<u>Notation</u>: For $\lambda, \mu \vdash n$, t a λ-tableau we define

$A_t \in \mathrm{End}_{\mathbb{C}}(M^\mu)$ by $A_t := \sum\limits_{\pi \in C_t} \operatorname{sgn}(\pi)\, \varphi^\mu_\pi$

Note that if $\lambda = \mu$, $A_t([t]) = e_t$, the polytabloid associated to t.

Now we are going to use the Dominance lemma.

<u>Lemma</u> Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$, t^λ a λ-tableau and s^μ a μ-tableau such that $A_{t^\lambda}([s^\mu]) \neq 0$. Then $\lambda \trianglerighteq \mu$.
Moreover, if $\lambda = \mu$, then $A_{t^\lambda}([s^\mu]) = \pm e_{t^\lambda}$.

Proof: We show that if $A_{t^\lambda}([s^\mu]) \neq 0$ then the assumption of the dominance lemma holds.
Suppose there are $i \neq j \in \{1, \dots, n\}$ such that i, j are located in the same row of s^μ and also in the same column of t^λ. Then the transposition $(i,j) \in C_{t^\lambda}$.
Then $H := \{id, (i,j)\} \leq C_{t^\lambda}$. Let $\sigma_1, \sigma_2, \dots, \sigma_k \in C_{t^\lambda}$ such that $C_{t^\lambda} = \dot{\bigcup}_{1 \leq r \leq k} \sigma_r H$. Compute

$$A_{t^\lambda}([s^\mu]) = \sum_{\pi \in C_{t^\lambda}} \operatorname{sgn}(\pi)\, \varphi^\mu_\pi([s^\mu]) =$$

$$= \cancel{\sum_{r=1}^{k} (\operatorname{sgn} \pi_{\sigma_r} [\sigma_r] \pi) * s^\mu]} = \sum_{r=1}^{k} \operatorname{sgn}(\sigma_r)[\sigma_r * s^\mu] + \operatorname{sgn}(\sigma_r (i,j)) \cdot [\sigma_r * (i,j) * s^\mu]$$

12.

$$= \sum_{r=1}^{h} \operatorname{sgn}(\sigma_r)\left[\sigma_r * s^{\mu}\right] - \operatorname{sgn}(\sigma_r)\left[\sigma_r * ((i,j) * s^{\mu})\right]$$

~~Protože~~ Since i,j are located in the same row of s^{μ}

$(i,j) * s^{\mu} \sim s^{\mu}$ and hence also

$\sigma_r * ((i,j) * s^{\mu}) \sim \sigma_r * s^{\mu}$

But then $A_{t^{\lambda}}([s^{\mu}]) = 0$. So if $A_{t^{\lambda}}([s^{\mu}]) \neq 0$ each pair of elements located in the same row of s^{μ} is not located in the same ~~ro~~ column of t^{λ}. The dominance lemma applies and $\lambda \trianglerighteq \mu$ follows.

Assume now $\lambda = \mu$ and $A_{t^{\lambda}}([s^{\mu}]) \neq 0$. We already know that $A_{t^{\lambda}}([s^{\mu}]) \neq 0$ implies that the assumption of the Dominance lemma is satisfied. In the proof of the Dominance lemma we constructed a λ-tableau u^{λ} satisfying a), b) on page 4. a) can be translated as $\exists \sigma \in C_{t^{\lambda}}$ $\sigma * [t^{\lambda}] = u^{\lambda}$ (cf Remark on page 7.)

