Decomposition of a regular representation

Assume G is a finite group, $\mathbb{F}$ alg. closed field, $\operatorname{char}\mathbb{F} \nmid |G|$

Wedderburn-Artin theorem implies that

$$\mathbb{F}G \simeq M_{n_1}(\mathbb{F}) \times M_{n_2}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F}) =: R$$

as $\mathbb{F}$-algebras.

In particular these $\mathbb{F}$-algebras have isomorphic centers.

In Homework #1 you were asked to find a basis of $Z(\mathbb{F}G) = \{f \in \mathbb{F}G \mid \forall h \in \mathbb{F}G \ fh = hf\}$. The most natural basis is this one: Let $O_1, O_2, \dots, O_\ell$ be the list of all conjugacy classes of G. That is, $G = O_1 \dot\cup O_2 \dot\cup \dots \dot\cup O_\ell$, where $O_i := \{gxg^{-1} \mid g \in G\}$ for every $x \in O_i$

Then $Z(\mathbb{F}G)$ has a basis $\{b_1, b_2, \dots, b_\ell\}$, where $b_i := \sum_{g \in O_i} \delta_g$. In particular, $\dim_{\mathbb{F}} Z(\mathbb{F}G) = \ell =$ # of conjugacy classes in G.

What about $Z(R) := \{r \in R \mid \forall s \in R \ \ rs = sr\}$?

<u>Exercise</u>: Let $e_i \in R$ be the element whose i-th component is the identity matrix and all the remaining components are zero. For example, $e_1 = (E_{n_1}, 0, \dots, 0)$, $E_{n_1} \in M_{n_1}(\mathbb{F})$ identity matrix

Then $\{e_1, e_2, \dots, e_k\}$ is a basis of $Z(R)$
In particular $k = \dim_{\mathbb{F}}(Z(R))$

Example: If $R = M_2(\mathbb{F}) \times M_3(\mathbb{F})$, then

$$Z(R) = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right) \middle| a, b \in \mathbb{F} \right\}$$

The number k is called # of isotypic components of R (each $M_{n_i}(\mathbb{F})$ in the product is an isotypic component of R). For us the following meaning of k is crucial

k = # of different simple (left) R-modules up to isomorphism

Example: If $R = M_2(\mathbb{F}) \times M_3(\mathbb{F})$, let

$$S_1 := \left\{ \left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \middle| a, b \in \mathbb{F} \right\}$$

$$S_2 := \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \right) \middle| a, b, c \in \mathbb{F} \right\}$$

These modules S_1, S_2 are simple R-modules $S_1 \not\simeq S_2$ and if S is a simple R-module, then $S \simeq S_1$ or $S \simeq S_2$

Since $\mathbb{F}G \simeq R$, k is also # different simple (left) $\mathbb{F}G$-modules. Recall that simple $\mathbb{F}G$-modules correspond to irreducible representations of G over $\mathbb{F}$

Overall $\dim_{\mathbb{F}}(\mathbb{F}G) = k = \ell$ has two different interpretations

ℓ = # conjugacy classes in G

k = # irreducible representations of G over $\mathbb{F}$ up to equivalence

Theorem Let G be a finite group, $\mathbb{F}$ algebraically closed field, char $\mathbb{F} \nmid |G|$. Then the number of all irreducible representations of G over $\mathbb{F}$ up to equivalence = the number of conjugacy classes in G. More precisely, if l is the number of conjugacy classes in G, there are $\varphi_1, \varphi_2, \dots, \varphi_l \in Rep_{\mathbb{F}}(G)$ such that: 1) each φ_i is irreducible

2) if φ_i and φ_j are equivalent, then $i=j$

3) if $\varphi \in Rep_{\mathbb{F}}(G)$ is irreducible, then $\exists i \in \{1, \dots, l\}$ such that φ is equivalent to φ_i

