Ideals in $\mathbb{C}G$, G finite

Example: Find all ideals (= 2-sided ideals) in $\mathbb{C}S_3$

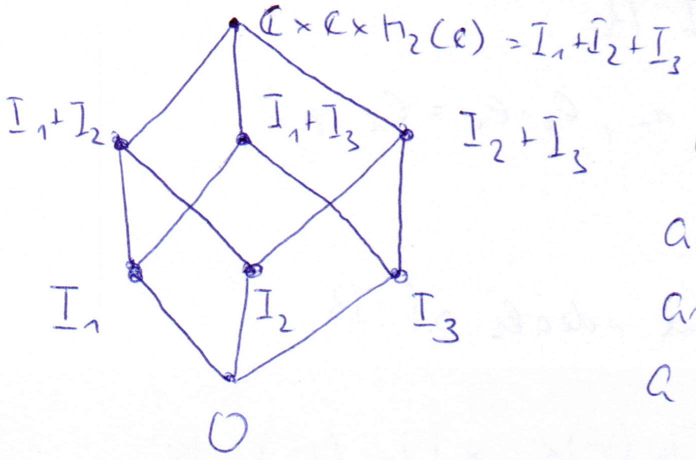
Recall $\mathbb{C}S_3 \simeq \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ as $\mathbb{C}$-algebras

The ideals of $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ can be easily found

$$I_1 := \mathbb{C} \times 0 \times 0, \ I_2 := 0 \times \mathbb{C} \times 0, \ I_3 := 0 \times 0 \times M_2(\mathbb{C})$$

are the minimal ideals of $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ and all ideals of this ring are sums of these ideals

The lattice of 2-sided ideals of $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ is this:



We can use the isomorphism $\varphi: \mathbb{C}S_3 \xrightarrow{\sim} \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ and find $\varphi^{-1}(I_1), \varphi^{-1}(I_2), \varphi^{-1}(I_3)$ and then every ideal of $\mathbb{C}S_3$ is a sum of these ideals

Recall that if $\Psi: S_3 \to GL(2, \mathbb{C})$ is an irreducible rep. of S_3 over $\mathbb{C}$ we can define φ by $\sigma_g \longmapsto (1, \operatorname{sgn} g, \Psi(g))$, $g \in S_3$. One can finish the example now easily (do it).

The aim of this lecture is to show how to find ideals of $\mathbb{C}G$ when G is a finite group provided a character table of G over $\mathbb{C}$ is known.

(2)

Let $\mathbb{F}$ be a field, $k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{N}$ and

$R := M_{n_1}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F})$. Let $I_1, \dots, I_k$

be the minimal ideals (= 2-sided ideals) of R

$I_1 := M_{n_1}(\mathbb{F}) \times 0 \times \dots \times 0$, $I_2 := 0 \times M_{n_2}(\mathbb{F}) \times 0 \times \dots \times 0$, $I_k := 0 \times \dots \times 0 \times M_{n_k}(\mathbb{F})$

For $i = 1, \dots, k$ let $e_i := (0, \dots, 0, \underset{\uparrow i}{E_i}, 0, \dots, 0)$ where E_i is the unit of $M_{n_i}(\mathbb{F})$. We want to define the set $\{e_1, \dots, e_k\}$ using properties which are preserved by ring isomorphisms. Note that $\{e_1, \dots, e_k\}$ is defined by:

- each e_i is in the center of R
- each e_i is nonzero idempotent, that is, $e_i \cdot e_i = e_i \neq 0$
- if $i \neq j$ then $e_i \cdot e_j = 0$
- $k = |\{e_1, \dots, e_k\}| = \#$ minimal ideals of R

Note that if $\varphi : \mathbb{F}G \xrightarrow{\simeq} M_{n_1}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F}) = R$ is an isomorphism of $\mathbb{F}$-algebras then

$I_i = e_i R$, $\varphi^{-1}(I_i) = \varphi^{-1}(e_i)\mathbb{F}G$. So if the set $X := \{\varphi^{-1}(e_1), \dots, \varphi^{-1}(e_k)\}$ is found, then the ideals of $\mathbb{F}G$ are $I_Y := \sum_{t \in Y} t\mathbb{F}G = (\sum_{t \in Y} t)\mathbb{F}G$, where $Y \subseteq X$

Now assume $\mathbb{F}$ alg. closed, char $\mathbb{F} \nmid |G|$, $k := \#$ conjugacy classes of G. Under these assumptions the isomorphism

$\varphi : \mathbb{F}G \simeq M_{n_1}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F})$ of $\mathbb{F}$-algebras exists (recall $n_1, \dots, n_k$ are degrees of irreducible reps from $Rep_{\mathbb{F}}(G)$)

