Character table of a finite group

<u>Notation</u>: G is a finite group, $G = \dot{\bigcup}_{1 \le i \le k} C_i$, where $C_1, C_2, \dots, C_k$ are the conjugacy classes of G

k always stands for the number of conjugacy classes

$m_i := |C_i|$, $1 \le i \le k$. In each C_i fix some $g_i \in C_i$

Assume $\mathbb{F}$ is an algebraically closed field, $\operatorname{char} \mathbb{F} \nmid |G|$

$\varphi_1, \varphi_2, \dots, \varphi_k$ is a fixed list of ~~as~~ irreducible reps of G over $\mathbb{F}$ (that is $\forall\, \varphi \in \operatorname{Rep}_{\mathbb{F}}(G)$ irreducible there $\exists!\ 1 \le i \le k$ such that φ is equivalent to φ_i)

The character table of G over $\mathbb{F}$ (under the notation introduced above) is $k \times k$ matrix over $\mathbb{F}$

$$A := \begin{array}{c} \\ i \\ \\ \\ \end{array}\!\!\begin{pmatrix} \cdots & \chi_{\varphi_i}(g_j) & \\ & \vdots & \\ & & \end{pmatrix}_{1 \le i, j \le k}$$
$$\qquad\qquad j$$

So if $g \in C_j$ the value of $\chi_{\varphi_i}(g)$ is in the (i,j)-th position of A.

<u>Proposition</u>: The character table is regular

<u>Proof</u>: Write χ_i instead of χ_{φ_i}

Recall orthogonality relations for characters

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g^{-1}) = \delta_{i,j} \cdot 1_{\mathbb{F}}$$

Observe that the map $x \mapsto x^{-1}$ is compatible with conjugation, i.e., if $x, y \in G$ are conjugated, then so are x^{-1}, y^{-1}. So the inverse "permutes" conjugacy classes of G. Therefore if $x, y \in C_t$, then $\chi_\varphi(x^{-1}) = \chi_\varphi(y^{-1})$ for any $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ of finite degree.

After this lengthy redundant explanation we may rewrite orthogonality relations to the form

$$\frac{1}{|G|} \sum_{t=1}^{k} n_t \chi_i(g_t) \chi_j(g_t^{-1}) = \delta_{i,j} \cdot 1_{\mathbb{F}}$$

or $\displaystyle\sum_{t=1}^{k} a_{i,t} \cdot \frac{n_t}{|G|} \chi_j(g_t^{-1}) = \delta_{i,j} \cdot 1_{\mathbb{F}}$, where

$a_{i,t}$ is the (i,t)-th entry of A

Write $\displaystyle b_{t,j} := \frac{n_t}{|G|} \chi_j(g_t^{-1}) \quad 1 \le t, j \le k$

Then $B := \begin{pmatrix} & b_{t,j} & \end{pmatrix}_{1 \le t,j \le k}$ is the inverse of A

$\square$

Recall that if G acts on X, then the size of any orbit divides $|G|$. In particular $\frac{|G|}{m_i} = \frac{|G|}{|C_i|} \in \mathbb{Z}$ $\forall 1 \leq i \leq k$

<u>Theorem</u> (second orthogonality relations)

Assume the introduced notation. Then

$$\sum_{i=1}^{k} \chi_{\varphi_i}(x)\chi_{\varphi_i}(y^{-1}) = \begin{cases} 0 & \text{if } x,y \text{ are not conjugated in } G \\ \frac{|G|}{|C_j|} & \text{if } x,y \in C_j \end{cases}$$

Proof*: In the previous proof we found a formula for $B = A^{-1}$. Since $B \cdot A = E$, we get

$$\sum_{i=1}^{k} b_{j,i} \cdot a_{i,l} = \delta_{j,l} \cdot 1_{\mathbb{F}} \quad \forall 1 \leq j,l \leq k, \text{ that is}$$

$$\sum_{i=1}^{k} \frac{m_j}{|G|} \chi_i(g_j^{-1})\chi_i(g_l) = \delta_{j,l} \cdot 1_{\mathbb{F}} \quad \forall 1 \leq j,l \leq k$$

Assume $x \in C_l$, $y \in C_j$, so $\chi_i(x) = \chi_i(g_l)$, $\chi_i(y^{-1}) = \chi_i(g_j^{-1})$

If x and y are not conjugated, then $\sum_{i=1}^{k} \chi_i(x)\chi_i(y^{-1}) = \sum_{i=1}^{k} \chi_i(g_l)\chi_i(g_j^{-1}) = 0$