If $\lambda = \mu$, the condition b) on page 4. can be stated briefly as ~~App~~ $s^{\mu} \sim u^{\lambda}$

Now compute $A_{t^{\lambda}}[s^{\mu}] = \sum_{\pi \in C_{t^{\lambda}}} \operatorname{sgn}(\pi)\, \varphi_{\pi}^{\lambda}[s^{\mu}] =$

$$= \sum_{\pi \in C_{t^{\lambda}}} \operatorname{sgn}(\pi)\, \varphi_{\pi}^{\lambda}[u^{\lambda}] = \sum_{\pi \in C_{t^{\lambda}}} \operatorname{sgn}(\pi)\, \varphi_{\pi}^{\lambda}[\sigma * t^{\lambda}] =$$

$$= \sum_{\pi \in C_{t^\lambda}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\sigma\pi)\,\varphi^\lambda_{\pi\sigma}([t^\lambda]) \underset{\sigma \in C_{t^\lambda}}{\overset{\uparrow}{=}} \mathrm{sgn}(\sigma) \sum_{\pi' \in C_{t^\lambda}} \mathrm{sgn}(\pi')\,\varphi^\lambda_{\pi'}([t^\lambda])$$

$$= \mathrm{sgn}(\sigma)\, e_t \quad \square$$

<u>Corollary</u> Let $m \in \mathbb{N}$, $\lambda \vdash m$, t a λ-tableau,

$$A_t := \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\, \varphi^\lambda_\pi \in \mathrm{End}_{\mathbb{C}}(M^\lambda). \text{ Then}$$

$$\mathrm{Im}\, A_t = \mathbb{C} e_t$$

Proof: $\mathrm{Im}\, A_t$ is the subspace of M^λ generated by $\{A_t([s]) \mid s \text{ is a } \lambda\text{-tableau}\}$. If $A_t([s]) \neq 0$, then $A_t([s]) = \pm e_t \in \mathbb{C} e_t$. By the definition of polytabloid $A_t([t]) = e_t$. Thus $\mathrm{Im}\, A_t = \mathbb{C} e_t$ $\square$

On M^λ we consider a "standard" scalar product $\langle -,- \rangle$ given by

$$\langle [t],[s] \rangle = \begin{cases} 0 & t \text{ and } s \text{ are not equivalent} \\ 1 & t \sim s \end{cases}$$

where t, s are tableaux of shape λ

The operator A_t considered in the Corollary is self-adjoint, since if $[u],[v]$ are λ-tabloids, then

$$\langle A_t([u]),[v] \rangle = \Big\langle \sum_{\pi \in C_t} \mathrm{sgn}(\pi)[\pi * u],[v] \Big\rangle =$$

$$= \begin{cases} \mathrm{sgn}(\pi) & \text{if } [v] = \pi * [u] \text{ for some } \pi \in C_t \\ 0 & \text{if } [v] \notin \{\pi * [u] \mid \pi \in C_t\} \end{cases}$$

Similarly $\langle [u], A_t([v]) \rangle = \begin{cases} \operatorname{sgn}(\sigma) & \text{if } \exists \sigma \in C_t \; \sigma * [v] = [u] \\ 0 & \text{if } [u] \notin \{\sigma * [v] \mid \sigma \in C_t\} \end{cases}$

So $\langle A_t([u]), [v] \rangle = \langle [u], A_t([v]) \rangle$ for every λ-tabloids $[u], [v]$.

Consequently $\langle A_t(x), y \rangle = \langle x, A_t(y) \rangle \; \forall x, y \in M^\lambda$.

<u>Subrepresentation theorem</u> Let $n \in \mathbb{N}$, $\lambda \vdash n$

Let $V \subseteq M^\lambda$ be a φ^λ-invariant subspace of M^λ.

Then either $S^\lambda \subseteq V$ or $V \subseteq (S^\lambda)^\perp$

<u>Remark</u>: Recall S^λ is the subspace of M^λ spanned by $\{e_t \mid t \text{ is a } \lambda\text{-tableau}\}$. And we proved that S^λ is φ^λ-invariant.

Proof: a) Assume $\exists t$ a λ-tableau and $v \in V$ such that $A_t(v) \neq 0$, where $A_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \varphi^\lambda_\pi$

Then $A_t(v) \in V$, since V is φ^λ-invariant.

