Representations of Symmetric groups over $\mathbb{C}$

The final part of the lecture describes irreducible complex representations of symmetric groups. This approach goes back to reverend Alfred Young (1873 - 1940). Another "standard" approach uses idempotents in group algebras of symmetric groups and can be found in the book by Curtis and Reiner "Representation theory of Finite Groups and Associative Algebras".

The first part is rather boring introduction of notions and notations.

<u>Definition</u> Let $n \in \mathbb{N}$. A <u>partition of n</u> is a tuple

$\lambda = (\lambda_1, \dots, \lambda_\ell)$, where $\lambda_1, \dots, \lambda_\ell \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$

and $\lambda_1 + \dots + \lambda_\ell = n$. We write $\underline{\lambda \vdash n}$

Note there is a canonical bijection between the set of all partitions of n and the set of conjugacy classes of S_n. For example, if $n = 3$, then

$$(3) \longleftrightarrow \{(1,2,3), (1,3,2)\}$$
$$(2,1) \longleftrightarrow \{(1,2), (1,3), (2,3)\}$$
$$(1,1,1) \longleftrightarrow \{id\}$$

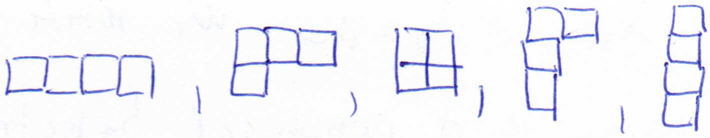
partitions of 3 $\qquad$ conjugacy classes of S_3

For each $\lambda \vdash n$ we define $\Psi^\lambda \in \mathrm{Rep}_{\mathbb{C}}(S_n)$ irreducible such that if $\lambda \neq \mu \vdash n$ Ψ^λ and Ψ^μ are not equivalent. Then $\{\Psi^\lambda \mid \lambda \vdash n\}$ is a complete list of all irreducible reps of S_n over $\mathbb{C}$ up to equivalence.

Before the construction of Ψ^λ we have to introduce quite a lot of new notions.

<u>Definition</u> Let $n \in \mathbb{N}$, $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\lambda \vdash n$. The <u>Young diagram of λ</u> consists of n boxes placed into ℓ rows such that the i-th row contains exactly λ_i boxes.

<u>Example</u> For $n=4$ there are 5 partitions of n: $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$. The corresponding Young diagrams are



(maybe you know an old-school computer game "Tetris" invented by A. Pažitnov in 1984. I wonder if he was inspired by representation theory of S_4.)

<u>Definition</u> (Dominance order) Let $n \in \mathbb{N}$, $\lambda, \mu \vdash n$. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_m)$ we say that <u>λ dominates μ</u> (<u>$\lambda \trianglerighteq \mu$</u>) if $\forall i \in \mathbb{N}$

3.

$\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$, where $\lambda_j = 0$ whenever $j > l$ and $\mu_j = 0$ if $j > m$.

<u>Remark</u>: It is easy to verify that $\trianglerighteq$ is indeed a partial order on the set of all partitions of n. In general, $\trianglerighteq$ is not linear: If $n=7$, $\lambda = (3,3,1)$, $\mu = (4,1,1,1)$ then $\lambda \ntrianglerighteq \mu$ $(3<4)$ and $\mu \ntrianglerighteq \lambda$ $(3+3 > 4+1)$

<u>Definition</u> (Young tableau) Let $n \in \mathbb{N}$, $\lambda \vdash n$. A λ-tableau (or a Young tableau of shape λ) is a Young diagram with ~~numbers~~ numbers $\{1, 2, \dots, n\}$ placed into the boxes.

<u>Remark</u>: For a given shape $\lambda \vdash n$ there are $n!$ different ~~λ~~ λ-tableaux. For example, if $n=3$ $\lambda = (2,1)$, then $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$, are all tableaux of shape λ.

<u>Dominance lemma</u> Let $n \in \mathbb{N}$, $\lambda \vdash n$, $\mu \vdash n$. Suppose t^λ and s^μ are tableaux satisfying

1) t^λ is a λ-tableau, 2) s^μ is a μ-tableau
3) every pair of different numbers located in the same row of s^μ is not located in the same column of t^λ

Then $\lambda \trianglerighteq \mu$.

Proof: let $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $\mu = (\mu_1, \dots, \mu_m)$

We show that there exists a λ-tableau u^λ such that

$\bar{\ell} := \#$ of columns in t^λ

a) $\forall i \in \{1, \dots, \bar{\ell}\}$ the i-th column of u^λ contains the same numbers as the i-th column of t^λ

b) If $k \in \{1, \dots, n\}$ is located in the i-th row of s^μ, then k is located in the j-th row of u^λ for some $j \leq i$

Let us show how $\lambda \trianglerighteq \mu$ follows from the existence of u^λ. Let X_i be the set of numbers located in the i-th row ~~as~~ of s^μ. Then every element of X_1 is located in the first row of u^λ. So $\lambda_1 \geq |X_1| = \mu_1$

Every element of $X_1 \cup X_2$ is located in the first two rows of ~~t^λ~~ u^λ, hence $\lambda_1 + \lambda_2 \geq |X_1 \cup X_2| = \mu_1 + \mu_2$.

