# Orthogonality of characters

For representations of finite groups over $\mathbb{C}$ characters are useful tool. For a given representation we can check whether the representation is irreducible, for 2 representations we can check whether they are equivalent, we can compute $\dim_{\mathbb{C}} \mathrm{Mor}(\varphi, \psi)$ and so on.

First let us state Schur's lemma in the form we need:

**Lemma (Schur)** Let $G$ be a finite group, $\mathbb{F}$ a field

(a) Assume $\varphi, \psi \in \mathrm{Rep}_{\mathbb{F}}(G)$ are irreducible. Then every $\Theta \in \mathrm{Mor}(\varphi, \psi)$ is either $0$ or an isomorphism

(b) Assume $\mathbb{F}$ algebraically closed, $\varphi \in \mathrm{Rep}_{\mathbb{F}}(G)$ irreducible. Then every $\Theta \in \mathrm{Mor}(\varphi, \varphi)$ is a scalar multiple of identity

**Proof** (a) We have $\varphi : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $\psi : G \to \mathrm{Aut}_{\mathbb{F}}(W)$

$\Theta \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ such that $\psi(g)\Theta = \Theta\varphi(g) \ \forall g \in G$

$$\begin{tikzcd} V \arrow[r, "\Theta"] \arrow[d, "\varphi(g)"'] \arrow[dr, phantom, "\circ"] & W \arrow[d, "\psi(g)"] \\ V \arrow[r, "\Theta"] & W \end{tikzcd}$$

Let $\psi_g := \psi(g)$, $\varphi_g := \varphi(g)$, $g \in G$

Observe that $\mathrm{Ker}\,\Theta$ is $\varphi$-invariant subspace of $V$

$\Theta(v) = 0 \Rightarrow \psi_g \Theta(v) = 0 = \Theta\varphi_g(v) \Rightarrow \varphi_g(v) \in \mathrm{Ker}\,\Theta$

Also $\mathrm{Im}\,\Theta$ is $\psi$-invariant: $w = \Theta(v) \Rightarrow \psi_g(w) = \psi_g\Theta(v) = \Theta\varphi_g(v)$

Since $\varphi, \psi$ are irreducible $\operatorname{Ker}\Theta \in \{0, V\}$, $\operatorname{Im}\Theta \in \{0, W\}$. If $\Theta \neq 0$, then $\operatorname{Ker}\Theta = 0$ and $\operatorname{Im}\Theta = W$, so $\Theta$ is an isomorphism.

(b) Assume $\varphi : G \to \operatorname{Aut}_{\mathbb{F}}(V)$

$$\begin{array}{ccc} V & \xrightarrow{\Theta} & V \\ {\scriptstyle \varphi(g)}\downarrow & & \downarrow{\scriptstyle \varphi(g)} \\ V & \xrightarrow{\Theta} & V \end{array}$$

Since $\mathbb{F}$ alg. closed, $\Theta$ has an eigenvalue $\lambda \in \mathbb{F}$, that is $\exists v \neq 0, v \in V$ $\Theta(v) = \lambda \cdot v$

Consider ~~$\Psi : V \to V$~~ $\bar{\Theta} : V \to V$, $\bar{\Theta}(v) = \Theta(v) - \lambda(v)$

that is $\bar{\Theta} = \Theta - \lambda \operatorname{id}_V$

Then $\bar{\Theta} \in \operatorname{Mor}(\varphi, \varphi)$, $\bar{\Theta}(v) = 0$, by (a), $\bar{\Theta}$ has to be zero. That is, $\Theta = \lambda \operatorname{id}_V$ $\square$

We need this lemma for matrix representations

<u>Lemma</u> (Schur) Let $G$ be a finite group, $\mathbb{F}$ a field

(a) Assume $\varphi : G \to GL(n, \mathbb{F})$, $\psi : G \to GL(m, \mathbb{F})$ are irreducible matrix reps of $G$ over $\mathbb{F}$, $C \in M_{m,n}(\mathbb{F})$ such that $\forall g \in G$ $C\varphi(g) = \psi(g) C$

Then either $C = 0$ or $n = m$ and $C$ is invertible

(b) Assume $\mathbb{F}$ algebraically closed, $\varphi : G \to GL(n, \mathbb{F})$ irreducible, $C \in M_n(\mathbb{F})$ such that $\forall g \in G$ $C\varphi(g) = \varphi(g) C$

Then $\exists \lambda \in \mathbb{F}$ such that $C = \lambda \cdot E$

Proof: Apply the previous lemma to linear representations induced by $\varphi, \psi$.

(a) $V = \mathbb{F}^n$, $W = \mathbb{F}^m$, $\varphi' : G \to \mathrm{Aut}_{\mathbb{F}}(V)$, $\psi' : G \to \mathrm{Aut}_{\mathbb{F}}(W)$

$$\varphi'(g) : v \longmapsto \varphi(g) \times v \quad \text{(matrix } \times \text{ vector)}$$

$$\psi'(g) : w \longmapsto \psi(g) \times w$$

By definition $\varphi, \psi$ are irreducible means $\varphi', \psi'$ are irreducible. Define

$$\Theta : V \longrightarrow W \qquad \Theta : v \longmapsto C \times v$$

The equation $C\varphi(g) = \psi(g)C$ can be read as

$$\Theta\varphi'(g) = \psi'(g)\Theta \quad \forall g \in G$$

From the previous lemma is either $\Theta = 0$ or $\Theta$ iso.

