{m_2} \oplus \dots \oplus \underbrace{\varphi_k \oplus \dots \oplus \varphi_k}_{m_k}$$

where $n_1, \dots, n_k, m_1, \dots, m_k \in \mathbb{N}_0$

The same argument as above gives

$$b(\chi_\varphi, \chi_\psi) = b\Big(\sum_{i=1}^{k} n_i \chi_{\varphi_i}, \sum_{j=1}^{k} m_j \chi_{\psi_j}\Big) = \sum_{i=1}^{k} n_i m_i 1_{\mathbb{F}}$$

We want to compare $b(\chi_\varphi, \chi_\psi)$ with $\dim_{\mathbb{F}} \operatorname{Mor}(\varphi, \psi)$

We use the language of $\mathbb{F}G$-Mod to have the formulas more familiar

Assume $S_1, \dots, S_k$ are simple $\mathbb{F}G$-modules corresponding to reps $\varphi_1, \dots, \varphi_k$. Then

$S_1^{n_1} \oplus S_2^{n_2} \oplus \dots \oplus S_k^{n_k} \simeq \mathbb{F}G$-module corresponding to φ

$S_1^{m_1} \oplus S_2^{m_2} \oplus \dots \oplus S_k^{m_k} \simeq$ ——"—— ψ

and $\mathrm{Hom}_{\mathbb{F}G}(S_1^{n_1} \oplus \dots \oplus S_k^{n_k}, S_1^{m_1} \oplus \dots \oplus S_k^{m_k})$ "corresponds" to $\mathrm{Mor}(\varphi, \psi)$

$$\mathrm{Hom}_{\mathbb{F}G}\left(\bigoplus_{i=1}^{k} S_i^{n_i}, \bigoplus_{i=1}^{k} S_i^{m_i}\right) \simeq \bigoplus_{i=1}^{k} \mathrm{Hom}_{\mathbb{F}G}(S_i^{n_i}, S_i^{m_i}) \simeq$$

Schur's lemme

$$\simeq \bigoplus_{i=1}^{k} \mathrm{Hom}_{\mathbb{F}G}(S_i, S_i)^{n_i m_i} \simeq \bigoplus_{i=1}^{k} \cancel{\mathrm{Hom}}\, \mathbb{F}^{n_i m_i}$$

All these isomorphisms are $\mathbb{F}$-linear therefore

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}G}\left(\bigoplus_{i=1}^{k} S_i^{n_i}, \bigoplus_{i=1}^{k} S_i^{m_i}\right) = \sum_{i=1}^{k} n_i m_i$$

In representation theory $\dim_{\mathbb{F}} \mathrm{Mor}(\varphi, \psi)$ is caled intertwining number of φ and ψ. We will not use this notion. Note that under our assumptions $\dim_{\mathbb{F}} \mathrm{Mor}(\varphi, \psi) = \dim_{\mathbb{F}} \mathrm{Mor}(\psi, \varphi)$ and when char $\mathbb{F} = 0$ we may compute this from χ_φ, χ_ψ

From the formula $\dim_{\mathbb{F}} \mathrm{Mor}(\varphi, \varphi) = \sum_{i=1}^{k} n_i^2$ we get φ is irreducible $\iff \dim_{\mathbb{F}} \mathrm{Mor}(\varphi, \varphi) = 1$ so if char $\mathbb{F} = 0$ we get a way to test φ for irreducibility from χ_φ.

The following theorem summarizes important conclusions of the considerations above

<u>Theorem</u> Let G be a finite group, $\mathbb{F}$ algebraically closed field of characteristic 0

(a) If $\varphi, \psi \in Rep_{\mathbb{F}}(G)$ are of finite degree, then φ and ψ are equivalent $\iff \chi_\varphi = \chi_\psi$

(b) If $\varphi \in Rep_{\mathbb{F}}(G)$ is of finite degree, then φ is irreducible $\iff \frac{1}{|G|} \sum\limits_{g \in G} \chi_\varphi(g)\chi_\varphi(g^{-1}) = 1_{\mathbb{F}}$

If $\mathbb{F} = \mathbb{C}$, then φ is irreducible $\iff \frac{1}{|G|} \sum\limits_{g \in G} \chi_\varphi(g)\overline{\chi_\varphi(g)} = \cancel{1_{\mathbb{F}}}\, 1$