In general every element of $X_1 \cup X_2 \cup \dots \cup X_i$ is located in the first i rows of u^λ, ~~hence~~ so $\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$

Therefore $\lambda \trianglerighteq \mu$.

Let us show how to find u^λ. It is easier to understand ~~it~~ via an example

Assume $n=8$, $\lambda=(5,2,1)$, $\mu=(4,2,2)$,

$$t^\lambda = \begin{array}{|c|c|c|c|c|} \hline 8&5&4&2&7 \\ \hline 1&3 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}, \quad S^\mu = \begin{array}{|c|c|c|c|} \hline 1&2&3&4 \\ \hline 5&6 \\ \cline{1-2} 7&8 \\ \cline{1-2} \end{array}$$

Elements of $X_1=\{1,2,3,4\}$ are located in different columns of t^λ. 2 and 4 are in the first row of t^λ but 1 and 3 are not. We therefore exchange 1 and 8, 3 and 5 in t^λ and get

$$t_1^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1&3&4&2&7 \\ \hline 8&5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Note that the columns of t^λ and t_1^λ contains the same sets of numbers.

Now look at $X_2=\{5,6\}$. Elements of this set are located in different columns of t^λ, hence also in different columns of t_1^λ. 5 is in the second row – which is OK. But 6 is in the third row of t_1^λ, we exchange 6 and 8

$$t_2^\lambda = \begin{array}{|c|c|c|c|c|} \hline 1&3&4&2&7 \\ \hline 6&5 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} =: u^\lambda$$

I hope it is clear how to find u^λ in general, since the formal proof is somewhat "a lot of words saying nothing"

We construct inductively $t_1^\lambda, t_2^\lambda, \dots, t_{m-1}^\lambda$ such that

- for $i = 1, \dots, \bar{l}$ the i-th ~~row~~ column of t_j^λ contains the same set of numbers as the i-th column of t^λ
- for $j \le i \le m-1$ if k is located in the j-th row of s^m, then it is located also in the first j rows of t_i^λ

t_1^λ is obtained from t^λ by permuting numbers in columns of t^λ to move elements of X_1 into the first row.

Assume $t_1^\lambda, t_2^\lambda, \dots, t_{i-1}^\lambda$ were found and there is $x \in X_i$ which is located in j-th row of t_{i-1}^λ for some $j > i$. So t_{i-1}^λ contains column of the form

$$\begin{array}{c} \boxed{y_1} \\ \boxed{y_2} \\ \vdots \\ \boxed{y_i} \\ \vdots \\ \boxed{x} \\ \vdots \end{array}$$

Note that $y_i \notin X_1 \cup X_2 \cup \dots \cup X_{i-1}$, hence we can exchange y_i and x. When such an exchange is done for all $x \in X_i$ which are not located in the first i rows of t_{i-1}^λ, we obtain t_i^λ

When t_{m-1}^λ is defined we set $s^\lambda := t_{m-1}^\lambda$ $\square$

For $m \in \mathbb{N}$ and $\lambda \vdash m$ let X^λ be the set of all Young tableaux of shape λ. The group S_m has a natural action on X^λ: If $\pi \in S_m$, $t^\lambda \in X^\lambda$ then $\pi * t^\lambda$ is the tableau obtained from t^λ by changing numbers in boxes by π. For example if $m = 3$, $\lambda = (2,1)$, $t^\lambda = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$

then $\pi * t^\lambda = \begin{array}{|c|c|} \hline \pi(1) & \pi(2) \\ \hline \pi(3) & \\ \cline{1-1} \end{array}$

<u>Definition</u> (Column stabilizer) Let $n \in \mathbb{N}$, $\lambda \vdash n$, $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $t \in X^\lambda$. The column stabilizer C_t is a subgroup of S_n given by $C_t := \{\pi \in S_n \mid \forall 1 \leq i \leq \ell$ the i-th column of t contains the same set of numbers as the i-th column of $\pi * t\}$

<u>Example</u> Let $n = 7$, $\lambda = (3,3,1)$, $t^\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1} \end{array}$ then

$$C_{t^\lambda} = S(\{1,4,7\}) \times S(\{2,5\}) \times S(\{3,6\})$$

where $S(\{1,4,7\})$, $S(\{2,5\})$ and $S(\{3,6\})$ are considered as subgroups of S_7

<u>Remark</u> (!) Condition a) from the proof of Dominance lemma could be written as $\exists \pi \in \cancel{S_t} C_{t^\lambda}$ such that

$$u^\lambda = \pi * t^\lambda$$

On the set of λ-tableaux we consider an equivalence relation $\sim$: $t_1^\lambda, t_2^\lambda \in X^\lambda$ are equivalent ($t_1^\lambda \sim t_2^\lambda$) if