If x and y are conjugated $l=j$ and

$$\sum_{i=1}^{k} \chi_i(x)\chi_i(y^{-1}) = \sum_{i=1}^{k} \chi_i(g_l)\chi_i(g_j^{-1}) = \frac{|G|}{m_j} \cdot 1_{\mathbb{F}} \qquad \square$$

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* Again we define $\chi_i := \chi_{\varphi_i}$ $\forall 1 \leq i \leq k$

Recall that if $\mathbb{F} = \mathbb{C}$, $\varphi \in Rep_{\mathbb{C}}(G)$ of finite degree then $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$. The second orthogonality relations then are of the form $\sum_{i=1}^{k} \chi_{\varphi_i}(x)\overline{\chi_{\varphi_i}(y)} = \begin{cases} 0 & x,y \text{ not conjugated} \\ \frac{|G|}{|C_j|} & x,y \in C_j \end{cases}$

In other words, colums of the character table are orthogonal with respect to the standard scalar product of $\mathbb{C}^k$.

Character table of a finite group over $\mathbb{C}$

Assume now that we know a $k \times k$ matrix over $\mathbb{C}$ which is a character table of a finite group G with $\mathbb{F} = \mathbb{C}$. We wonder, which properties of G are determined by this matrix.

It is easy to see that the quaternion group Q and D_8 have indistinguaishable character tables. So the character table over $\mathbb{C}$ does not determine the group up to isomorphism.

In fact, the information stored in the character table is exactly the same as information about multiplication in $Z(\mathbb{C}G)$ with respect to the basis found in Homework #1. We will see more on this later.

Assume we know $A = (a_{i,j})_{1\le i,j\le k}$ a $k\times k$ matrix over $\mathbb{C}$ which is a character table of a finite group G over $\mathbb{C}$. What can we say about G?

First of all G has k conjugacy classes, call them as above $C_1, \dots, C_k$; $G = \dot{\bigcup}_{1\le i\le k} C_i$. The group has k different irreducible reps over $\mathbb{C}$, $\varphi_1, \dots, \varphi_k$ such that $a_{i,j} = \chi_{\varphi_i}(g)$ $\forall g \in C_j$ $\forall 1\le i\le j\le k$

Fix elements $g_1, \dots, g_k$, $g_i \in C_i$ $1\le i\le k$ $\Big\}$ so $a_{i,j} = \chi_i(g_j)$

Instead of χ_{φ_i} we write χ_i $1\le i\le k$

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1. A determines $j \in \{1, \dots, k\}$ satisfying $C_j = \{1_G\}$

note that if $g_j = 1_G$ then $\chi_i(g_j) = d_{\varphi_i}$ the degree of φ_i. In particular $\chi_i(g_j) \in \mathbb{N}$ $\forall 1\le i\le k$

[On the other hand recall $\chi_i(g)$ is the sum of $\lambda_1 + \lambda_2 + \dots + \lambda_{d_i}$, where $\lambda_1, \dots, \lambda_{d_i}$ are the eigenvalues of $\varphi_i(g)$, so $|\chi_i(g)| \le d_i = \chi_i(g_j)$]

The second orthogonality relations imply A can have at most one column consisting of positive integers. Of course, this is the column determined by 1_G

2. A determines degrees of all complex irreducible reps of G.

If $C_j = \{1_G\}$ then $a_{1,j}, a_{2,j}, \dots, a_{k,j}$ are degrees of $\varphi_1, \varphi_2, \dots, \varphi_k$.

3. A determines $|G|$

If d_i = degree of φ_i, we know $|G| = d_1^2 + d_2^2 + \dots + d_k^2$

4. Sizes of $C_1, C_2, \dots, C_k$ are determined by A

Use the second orthogonality relations

$$\sum_{i=1}^{k} a_{ij}\overline{a_{ij}} = \sum_{i=1}^{k} \chi_i(g_j)\chi_i(g_j^{-1}) = \frac{|G|}{|C_j|}$$

So $|C_j| = \dfrac{|G|}{\|a_j\|^2}$, where a_j is the j-th column of A

5. A determines $Z(G)$ as a union of some conjugacy classes

Note that $\forall g \in G$ $g \in Z(G) \iff |\{xgx^{-1} \mid x \in G\}| = 1$

So $Z(G) = \bigcup\limits_{\substack{1 \le j \le k \\ |C_j| = 1}} C_j$. By 4., A determines sizes of $C_1, \dots, C_k$ and hence also $Z(G)$.