Proof (already seen above):

(a) $(\Rightarrow)$ holds for any reps of finite degree

$(\Leftarrow)$ Fix the list $\varphi_1, \dots, \varphi_r$ of all different irreducible reps of G over $\mathbb{F}$ up to equivalence

$\chi_\varphi = \chi_\psi \Rightarrow \forall i$ the multiplicity of φ_i in φ = — " — ψ

Therefore $\varphi \simeq \psi$

(b) $(\Rightarrow)$ orthogonality relations of characters

$(\Leftarrow)$ If $\varphi_1, \dots, \varphi_r$ are as above and $\varphi \simeq \underbrace{\varphi_1 \oplus \dots \oplus \varphi_1}_{m_1} \oplus \dots \oplus \underbrace{\varphi_r \oplus \dots \oplus \varphi_r}_{m_r}$

Then $1_{\mathbb{F}} = b(\varphi, \varphi) = \sum\limits_{i=1}^{r} m_i^2 \cdot 1_{\mathbb{F}}$. Since $\operatorname{char} \mathbb{F} = 0$

$\varphi \simeq \varphi_i$ for some i $\square$

<u>Example</u> Find characters of all irreducible reps of S_4 over $\mathbb{C}$

- There are 5 conjugacy classes in S_4

$C_1 = \{id\}$, $C_2 = \{2\text{-cycles}\}$, $C_3 = \{3\text{-cycles}\}$, $C_4 = \{4\text{-cycles}\}$

$C_5 = \{(12)(34), (13)(24), (14)(23)\}$

The normal subgroups of S_4 are $\{id\}$, $V = C_1 \cup C_5$,

$A_4 = C_1 \cup C_3 \cup C_5$

You should know that $S_4/V \simeq S_3$ (for example via

$\pi V \longleftarrow \pi$, $\pi \in S_3$)

Therefore $A_4 = [S_4, S_4]$ and there are 2 reps

of degree 1: $\varepsilon : \pi \longmapsto 1$

$sgn : \pi \longmapsto sgn(\pi)$

It is easy to "compute" their characters

	C_1	C_2	C_3	C_4	C_5
χ_ε	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1

(recall characters are constant on conjugacy classes, so the table determines corresponding characters)

Note that if $G \xrightarrow{f} H$ is an onto homomorphism of groups and $H \xrightarrow{\varphi} Aut_{\mathbb{F}}(W)$ an irreducible rep of H over $\mathbb{F}$, then $\varphi \circ f$ is an irreducible rep of G over $\mathbb{F}$ since every $\varphi \circ f$-invariant subspace of W is also φ-invariant

We apply this observation for the canonical map

$\pi : S_4 \longrightarrow S_4/V$

Since $S_4/V \simeq S_3$, there is an irreducible complex rep of S_3.

S_4/V of degree 2

Recall how we found irred. rep of S_3 of degree 2:

We considered linear action of S_3 on $\mathbb{C}^3$

$$\sigma * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ x_{\sigma^{-1}(3)} \end{pmatrix}$$

Let $l : S_3 \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^3)$ be the corresponding rep of S_3

$\mathbb{C}^3 = U \oplus W$ where $U = \left\{ \begin{pmatrix} x \\ x \\ x \end{pmatrix} \middle| x \in \mathbb{C} \right\}$, $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| x_1 + x_2 + x_3 = 0 \right\}$

are l-invariant subspaces of $\mathbb{C}^3$

~~Define $l_1 := l|_U$,~~

χ_l was determined in the exercises is

$\chi_l(\sigma) = \#$ fixed points of σ (hope you did it)

$\chi_{l_U}(\sigma) = 1$ since $l_U : S_3 \longrightarrow \mathrm{Aut}_{\mathbb{C}}$

$\chi_l = \chi_{l_U} + \chi_{l_W}$ gives an easy way to compute χ_{l_W}

$$\chi_{l_W}(\sigma) = \#\text{ fixed points of } \sigma - 1$$

So $\chi_{l_W}(\sigma) = \begin{cases} 2 & \sigma = id \\ 0 & \sigma \text{ is a transposition} \\ -1 & \sigma \text{ is a 3-cycle} \end{cases}$

We can also check that l_W is irreducible:

$$\frac{1}{6}\sum_{\sigma\in S_3} \chi_{\ell_W}(\sigma)\overline{\chi_{\ell_W}(\sigma)} = \frac{1}{6}\left(2\cdot 2 + 3\cdot 0\cdot 0 + 2\cdot(-1)\cdot(-1)\right) = 1$$ (8.)