$\forall 1 \leq i \leq \ell$ the i-th row of t_1^λ contains the same set of numbers as the i-th row of t_2^λ

<u>Example</u> Assume $\lambda = (3,1) \vdash 4$ $t_1^\lambda = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d \\ \cline{1-1} \end{array}$, $t_2^\lambda = \begin{array}{|c|c|c|} \hline e & f & g \\ \hline h \\ \cline{1-1} \end{array}$

Then $t_1^\lambda \sim t_2^\lambda \iff d = h$

<u>Definition (Tabloid)</u> Let $n \in \mathbb{N}$, $\lambda \vdash n$. An equivalence class of a λ-tableau is called a <u>λ-tabloid</u>. If $t \in X^\lambda$, $[t] := \{ s \in X^\lambda \mid s \sim t \}$ is called the <u>tabloid of t</u>. The set of all λ-tabloids is denoted by T^λ, that is, $T^\lambda := X^\lambda / \sim$.

It is easy to see that $\forall t_1, t_2 \in X^\lambda$, $\pi \in S_n$ $t_1 \sim t_2 \Leftrightarrow \pi * t_1 \sim \pi * t_2$. Hence S_n has an action on the set T^λ given by $\pi * [t] := [\pi * t]$, $\pi \in S_n$, $[t] \in T^\lambda$

Now we can start with the construction of irreducible reps of S_n over $\mathbb{C}$.

Firstly, the action of S_n on T^λ induces a permutation representation: For $n \in \mathbb{N}$ and $\lambda \vdash n$, let M^λ be ~~the~~ a $\mathbb{C}$-vector space with basis T^λ and $\varphi^\lambda : S_n \to \mathrm{Aut}_{\mathbb{C}}(M^\lambda)$ the representation of S_n induced by the action of S_n on T^λ. If $\pi \in S_n$, $[t] \in T^\lambda$ then

$$\varphi^\lambda(\pi) = \varphi^\lambda_\pi, \quad \varphi^\lambda_\pi([t]) = [\pi * t]$$

Of course, φ^λ is not always irreducible but M^λ contains an important φ^λ-invariant subspace

Definition (Polytabloid) Let $n \in \mathbb{N}$, $\lambda \vdash n$ and $t \in X^\lambda$

The element $e_t := \sum_{\pi \in C_t} \operatorname{sgn}(\pi)[\pi * t] \in M^\lambda$ is called the polytabloid associated to t.

Proposition If $\sigma \in S_n$, $t \in X^\lambda$, then $\varphi_\sigma^\lambda(e_t) = e_{\sigma * t}$

Proof: Note that $C_{\sigma * t} = \sigma C_t \sigma^{-1}$. You can verify it either directly or one can consider an equivalence relation $\wr$ on X^λ: t_1 & t_2 are equivalent if for every $1 \leq i \leq \bar{l}$ — the number of columns in t_1, t_2 — t_1 and t_2 contain in the i-th column the same sets of numbers. Then S_n has a canonical action on $X^\lambda/\wr$ and C_t is just the stabilizer of $[t]_\wr$ in this action. So $C_{\sigma * t} = \sigma C_t \sigma^{-1}$ is then the usual formula for stabilizers of elements in the same orbit of an action.

$$\varphi_\sigma^\lambda e_t = \sum_{\pi \in C_t} \operatorname{sgn}\pi \, \varphi_\sigma^\lambda \varphi_\pi^\lambda([t]) = \sum_{\pi \in C_t} \operatorname{sgn}\pi \, \varphi^\lambda_{\sigma\pi\sigma^{-1}}([\sigma * t])$$

$$= \sum_{\pi \in C_t} \operatorname{sgn} \sigma\pi\sigma^{-1} \, \varphi^\lambda_{\sigma\pi\sigma^{-1}}([\sigma * t]) = \sum_{\pi \in C_{\sigma * t}} \operatorname{sgn}\pi \, \varphi_\pi^\lambda([\sigma * t])$$

$$= e_{\sigma * t}$$

$\square$

<u>Corollary</u>: If $n \in \mathbb{N}$, $\lambda \vdash n$ and S^λ is the subspace of M^λ spanned by $\{ e_t \mid t \in X^\lambda \}$, then S^λ is a φ^λ-invariant subspace of M^λ.

<u>Definition</u> (Specht representation) If $n \in \mathbb{N}$, $\lambda \vdash n$ and S^λ is the subspace of M^λ generated by all λ-polytabloids, then $\psi^\lambda = \varphi^\lambda_{S^\lambda} : S_n \to \mathrm{Aut}_{\mathbb{C}}(S^\lambda)$ is called the <u>Specht representation associated to λ</u>

So we finally defined representation ψ^λ for every $\lambda \vdash n$. It remains to show that these reps are irreducible and that $\lambda \neq \mu \vdash n \Rightarrow \psi^\lambda$ and ψ^μ are not equivalent.