To find the set X we need elements $\varepsilon_1, \dots, \varepsilon_k \in \mathbb{F}G$ such that • each $\varepsilon_i \in Z(\mathbb{F}G)$, • each $\varepsilon_i \times \varepsilon_i = \varepsilon_i \neq 0$
• if $1 \leq i \neq j \leq k$ then $\varepsilon_i \times \varepsilon_j = 0$

Since k = # conjugacy classes of G and $k = |\{\varepsilon_1, \dots, \varepsilon_k\}|$, the condition $|\{\varepsilon_1, \dots, \varepsilon_k\}|$ = # minimal ideals of $\mathbb{F}G$ is satisfied

Using the description of $\{e_1, \dots, e_k\}$ on page 2 we see $\varphi(\{\varepsilon_1, \dots, \varepsilon_k\}) = \{e_1, \dots, e_k\}$

<u>Remark</u>: The condition $|\{e_1, \dots, e_k\}|$ = # minimal ideals of R of R can be replaced by $e_1 + e_2 + \dots + e_k = R$

Now we are going to use the exercise 3 from the set 'Characters of representations' posed on March 26

<u>Lemma</u>: Let G be a finite group, $\mathbb{F}$ algebraically closed, char $\mathbb{F} \nmid |G|$. Assume $\varphi, \psi \in Rep_{\mathbb{F}}(G)$ are irreducible and let $h \in G$. Then $\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(hg) \chi_\psi(g^{-1}) = 0$ if φ, ψ are not equivalent and $\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(hg) \chi_\psi(g^{-1}) = \frac{\chi_\varphi(h)}{\chi_\varphi(1_G)}$ if φ and ψ are equivalent.

Let $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible, $\chi := \chi_\varphi$ be its character

define $$e_\chi = \frac{\chi(1_G)}{|G|} \sum_{g\in G} \chi(g^{-1})\, \delta_g \in \mathbb{F}G$$

Since $\chi(g) = \chi(h)$ whenever $g, h \in G$ are conjugated, we have $e_\chi \in Z(\mathbb{F}G)$ (cf Homework #1)

Compute $$e_\chi * e_\chi = \left(\frac{\chi(1_G)}{|G|}\right)^2 \sum_{g,h\in G} \chi(g^{-1})\chi(h^{-1})\, \underbrace{\delta_g * \delta_h}_{\delta_{gh}} =$$

$$= \left(\frac{\chi(1_G)}{|G|}\right)^2 \sum_{g\in G} \left(\sum_{h\in G} \chi(hg^{-1})\chi(h^{-1})\right) \delta_g \overset{\text{Lemma}}{=}$$

$+\chi(hg^{-1}) = \chi(g^{-1}h)$

$$= \left(\frac{\chi(1_G)}{|G|}\right)^2 \sum_{g\in G} \left(\frac{|G|}{\chi(1_G)} \cdot \chi(g^{-1})\right) \delta_g = \frac{\chi(1_G)}{|G|} \sum_{g\in G} \chi(g^{-1})\,\delta_g =$$

$$= e_\chi$$

Recall $\chi(1_G) = d \cdot 1_{\mathbb{F}} \neq 0$, where d is the degree of φ

the coefficient at δ_{1_G} in e_χ is $\frac{\chi(1_G)^2}{|G|} \neq 0$. So $e_\chi \neq 0$ is an idempotent

Let $\varphi' \in \mathrm{Rep}_{\mathbb{F}}(G)$ be irreducible, $\chi' := \chi_{\varphi'}$,

$$e_{\chi'} = \frac{\chi'(1_G)}{|G|} \sum_{g\in G} \chi'(g^{-1})\,\delta_g \in \mathbb{F}G$$

Assume φ and φ' are not equivalent. Then

$$e_\chi * e_{\chi'} = \frac{\chi(1_G)\chi'(1_G)}{|G|^2} \sum_{g,h\in G} \chi(g^{-1})\chi'(h^{-1})\, \delta_{gh} =$$

$$= \frac{\chi(1_G)\chi'(1_G)}{|G|^2} \sum_{g\in G} \Big(\sum_{h\in G} \chi(hg^{-1})\chi'(h^{-1}) \Big) \delta_g \overset{\text{Lemma}}{=} 0$$

Having this, we can conclude

Char $\mathbb{F} \nmid |G|$

<u>Theorem</u>: Let G be a finite group, $\mathbb{F}$ algebraically closed, field, $\varphi_1, \varphi_2, \dots, \varphi_k \in \operatorname{Rep}_{\mathbb{F}}(G)$ a list of all different irreduc. reps of G over $\mathbb{F}$ up to equivalence. For $1 \le i \le k$ let χ_i be the character of φ_i and let

$$\varepsilon_i := \frac{\chi_i(1_G)}{|G|} \sum_{g\in G} \chi_i(g^{-1})\,\delta_g \in \mathbb{F}G$$

Then
- $0 \neq \varepsilon_i = \varepsilon_i * \varepsilon_i \in Z(\mathbb{F}G)$ for each $1 \le i \le k$
- $\varepsilon_i * \varepsilon_j = 0$ whenever $1 \le i \neq j \le k$
- $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_k = \delta_{1_G}$