ℓ_W is an irreducible rep of S_3 over $\mathbb{C}$

Fixing an isomorphism $i: S_4/V \xrightarrow{\sim} S_3$ we get an irreducible rep of S_4 $\varphi_2 := \ell_W \circ i \circ \pi : S_4 \to \mathrm{Aut}_{\mathbb{C}}(W)$

$$\chi_{\varphi_2} = \chi_{\ell_W} \circ i \circ \pi$$

	C_1	C_2	C_3	C_4	C_5
χ_{φ_2}	2	0	-1	0	2

Let's summarize what we have

- have to find 5 different irred. reps of S_4 over $\mathbb{C}$
- we found 3 of them of degrees $d_1 = 1, d_2 = 1, d_3 = 2$

if d_4, d_5 are degrees of the remaining 2 irred. reps then

$$24 = 1^2 + 1^2 + 2^2 + d_4^2 + d_5^2$$

The only way how to write 18 as a sum of 2 squares is $18 = 3^2 + 3^2$, so $d_4 = d_5 = 3$

We want to guess 3-dimensional irreducible rep of S_4 as we did in the case of 2-dim. -"- S_3

Consider an action of S_4 on $\mathbb{C}^4$

$$\sigma * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} := \begin{pmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ x_{\sigma^{-1}(3)} \\ x_{\sigma^{-1}(4)} \end{pmatrix} \quad \sigma \in S_4$$

Again, $\ell : S_4 \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^4)$ is the corresponding rep. of S_4 over $\mathbb{C}$

(9.)

$\mathbb{C}^4 = U \oplus W$, where $U = \left\{ \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} \middle| x \in \mathbb{C} \right\}$, $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| x_1 + x_2 + x_3 + x_4 = 0 \right\}$

U and W are ℓ-invariant subspaces of S_4

$\ell = \ell_U \oplus \ell_W$ and the same arguments as for S_3

give $\chi_{\ell_W}(\sigma) = \#$ fixed points of $\sigma - 1$

	C_1	C_2	C_3	C_4	C_5
χ_{ℓ_W}	3	1	0	-1	-1

We check the irreducibility of ℓ_W

$$\frac{1}{24}\left(|C_1| \cdot 9 + |C_2| \cdot 1 + |C_4| \cdot 1 + |C_5| \cdot 1 \right)$$

$$= \frac{1}{24}\left(9 + 6 + 6 + 3 \right) = 1 \checkmark$$

Položme $\varphi_3 = \ell_W$. Chybí nám jenom 1 reprezentace stupně 3, označme ji φ_3'. V zadání chceme určit $\chi_{\varphi_3'}$ nikoliv matice φ_3'. To můžeme udělat pomocí rozkladu regulární reprezentace. Připomeňme, že

$\mathrm{reg}: G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}G)$ je reprezentace odpovídající modulu $\mathbb{F}G$ nebo také permutační reprezentace daná akcí G na G levou translací (viz cvičení)

Za určitých předpokladů je násobnost ireduc. rep. v reg rovna stupni dotyčné ireducibilní representace

(I am sorry, after a short break I somehow switched to Czech. The previous paragraph recalls what a regular representation reg is and that multiplicities of irreducible representations in reg are given by degrees if the usual assumptions on G and F are satisfied)

$$\chi_{reg} = \chi_{\varepsilon} + \chi_{sgn} + 2 \cdot \chi_{\varphi_2} + 3\chi_{\varphi_3} + 3\chi_{\varphi_3'} \quad \text{since}$$

$$reg \simeq \varepsilon \oplus sgn \oplus \varphi_2 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_3 \oplus \varphi_3 \oplus \varphi_3' \oplus \varphi_3' \oplus \varphi_3'$$

In general $reg(1_G) = |G| \cdot 1_F$, $reg(g) = 0$ $g \neq 1_G$

The result is summarized in the table

	C_1	C_2	C_3	C_4	C_5
χ_{ε}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_{φ_2}	2	0	-1	0	2
χ_{φ_3}	3	1	0	-1	-1
$\chi_{\varphi_3'}$	3	-1	0	1	-1
χ_{reg}	24	0	0	0	0