Back to matrices: $C = 0$ or $n = m$ & $C$ regular

(b) Similar as (a), or we can apply the first part of this lemma to a singular matrix $\bar{C} := C - \lambda E$, where $\lambda$ is an eigenvalue of $C$. $\square$

Notation Assume $\varphi : G \longrightarrow GL(n, \mathbb{F})$ is a map

(for example a matrix representation of $G$)

For $1 \leq i,j \leq n$ define $\varphi_{i,j}(g) \in \mathbb{F}$ to be the value in the $(i,j)$-th entry of $\varphi(g)$. Thus

$$\varphi(g) = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} \cdots & \varphi_{i,j}(g) \\ & \vdots \end{pmatrix}_{1 \leq i,j \leq n} \end{array} \qquad \forall g \in G$$

**Corollary** Let $G$ be a finite group, $\varphi : G \to GL(n, \mathbb{F})$
$\psi : G \to GL(m, \mathbb{F})$ irreducible matrix reps of $G$ over $\mathbb{F}$. If $\varphi$ and $\psi$ are not equivalent, then

$$\sum_{g \in G} \varphi_{i,j}(g)\, \psi_{k,l}(g^{-1}) = 0$$

for every $1 \le i, j \le n$, $1 \le k, l \le m$

**Proof** Let $A \in M_{n,m}(\mathbb{F})$ be an arbitrary matrix

Define $C_A := \sum_{g \in G} \varphi(g) A \psi(g^{-1})$

if $h \in G$, then $\varphi(h) C_A = \sum_{g \in G} \varphi(hg) A \psi(g^{-1}h^{-1}h)$

$$= C_A \psi(h)$$

Since $\varphi, \psi$ are non-equivalent irreducible, Schur's lemma imply $C_A = 0$

Choose $A = \begin{array}{cc} j & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ & k \end{array}$ (the only non-zero entry is in the ~~(k,j)~~ $(j,k)$-th position)

Recall: if $X, Y, Z$ are matrices whose product $XYZ$ is defined, the element $(XYZ)_{i,l}$ in the $(i,l)$-th entry of $XYZ$ is $\sum_{j',k'} x_{i,j'} \cdot y_{j',k'} \cdot z_{k',l}$,
where the sum ranges over all possible values of $j', k'$

So the $(i,l)$-th entry of $C_A$ is

$$(C_A)_{i,l} = \sum_{g \in G} \sum_{\substack{1 \le j' \le m \\ 1 \le k' \le m}} \varphi_{i,j'}(g) A_{j',k'} \psi_{k',l}(g^{-1})$$

By our choice of $A$, $A_{j',k'} \neq 0$ only if $j'=j$, $k'=k$

Therefore $(C_A)_{i,l} = \sum_{g \in G} \varphi_{i,j}(g) \psi_{k,l}(g^{-1})$

As we noticed $C_A = 0$, so $\sum_{g \in G} \varphi_{i,j}(g) \psi_{k,l}(g^{-1}) = 0$ $\square$

<u>Corollary</u> Let $G$ be a finite group, $\mathbb{F}$ alg. closed field, $\varphi : G \to GL(n, \mathbb{F})$ irreducible matrix rep of $G$ over $\mathbb{F}$

char $\mathbb{F} \nmid |G|$

Then $n \sum_{g \in G} \varphi_{i,j}(g) \varphi_{k,l}(g^{-1}) = \delta_{i,l} \cdot \delta_{j,k} |G| \cdot 1_{\mathbb{F}}$

(this is a relation in $\mathbb{F}$)

Proof: For an arbitrary matrix $A \in M_n(\mathbb{F})$ define

$$C_A = \sum_{g \in G} \varphi(g) A \varphi(g^{-1})$$

Again $\forall h \in G$ $\varphi(h) C_A = C_A \varphi(h)$, part (5) of Schur's lemma imply $C_A = \lambda_A E$ for some $\lambda_A \in \mathbb{F}$

if ~~[illegible]~~ $i \neq l$ we continue as in the previous corollary:

Choose
$$A = \begin{array}{c} \\ j \\ \\ \end{array}\begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \\ 0 & \vdots & 0 \end{pmatrix}$$
$$\qquad\qquad\quad k$$

$$(C_A)_{i,l} = \sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,l}(g^{-1})$$

$i \neq l$ imply $(C_A)_{i,l} = 0$, so we are done.