6. A determines kernels of irreducible complex reps of G as unions of some conjugacy classes.

Note if $g \in \operatorname{Ker} \varphi_i$, then $\chi_i(g) = \operatorname{Tr} \varphi_i(g) = d_i$

Conversely, assume $\chi_i(g) = d_i$ for some g

As already recalled $|\chi_i(g)| \leq d_i$, because $\varphi_i(g)$ is a diagonalizable operator (or matrix) whose eigenvalues lies in $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

If $\chi_i(g) = \lambda_1 + \lambda_2 + \dots + \lambda_{d_i}$ where $\lambda_1, \dots, \lambda_{d_i}$ have abs value 1, then $|\chi_i(g)| = d_i \Leftrightarrow \lambda_1 = \lambda_2 = \dots = \lambda_{d_i} \Leftrightarrow \varphi_i(g)$ is a scalar multiple of the identity. Now if $\chi_i(g) = d_i$, then $\lambda_1 = \lambda_2 = \dots = \lambda_{d_i} = 1$ which means $\varphi_i(g)$ is the identity.

So we proved: $g \in \ker \varphi_i \Leftrightarrow \chi_i(g) = d_i$. Since d_i can be found in A (see 2.) we have

$$\ker \varphi_i = \bigcup_{\substack{1 \leq j \leq k \\ a_{i,j} = d_i}} C_j$$

7. A determines $[G,G]$ as a union of some conjugacy classes.

<u>Lemma</u> $[G,G] = \bigcap_{f \in \mathrm{Hom}(G, \mathbb{C}^*)} \ker f$, if G is a finite group.

Proof: Recall $[G,G] = \langle \{[x,y] \mid x,y \in G\} \rangle$ is the smallest element of $\{N \mid N \trianglelefteq G \ \& \ G/N \text{ abelian}\}$

If $f \in \mathrm{Hom}(G, \mathbb{C}^*)$, then $\mathrm{Im}\, f \simeq G/\ker f$ is a subgroup of $\mathbb{C}^*$, hence abelian.

Therefore $[G,G] \subseteq \bigcap_{f \in \mathrm{Hom}(G, \mathbb{C}^*)} \ker f$

Conversely, assume $g \in G \setminus [G,G]$. Then $\bar{g} := g[G,G]$ is not the unit of $G/[G,G]$. Therefore there exists $h_g \in \mathrm{Hom}(G/[G,G], \mathbb{C}^*)$ such that $h(\bar{g}) \neq 1$. Indeed, consider the character table of $G/[G,G]$ over $\mathbb{C}$. Since $G/[G,G]$ is abelian, every irreducible rep of $G/[G,G]$ over $\mathbb{C}$ is 1-dimensional, so the column of the character table of $G/[G,G]$ indexed by $\{1_{G/[G,G]}\}$ is $(\underbrace{1,1,\dots,1}_{|G/[G,G]|})^T$. But if $h(\bar{g}) = 1$ $\forall h \in \mathrm{Hom}(G/[G,G], \mathbb{C}^*)$, then the column indexed by the conjugacy class of $G/[G,G]$ containing $\bar{g}$ is also $(1,\dots,1)^T$. But this contradicts 2nd orth. relations

What's going on:
Char table of $G/[G,G]$ over $\mathbb{C}$

	$\{1\}$	$\{x\bar{g}x^{-1} \mid x \in \bar{G}\}$
χ_1	1	1
χ_2	1	$\vdots$
$\vdots$	$\vdots$	$\vdots$
χ_k	1	1

not ~~por~~ orthogonal with resp. to std scalar product

Now if $\pi : G \to G/[G,G]$ is the canonical projection and $h : G/[G,G] \to \mathbb{C}^*$ is a hom. satisfying $h(\bar{g}) \neq 1$, then $h \circ \pi \in \mathrm{Hom}(G, \mathbb{C}^*)$ satisfies $h(\bar{g}) \neq 1$ □

Now we can identify $[G,G]$ from A. For simplicity assume $C_1 = \{1_G\}$, $a_{1,1} = a_{2,1} = \dots = a_{\ell,1} = 1$, $a_{\ell+1,1}, \dots, a_{k,1} \neq 1$. That is in the first column of A we see degrees of $\varphi_1, \varphi_2, \dots, \varphi_k$ and $\varphi_1, \dots, \varphi_\ell$ are of degree 1, $\varphi_{\ell+1}, \dots, \varphi_k$ are of degree > 1.

Recall $\mathrm{Hom}(G,\mathbb{C}^*) = \{\varphi_1, \dots, \varphi_\ell\}$ and by 6.
we can identify $\ker \varphi_1, \dots, \ker \varphi_\ell$. In fact, for
degree 1 reps we have $\chi_\varphi = \varphi$ so if $1 \le i \le \ell$, then
$\ker \varphi_i = \bigcup_{\substack{1\le j\le k \\ a_{i,j}=1}} C_j$. By the lemma $[G,G] = \bigcap_{1\le i\le \ell} \ker \varphi_i$.