Example: Find irreducible representations of S_3 over $\mathbb{C}$

Since $S_3 = \{id\} \dot\cup \{(1,2), (1,3), (2,3)\} \dot\cup \{(1,2,3), (1,3,2)\}$ there are 3 conjugacy ~~cl~~classes in S_3. So we have to find 3 different irreducible reps in $Rep_{\mathbb{C}}(S_3)$

First we search for degree 1 representations, that is elements of Hom $(S_3, \mathbb{C}^*)$. There are 2 such reps:

- Trivial $\varepsilon : S_3 \to \mathbb{C}^* = GL(1, \mathbb{C})$ $\varepsilon(\pi) = 1 \ \forall \pi \in S_3$
- Sign $\sigma : S_3 \to \mathbb{C}^* = GL(1, \mathbb{C})$ $\sigma(\pi) = sgn(\pi) \ \forall \pi \in S_3$

The remaining irreducible representation should be familiar from the exercises: Assume $V = \mathbb{C}^3$, $\varphi : S_3 \to Aut_{\mathbb{C}}(V)$

$$\varphi(\pi)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} := \begin{pmatrix} x_{\pi^{-1}(1)} \\ x_{\pi^{-1}(2)} \\ x_{\pi^{-1}(3)} \end{pmatrix}, \pi \in S_3$$

$$W := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V \mid x_1 + x_2 + x_3 = 0 \right\}.$$ Then W is φ-invariant subspace of V

Let us check that $\varphi_W : S_3 \to \mathrm{Aut}_{\mathbb{C}}(W)$ is irreducible.

Let $U \leq W$ be an φ_W-invariant subspace of W, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in U$ nonzero. Since $x_1+x_2+x_3=0$, $x_1=x_2=x_3$ cannot hold. WLOG assume $x_1 \neq x_2$. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in U, \quad \varphi_W((1,2))\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix} \in U, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix} =$$

$$= \begin{pmatrix} x_1-x_2 \\ x_2-x_1 \\ 0 \end{pmatrix} \in U, \quad \varphi_W((2,3))\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix} \in U$$

Since x_2 and x_3 cannot be both zero $\left\langle \begin{pmatrix} x_1-x_2 \\ x_2-x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix} \right\rangle = W$, that is $U = W$

<u>Conclusion</u>: Every irreducible representation of S_3 over $\mathbb{C}$ is equivalent to ε, σ or φ_W. Every representation of S_3 over $\mathbb{C}$ is equivalent to a direct sum of $\varepsilon, \sigma, \varphi_W$, say if $\varphi : S_3 \to \mathrm{Aut}_{\mathbb{F}}(X)$ is a rep. of finite degree, then

$$\varphi \cong \underbrace{\varepsilon \oplus \varepsilon \oplus \dots \oplus \varepsilon}_{n_1} \oplus \underbrace{\sigma \oplus \dots \oplus \sigma}_{n_2} \oplus \underbrace{\varphi_W \oplus \dots \oplus \varphi_W}_{n_3}$$

(to be precise ε and σ should be considered as $\varepsilon, \sigma : S_3 \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$)

The numbers n_1, n_2, n_3 describe the rep. up to equivalence that is if $\underbrace{\varepsilon \oplus \dots \oplus \varepsilon}_{n_1} \oplus \underbrace{\sigma \oplus \dots \oplus \sigma}_{n_2} \oplus \underbrace{\varphi_W \oplus \dots \oplus \varphi_W}_{n_3}$

and $\underbrace{\varepsilon \oplus \dots \oplus \varepsilon}_{m_1} \oplus \underbrace{\sigma \oplus \sigma \oplus \dots \oplus \sigma}_{m_2} \oplus \underbrace{\varphi_W \oplus \dots \oplus \varphi_W}_{m_3}$

are equivalent, then $n_1 = m_1$, $n_2 = m_2$, & $n_3 = m_3$.