In particular, (ideals (=2-sided ideals) of $\mathbb{F}G$ are of the form $\sum_{i\in Y} \varepsilon_i \mathbb{F}G$, where $Y \subseteq \{1, \dots, k\}$

(so $\mathbb{F}G$ contains exactly 2^k different ideals)

Proof (summary of the discussion above):

Since $k = \#$ Conjugacy classes of G, we know

$$\varphi : \mathbb{F}G \simeq M_{n_1}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F}) =: R$$

The properties $\varepsilon_i \in Z(\mathbb{F}G)$, $\varepsilon_i \times \varepsilon_i = \varepsilon_i \neq 0$, $\varepsilon_i \times \varepsilon_j = 0$ if $i \neq j$

imply that the isomorphism φ maps $\{\varepsilon_1, \varepsilon_1, \dots, \varepsilon_k\}\}$

to $\{e_1, \dots, e_k\}$ described on page 2. Since $e_1 + e_2 + \dots + e_k = 1_R$,

$\varepsilon_1 + \dots + \varepsilon_k = \sigma_{1_G} = 1_{\mathbb{F}G}$ follows. The last statements follows

from the known structure of ideals in R $\square$

As an easy application we present an alternative proof of the "degree theorem"

<u>Lemma</u> Let V be a vector space over $\mathbb{Q}$, $0 \neq H$ a finitely generated subgroup of $(V, +)$, $q \in \mathbb{Q}$. If $qH \subseteq H$, then $q \in \mathbb{Z}$.

Proof: Note H is a f.g. torsion-free abelian group, hence H is free.

Let $\{b_1, \dots, b_n\}$ be a free basis of H.

Since $q b_1 \in H$, there are $z_1, \dots, z_n \in \mathbb{Z}$ such that

$q b_1 = z_1 b_1 + z_2 b_2 + \dots + z_n b_n$. Write $q = \frac{a}{b}$ $a, b \in \mathbb{Z}$,

then $a b_1 = (b z_1) b_1 + (b z_2) b_2 + \dots + (b z_n) b_n$

Hence $a = b z_1$, (also $b z_2 = \dots = b z_n = 0$ which we do not need). Therefore $q = \frac{a}{b} = z_1 \in \mathbb{Z}$ $\square$

<u>Theorem</u>: Let G be a finite group, $\varphi \in \mathrm{Rep}_{\mathbb{C}}(G)$ irreducible.

this is not a representation in ecuros but rep. over $\mathbb{C}$

Then the degree of φ divides $|G|$.

Proof: Apply the Lemma with $V = \mathbb{C}G$ considered as a vector space over $\mathbb{Q}$ and $q = \frac{|G|}{d}$, where $d := \chi_\varphi(1)$ is the degree of φ. Of course, the problem is to define H.

Let $e := \frac{\chi_\varphi(1)}{|G|} \sum_{g \in G} \chi_\varphi(g^{-1}) \delta_g$, $\xi := e^{\frac{2\pi i}{|G|}}$

Let H be the subgroup of $(V, +)$ generated by

$$S := \{ \xi^k \delta_g * e \mid 0 \le k < |G|, g \in G \}$$

since $|S| \le 2|G|$, H is finitely generated

In order to verify $q \cdot H \subseteq H$, it is sufficient to show $q \cdot s \in H$ for every $s \in S$. ~~Sin~~

For $0 \le k < |G|$, $g \in G$ compute

$$q \cdot \xi^k \delta_g * e = \frac{|G|}{d} \cdot (\xi^k \delta_g) * (e * e)$$

$$= \xi^k \delta_g * \Big(\sum_{hg \in G} \chi_\varphi(h^{-1}) \delta_h \Big) * e$$

$$= \sum_{h \in G} \chi_\varphi(h^{-1}) \cdot \xi^k \delta_{gh} * e$$

for each $h \in G$ $\chi_\varphi(h^{-1}) = \lambda_1 + \lambda_2 + \dots + \lambda_d$, where

$\lambda_1, \dots, \lambda_d \in \{ \xi^l \mid 0 \le l < |G| \}$

hence $\chi_\varphi(h^{-1}) \cdot \xi^k = \xi^{l_1} + \xi^{l_2} + \dots + \xi^{l_d}$ for

some $l_1, l_2, \dots, l_d \in \{0, \dots, |G|-1\}$

Therefore also $\chi_\varphi(h^{-1}) \xi^k \delta_{gh} * e$ is a sum of elements from S and hence $q \cdot \xi^k \delta_g * e \in H$

Now the Lemma implies $q \in \mathbb{Z}$, so $d \mid |G|$ $\square$

There should be a remark about the general ring theoretical point of view on what we did in this lecture. It is <u>not</u> going to be a subject of the exam, so I will add a seen set of exercises for those who are interested.