Thus $[G,G] = \bigcup_{\substack{a_{i,j}=1 \\ \forall 1\le i\le \ell}} C_j$

You can try to solve the following exercises

8. The character table A determines all normal subgroups of G.

9. The character table A determines whether the group is nilpotent or not.

10. A determines structural constants of $Z(\mathbb{C}G)$ with respect to "homework basis"

Consider $i,j,\ell \in \{1,\dots,k\}$ and for $g \in C_\ell$
consider the number of ways how g can be written
as a product of the set $C_i \cdot C_j$, i.e.

$$|\{(x,y) \mid x \in C_i, y \in C_j\ xy = g\}|$$

Note that this quantity does not depend on the choice of g;

If $g, g' \in C_\ell$, $g' = hgh^{-1}$, then we have a bijection

$$\{(x,y) \mid x \in C_i, y \in C_j, xy = g\} \longleftrightarrow \{(x,y) \mid \begin{smallmatrix} x \in C_i \\ y \in C_j \end{smallmatrix}\ xy = g'\}$$

$$(x,y) \longmapsto (hxh^{-1}, hyh^{-1})$$

$$(h^{-1}xh, h^{-1}yh) \longleftarrow (x,y)$$

For $i, j, \ell \in \{1, \dots, k\}$ we define

$$h_{i,j,\ell} := |\{(x,y) \mid x \in C_i, y \in C_j, xy = g\}|$$

where g is a chosen element of C_ℓ (and the size of the set is independent of the ~~co~~ choice)

These numbers are sometimes called constants for multipl. of conjugacy classes. But let us try to understand them in the context of the group algebra.

First of all $\mathbb{C}G \simeq M_{d_1}(\mathbb{C}) \times \dots \times M_{d_k}(\mathbb{C})$ as $\mathbb{C}$-algebras where $d_1, \dots, d_k$ are the degrees of $\varphi_1, \dots, \varphi_k$

From this isomorphism, $Z(\mathbb{C}G) \simeq \underbrace{\mathbb{C} \times \mathbb{C} \times \dots \times \mathbb{C}}_{k}$.

If A is a finite dimensional $\mathbb{F}$-algebra with basis $b_1, \dots, b_m$. Its structure is given by $c_{i,j,\ell} \in \mathbb{F}$ where

$$b_i \cdot b_j = \sum_{\ell=1}^{m} c_{i,j,\ell}\, b_\ell \qquad 1 \le i, j \le m$$

Elements $c_{i,j,\ell}$ are called <u>structural constants</u> of A w.r.t. $b_1, \dots, b_m$

11.

Now back to $Z(\mathbb{C}G)$. I hope most of you found or verified that elements

$$b_i = \sum_{g \in C_i} \delta_g, \quad 1 \le i \le k, \text{ form a basis of } Z(\mathbb{C}G)$$

Compute $b_i * b_j = \left(\sum_{x \in C_i} \delta_x\right) * \left(\sum_{y \in C_j} \delta_y\right) =$

$$= \sum_{x \in C_i, y \in C_j} \delta_{xy} = \sum_{l=1}^{k} h_{i,j,l}\, b_l$$

for each $g \in C_l$ the summand δ_g occurs on the LHS $h_{i,j,l}$-times

This should justify to call these numbers $h_{i,j,l}$ as the structural constants of $Z(\mathbb{C}G)$ w.r.t. the "homework basis"

We want to show that $h_{i,j,l}$ can be extracted from A

First let us have a look at another application of the Schur's lemma

<u>Lemma</u> Let $\psi: G \to GL(d, \mathbb{F})$ be an irreducible matrix rep. of a finite group G, where $\mathbb{F}$ is alg. closed field, char $\mathbb{F} \nmid |G|$. If $C \subseteq G$ is a conjugacy class in G then there exists $\lambda \in \mathbb{F}$ such that $\sum_{g \in C} \psi(g) = \lambda \cdot E$.

Moreover, $\lambda = \frac{|C|}{d} \chi_\psi(g)$, where $g \in C$

Proof: If $x \in G$, $\psi(x) \sum_{g \in C} \psi(g) = \sum_{g \in C} \psi(xgx^{-1} x) =$

$\left(\sum_{g \in C} \psi(g)\right) \cdot \psi(x)$ (note $g \mapsto xgx^{-1}$, $g \in C$ is a permut. of C)

Since ψ is irreducible and $\mathbb{F}$ alg. closed, Schur's lemma implies $\sum_{g \in C} \psi(g)$ is a scalar multiple of E

Compute traces of $\sum_{g \in C} \psi(g) = \lambda \cdot E$. $Tr(\lambda E) = d \cdot \lambda$

$Tr\left(\sum_{g \in C} \psi(g)\right) = |C| \chi_\psi(g)$ for any $g \in C$

since similar matrices have equal traces.