Multiplicity

Let R be any ring, M a finitely generated semisimple R-module, that is, $M = S_1 \oplus S_2 \oplus \cdots \oplus S_r$, where $S_1, \ldots, S_r$ are simple R-modules. Let S be a simple R-module. It follows from the Krull-Schmidt theorem (or also from the Jordan-Hölder theorem) that the value $|\{i \in \{1, \ldots, r\} \mid S \simeq S_i\}|$ depends only on S and M (and not on the way how we decompose M as a direct sum of simple modules).
This value is called the <u>multiplicity of S in M</u>.

<u>Example</u> Let $R := M_2(\mathbb{F}) \times M_3(\mathbb{F})$,

$$M := \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \;\middle|\; a, b, c, d \in \mathbb{F} \right\}$$

$$S_1 := \left\{ \left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \;\middle|\; a, b \in \mathbb{F} \right\}$$

$$S_2 := \left\{ \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \;\middle|\; a, b \in \mathbb{F} \right\}$$

$$S_3 := \left\{ \left(\begin{pmatrix} a & a \\ b & b \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \;\middle|\; a, b \in \mathbb{F} \right\}$$

$$T := \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \right) \middle| a, b, c \in \mathbb{F} \right\}$$

Then $S_1 \simeq S_2 \simeq S_3 \not\simeq T$

$M = S_1 \oplus S_2 = S_1 \oplus S_3 = S_2 \oplus S_3$

in any of these decompositions of M we have 2 summands isomorphic to S_1 and 0 summands isomorphic to T. Therefore the multiplicity of S_1 in M is 2 and the multiplicity of T in M is 0.

As always we can reformulate the notion of multiplicity in the language of group representations

<u>Def</u> Let G be a finite group, $\mathbb{F}$ a field ~~a~~, $\operatorname{char} \mathbb{F} \nmid |G|$, $\varphi \in \operatorname{Rep}_{\mathbb{F}}(G)$ irreducible, $\psi \in \operatorname{Rep}_{\mathbb{F}}(G)$ a representation of finite degree (acting on a finite dimensional space). Then ψ is equivalent to a direct sum of irreducible reps of G over $\mathbb{F}$, say $\psi \simeq \varphi_1 \oplus \varphi_2 \oplus \dots \oplus \varphi_k$, $\varphi_1, \dots, \varphi_k$ irreducible

Then $m(\varphi, \psi) := |\{ 1 \leq i \leq k \mid \varphi_i \text{ is equivalent to } \varphi \}|$ is called the <u>multiplicity</u> of φ ~~a~~ in ψ.

<u>Remark</u> Consider $R = M_{n_1}(\mathbb{F}) \times M_{n_2}(\mathbb{F}) \times \dots \times M_{n_k}(\mathbb{F})$, let $S_1, S_2, \dots, S_k$ be simple left modules

corresponding to components of R.

For example, $S_1 = ((*|0 \cdots 0), 0, \ldots, 0)$

$S_2 = (0, (*|0 \cdots 0), 0, \ldots, 0)$

$\vdots$

$S_k = (0, 0, \ldots, (*|0 \cdots 0))$

Then $_R R \simeq \underbrace{S_1 \oplus \cdots \oplus S_1}_{m_1} \oplus \underbrace{S_2 \oplus \cdots \oplus S_2}_{m_2} \oplus \cdots \oplus \underbrace{S_k \oplus \cdots \oplus S_k}_{m_k}$

The important point to note: m_i has 2 different meanings: a) $m_i = m(S_i, R)$ - the multiplicity of S_i in R

b) $m_i = \dim_{\mathbb{F}}(S_i)$

When $\mathbb{F}$ is algebraically closed the isomorphism between $\mathbb{F}G$ and $M_{n_1}(\mathbb{F}) \times \cdots \times M_{n_k}(\mathbb{F})$ gives

<u>Theorem</u>: Let G be a finite group, $\mathbb{F}$ alg. closed field, char $\mathbb{F} \nmid |G|$