If ~~$k$~~ $j \neq k$, then $\sum_{g \in G} \varphi_{i,j}(g)\varphi_{k,l}(g^{-1}) =$

$$= \sum_{g \in G} \varphi_{k,l}(g^{-1})\varphi_{i,j}(g) = \sum_{g \in G} \varphi_{k,l}(g)\varphi_{i,j}(g^{-1}) = 0.$$

It remains to prove

$$n \sum_{g \in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = |G| \cdot 1_{\mathbb{F}}$$

In this case we choose
$$A = \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \leftarrow j$$
$$\uparrow j$$

and $C_A = \begin{pmatrix} \lambda_j & & 0 \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$ for some $\lambda_j \in \mathbb{F}$

Therefore $\lambda_j = (C_A)_{i,i} = \sum_{g \in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1})$

Now compute $\operatorname{Tr} C_A$ in 2 ways:

1) $\operatorname{Tr} C_A = \operatorname{Tr} \operatorname{diag}(\lambda_j, \ldots, \lambda_j) = n \cdot \lambda_j$

2) $\operatorname{Tr} C_A = \operatorname{Tr} \sum_{g \in G} \varphi(g) A \varphi(g)^{-1} = \cancel{\operatorname{Tr}} \operatorname{Tr} \sum_{g \in G} A = |G| \cdot 1_{\mathbb{F}}$

Thus $n\lambda_j = n \sum_{g\in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = |G|.1_{\mathbb{F}}$ $\square$

Note the assumption char $\mathbb{F} \nmid |G|$ was not neccessary (but it will bea needed later)

<u>Remark</u> Assume $G$ finite group, $\mathbb{F}$ alg. closed field, char $\mathbb{F} \nmid |G|$, $\varphi: G \to GL(n, \mathbb{F})$ irreducible rep of $G$ over $\mathbb{F}$. Then char $\mathbb{F} \nmid n$.

Indeed, if char $\mathbb{F} | n$, then $n \sum_{g\in G} \varphi_{i,j}(g)\varphi_{j,i}(g^{-1}) = 0$ contradicts the previous Corollary.

<u>Theorem</u> (orthogonality of characters)

1. Let $G$ be a finite group, $\varphi: G \to GL(n, \mathbb{F})$, $\psi: G \to GL(m, \mathbb{F})$ irreducible reps of $G$ over $\mathbb{F}$ ~~st~~ which are not equivalent.

   Then $\sum_{g\in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = 0$

2. Let $G$ be a finite group, $\varphi: G \to GL(n, \mathbb{F})$, $\psi: G \to GL(m, \mathbb{F})$ equivalent irreducible reps of $G$ over $\mathbb{F}$. If $\mathbb{F}$ is algebraically closed, char $\mathbb{F} \nmid |G|$, then

   $$\frac{1}{|G|} \sum_{g\in G} \chi_\varphi(g)\chi_\psi(g^{-1}) = 1_{\mathbb{F}}$$

Proof: 1. Recall the definition of characters.

$$\chi_\varphi(g)=\sum_{i=1}^{n}\varphi_{i,i}(g),\quad \chi_\psi(g)=\sum_{i=1}^{m}\psi_{i,i}(g)$$

Then
$$\sum_{g\in G}\chi_\varphi(g)\chi_\psi(g^{-1})=\sum_{g\in G}\left(\sum_{i=1}^{n}\varphi_{i,i}(g)\right)\left(\sum_{j=1}^{m}\psi_{j,j}(g^{-1})\right)=$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{g\in G}\varphi_{i,i}(g)\psi_{j,j}(g^{-1})=0$$

(since $\varphi,\psi$ are irreducible not equivalent, the first corollary says the inner sum is zero)

2. $\varphi,\psi$ are equivalent, so $n=m$ and $\chi_\varphi=\chi_\psi$

Therefore we may assume $\varphi=\psi$

$$\sum_{g\in G}\chi_\varphi(g)\chi_\varphi(g^{-1})=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{g\in G}\varphi_{i,i}(g)\varphi_{j,j}(g^{-1})$$

$$\cancel{=\sum_{i=1}^{n}\sum_{g\in G}\varphi_{i,i}(g)\varphi_{j,j}(g^{-1})}$$

$$=\sum_{i=1}^{n}\sum_{g\in G}\varphi_{i,i}(g)\varphi_{i,i}(g^{-1})=\sum_{i=1}^{n}\frac{|G|}{n}\cdot 1_{\mathbb{F}}=|G|\cdot 1_{\mathbb{F}}$$

□

if $i\neq j$ the inner sum is zero by the second corollary

the second corollary

Note $1/n$ defined in $\mathbb{F}$

If $\mathbb{F} = \mathbb{C}$ $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$. So we get the orthogonality with respect to a standard scalar product:

<u>Corollary</u>: Let $G$ be a finite group, $\varphi: G \to GL(n, \mathbb{C})$

$\psi: G \to GL(m, \mathbb{C})$ irreducible reps of $G$ over $\mathbb{C}$

Then

$$\frac{1}{|G|} \sum_{g \in G} \chi_\varphi(g) \overline{\chi_\psi(g)} = \begin{cases} 0 & \text{if } \varphi \text{ and } \psi \text{ are not equivalent} \\ 1 & \text{if } \varphi \text{ and } \psi \text{ are equivalent} \end{cases}$$

For other applications and some examples see the next lecture.