On the other hand $\operatorname{Im} A_t = \mathbb{C} e_t$, so $e_t \in V$. Finally $\forall \sigma \in S_n$ $e_{\sigma * t} = \varphi^\lambda_\sigma(e_t) \in V$, for $\sigma \in S_n$ (again use V is φ^λ-invariant)

So in this case V contains all polytabloids associated to tableaux of shape λ. In particular $S^\lambda \subseteq V$.

b) $\forall$ λ-tableau t and $\forall$ $v \in V$ $A_t(v) = 0$

Then $\langle e_t, v\rangle = \langle A_t([t]), v\rangle = \langle [t], A_t(v)\rangle = 0$

for every $v \in V$, t a λ-tableau

$S^\lambda = \langle \{ e_t \mid t \in X^\lambda \} \rangle$, therefore $S^\lambda \subseteq V^\perp$, &

$V \subseteq (S^\lambda)^\perp$ $\square$

<u>Corollary</u> Let $n \in \mathbb{N}$, ~~$\lambda \in$~~ $\lambda \vdash n$, $\psi^\lambda : S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ the Specht representation associated to λ. Then ψ^λ is irreducible.

Proof: Assume ψ^λ is not irreducible. Then there exists a ψ^λ- invariant subspace of S^λ, $0 \neq V \subsetneq S^\lambda$.

Since $\psi^\lambda = \varphi^\lambda_{S^\lambda}$, V is also φ^λ-invariant.

Obviously $V \subsetneq S^\lambda$ and $V \not\subseteq (S^\lambda)^\perp$ contradicts the subrepresentation theorem $\square$

Finally we are left to check that if $n \in \mathbb{N}$, $\lambda, \mu \vdash n$ and $\lambda \neq \mu$, then ψ^λ and ψ^μ are not equivalent.

Assume ψ^λ and ψ^μ are equivalent. Then Maschke's theorem implies $M^\lambda = S^\lambda \oplus C^\lambda$, $M^\mu = S^\mu \oplus C^\mu$, where & C^λ is a φ^λ-invariant subspace of M^λ and C^μ is a φ^μ-invariant subspace of M^μ

Let $T \in \mathrm{Mor}(\psi^\lambda, \psi^\mu)$ be an isomorphism in $\mathrm{Rep}_{\mathbb{C}}(S_n)$. Then $\overline{T} := T \oplus 0 \in \mathrm{Mor}(\varphi^\lambda, \varphi^\mu)$ is a morphism in $\mathrm{Rep}_{\mathbb{C}}(S_n)$ with image S^μ. Compute for a λ-tableau t

$$\overline{T}(e_t) = \overline{T}\Big(\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\, \varphi^\lambda_\pi([t])\Big)$$

$$= \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\, \varphi^\mu_\pi(\overline{T}([t]))$$

Since $\overline{T} \neq 0$, the operator $\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\, \varphi^\mu_\pi \in \mathrm{End}_{\mathbb{C}}(M^\mu)$ is nonzero. Therefore $\exists\, s \in X^\mu$ such that $\sum_{\pi \in C_t} \mathrm{sgn}(\pi)\varphi^\mu_\pi([s]) \neq 0$

Therefore $\lambda \trianglerighteq \mu$ by Lemma on page 11.
Symmetric argument gives $\mu \trianglerighteq \lambda$ and we conclude that $\mu = \lambda$ if ψ^λ and ψ^μ are equivalent.

So we are done.

<u>Theorem</u> Let $n \in \mathbb{N}$, $\psi^\lambda : S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ the Specht representation defined on page 10.

(a) $\forall\, \lambda \vdash n$ $\psi^\lambda \in \mathrm{Rep}_{\mathbb{C}}(S_n)$ is irreducible

(b) $\forall\, \psi \in \mathrm{Rep}_{\mathbb{C}}(S_n)$ irreducible there exists unique $\lambda \vdash n$ such that ψ is equivalent to ψ^λ