Let $\varphi \in Rep_{\mathbb{F}}(G)$ be ~~an~~ irreducible,

$Reg \in Rep_{\mathbb{F}}(G)$ the regular repres. of G over $\mathbb{F}$

Then the degree of $\varphi = m(\varphi, Reg)$

<u>Proof</u>: Use the notation from the ~~rep~~ remark above

The regular rep. of G over $\mathbb{F}$ corresponds to $\mathbb{F}G$-module $\mathbb{F}G$, irreducible rep. $\varphi \in Rep_{\mathbb{F}}(G)$

corresponds to a simple module $S \in \mathbb{F}G$-mod
The degree of φ is then $\dim_{\mathbb{F}}(S)$, so
$m(\varphi, Reg)$ is the multiplicity of S in $\mathbb{F}G$
Use the isomorphism of $\mathbb{F}$-algebras $\mathbb{F}G \simeq M_{n_1}(\mathbb{F}) \times \dots \times M_{n_l}(\mathbb{F})$
to see that this multiplicity is $\dim_{\mathbb{F}}(S)$. □

! Summary Let G be a finite group, $\mathbb{F}$ alg. closed field char $\mathbb{F} \nmid |G|$. Let l be the number of conjugacy classes in G, $\varphi_1, \dots, \varphi_k$ all different irreducible reps of G over $\mathbb{F}$ up to equivalence, $\varphi_i : G \to Aut_{\mathbb{F}}(V_i)$, $d_i := \dim_{\mathbb{F}}(V_i)$ the degree of φ_i. Then

(a) $k = l$

(b) The regular rep of G over $\mathbb{F}$ is equivalent to

$$\underbrace{\varphi_1 \oplus \varphi_1 \oplus \dots \oplus \varphi_1}_{d_1} \oplus \underbrace{\varphi_2 \oplus \varphi_2 \oplus \dots \oplus \varphi_2}_{d_2} \oplus \dots$$
$$\dots \oplus \underbrace{\varphi_k \oplus \dots \oplus \varphi_k}_{d_k}$$

(c) $|G| = d_1^2 + d_2^2 + \dots + d_k^2$

We already ~~look at~~ proved (a), (b). (c) follows from (b)

The regular ~~rep.~~ rep of G over $\mathbb{F}$ acts on $\mathbb{F}G$

$$\underbrace{\varphi_1\oplus\varphi_1\oplus\cdots\oplus\varphi_1}_{d_1}\oplus\cdots\oplus\underbrace{\varphi_r\oplus\varphi_r\oplus\cdots\oplus\varphi_r}_{d_r}$$

acts on $V_1^{d_1}\oplus V_2^{d_2}\oplus\cdots\oplus V_r^{d_r}$

Since these representations are equivalent,

$\dim_{\mathbb{F}}(\mathbb{F}G) = \dim_{\mathbb{F}}(V_1^{d_1}\oplus V_2^{d_2}\oplus\cdots\oplus V_r^{d_r})$

equivalently $|G| = d_1^2 + d_2^2 + \cdots + d_r^2$

<u>Example</u> We found irreducible reps of S_3 over $\mathbb{C}$ $\varepsilon, \sigma, \varphi_W$. The theory tells us that $\mathbb{C}S_3 \simeq \mathbb{C}\times\mathbb{C}\times M_2(\mathbb{C})$ since ε, σ are of degree and φ_W has degree 2.

In fact, we can use these reps to construct the homomorphisms between these algebras

Let $\psi_W : S_3 \longrightarrow GL(2,\mathbb{C})$ be a matrix rep. corresponding to φ_W (see the exercises)

Let $\Theta : \mathbb{C}S_3 \longrightarrow \mathbb{C}\times\mathbb{C}\times M_2(\mathbb{C})$ be the $\mathbb{C}$-linear map satisfying

$$\Theta(\sigma_\pi) = (1, \operatorname{sgn}(\pi), \psi_W(\pi))$$

Then Θ is an isomorphism of algebras

This can be checked directly, for example by verifying independence of $\{\Theta(\sigma_\pi) \mid \pi\in S_3\}$. But there are better ways.