Recall char $\mathbb{F} \nmid |G| \Rightarrow$ char $\mathbb{F} \nmid d$, hence

$$\lambda = \frac{|C|}{d} \chi_\psi(g) \text{ for any } g \in C. \quad \square$$

Back to our situation with $C_1, \dots, C_k$ conjugacy classes and $\varphi_1, \dots, \varphi_k$ irreducible reps of G over $\mathbb{C}$,

$g_1, \dots, g_k \in G$ $g_i \in C_i$

For $1 \le i, j \le k$ define $\lambda_j^{\varphi_i} := \frac{|C_j|}{d_i} \underbrace{\chi_{\varphi_i}(g_j)}_{a_{i,j}}$, where

$d_i = \chi_{\varphi_i}(1_G)$ is the degree of φ_i

Note that values of $\lambda_j^{\varphi_i}$ are determined ~~from~~ by A since $\chi_{\varphi_i}(g_j) = a_{i,j}$, and $|C_j|, d_i$ can be found (see 2., 3.)

Moreover, $\sum_{g \in C_j} \varphi_i(g) = \lambda_j^{\varphi_i} \cdot Id$, where Id

is the identity on the space, where φ_i acts (or a unit matrix if φ_i is a matrix rep. of G)

Assume, for example, that $\varphi_i : G \to Aut_{\mathbb{C}}(V_i)$

In $End_{\mathbb{C}}(V_i)$ compute

$$\left(\sum_{g \in C_j} \varphi_i(g)\right) \cdot \left(\sum_{g \in C_l} \varphi_i(g)\right) = \sum_{\substack{x \in C_j \\ y \in C_l}} \varphi_i(xy) = \sum_{m=1}^{k} h_{j,l,m} \left(\sum_{g \in C_m} \varphi_i(g)\right)$$

$\Downarrow$

$$\lambda_j^{\varphi_i} \cdot \lambda_l^{\varphi_i} \, Id_{V_i} = \sum_{m=1}^{k} h_{j,l,m} \lambda_m^{\varphi_i} \, Id_{V_i} \text{, therefore}$$

$$(*) \quad \lambda_j^{\varphi_i} \cdot \lambda_l^{\varphi_i} = \sum_{m=1}^{k} h_{j,l,m} \lambda_m^{\varphi_i} \qquad \forall \, 1 \le j, l \le m \quad 1 \le i \le k$$

Fix $1 \le j, l \le k$ and consider $k \times k$ matrix Λ

$$\Lambda = \begin{array}{c} \\ m \end{array}\begin{pmatrix} & \vdots & \\ \cdots & \lambda_m^{\varphi_i} & \\ & & \end{pmatrix}_{1 \le m, i \le k}$$
$$\qquad\quad i$$

note that $\Lambda^T = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix}^{-1} \cdot A \cdot \begin{pmatrix} |C_1| & & 0 \\ & \ddots & \\ 0 & & |C_k| \end{pmatrix}$

In particular Λ ~~(*)~~ is regular.

For fixed j, l $(*)$ can be written in the matrix form:

$$\left(\lambda_j^{\varphi_1} \cdot \lambda_l^{\varphi_1}, \lambda_j^{\varphi_2} \cdot \lambda_l^{\varphi_2}, \dots, \lambda_j^{\varphi_k} \cdot \lambda_l^{\varphi_k}\right) = \left(h_{j,l,1}, h_{j,l,2}, \dots, h_{j,l,m_k}\right) \cdot \underline{\Lambda}$$

Or as a system of linear equations

$$\underline{\Lambda}^T \begin{pmatrix} h_{j,l,1} \\ \vdots \\ h_{j,l,k} \end{pmatrix} = \begin{pmatrix} \lambda_j^{\varphi_1} \cdot \lambda_l^{\varphi_1} \\ \vdots \\ \lambda_j^{\varphi_k} \cdot \lambda_l^{\varphi_k} \end{pmatrix}$$

Note that $\underline{\Lambda}$ and the RHS are determined by A. Since $\underline{\Lambda}$ is regular $h_{j,l,1}, \dots, h_{j,l,k}$ are determined by A too.

<u>Exercise</u> Use for example second orthogonality relations and derive precise formulas for $h_{j,l,m}$.