5. série — Taylorovy polynomy

1. Vypočtěte Taylorovy polynomy následujících funkcí stupně k v bodě a:

1.1. $\operatorname{arctg} x,\ k=3,\ a=1,$ **1.2.** $\operatorname{tg} x,\ k=3,\ a=\pi/4,$ **1.3.** $e^x,\ k=5,\ a=2.$

2. Vypočtěte Taylorovy polynomy následujících funkcí stupně k v bodě a:

2.1. $\sin x \cdot \cos x,\ k=4,\ a=0,$ **2.2.** $e^{x^2},\ k=6, a=0,$ **2.3.** $\cos(x^3-1),\ k=3,\ a=1,$

2.4. $\sqrt[6]{\cos x},\ k=5,\ a=0,$ **2.5.** $x^7\sin(x^2),\ k=10,\ a=0,$ **2.6.** $\ln\cos x,\ k=6,\ a=0,$

3. Vypočtěte Taylorovy polynomy následujících funkcí stupně k v bodě a:

3.1. $x^2(\sin(x^2+x)+e^{\sin x}),\ k=5, a=0,$ **3.2.** $\sin(x^2-1)\left(\sqrt[5]{x}-\sqrt[5]{x^2}\right),\ k=5, a=1,$

3.3. $\operatorname{tg}(\sin x)-\sin(\operatorname{tg} x),\ k=7, a=0,$ **3.4.** $(x^2+1)^x-1,\ k=5,\ a=0.$

4. Vypočtěte následující limity s využitím Taylorových polynomů:

4.1. $\displaystyle\lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3},$ **4.2.** $\displaystyle\lim_{x\to 0}\frac{(x^2+1)^x-1}{x^5+x^3},$ **4.3.** $\displaystyle\lim_{x\to 0}\frac{1}{x}-\frac{1}{\sin x},$

4.4. $\displaystyle\lim_{x\to 0}\frac{\sin(\cos(\sin x)-1)+\frac{1}{2}x^2}{x^2\sin^2 x},$ **4.5.** $\displaystyle\lim_{x\to 0}\frac{\operatorname{tg}(\sin x)-\sin(\operatorname{tg} x)}{x^7},$

4.6. $\displaystyle\lim_{x\to+\infty}\left(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}\right),$ **4.7.** $\displaystyle\lim_{x\to+\infty}\left(\left(x^3-x^2+\frac{x}{2}\right)e^{\frac{1}{x}}-\sqrt{x^6+1}\right),$

4.8. $\displaystyle\lim_{x\to 1}\frac{(x-1)^2}{\sin(x-1)-\sqrt{x}\ln x}+\frac{8}{(x-1)},$ **4.9.** $\displaystyle\lim_{x\to 0}\frac{(\cos x)^{\sin x}-(\cos x)^x}{x^2(x-\sin x)},$

Výsledky a návody

1.1. $\frac{1}{4}\pi+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\frac{1}{12}(x-1)^3,$ **1.2.** $1+2(x-\frac{1}{4}\pi)+2(x-\frac{1}{4}\pi)^2+\frac{8}{3}(x-\frac{1}{4}\pi)^3,$ **1.3.** $e^2+e^2(x-2)+\frac{1}{2}e^2(x-2)^2+\frac{1}{6}e^2(x-2)^3+\frac{1}{24}e^2(x-2)^4+\frac{1}{120}e^2(x-2)^5,$
2.1. $x-\frac{2}{3}x^3,$ **2.2.** $1+x^2+\frac{1}{2}x^4+\frac{1}{6}x^6,$ **2.3.** $1-\frac{9}{2}(x-1)^2-9(x-1)^3,$ **2.4.** $1-\frac{1}{12}x^2-\frac{1}{96}x^4,$
2.5. $x^9,$ **2.6.** $-\frac{1}{2}x^2-\frac{1}{12}x^4-\frac{1}{45}x^6.$
3.1. $x^2+2x^3+\frac{3}{2}x^4-\frac{1}{6}x^5,$ **3.2.** $-\frac{2}{5}(x-1)^2-\frac{3}{25}(x-1)^3+\frac{103}{375}(x-1)^4+\frac{26}{75}(x-1)^5,$ **3.3.** $\frac{1}{30}x^7,$
3.4. $x^3-\frac{1}{2}x^5.$
4.1. $\frac{1}{3}$, , **4.2.** 1, **4.3.** 0, **4.4.** $\frac{5}{24},$ **4.5.** $\frac{1}{30},$ **4.6.** $\frac{1}{3},$ **4.7.** $\frac{1}{6},$ **4.8.** $\frac{8}{3},$ **4.9.** $\frac{1}{2}.$

1.1 $f(x) = \operatorname{arctg} x$ $\quad T_3^{\operatorname{arctg},1}(x) = \operatorname{arctg} 1 + (\operatorname{arctg}' 1)(x-1) + \ldots$

$f'(x) = \frac{1}{1+x^2}$ $\quad = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}\cdot\left(\frac{-2}{4}\right)(x-1)^2$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\quad + \underbrace{\frac{1}{6}\cdot\left(\frac{-2}{4} + \frac{8}{8}\right)}_{\frac{1}{12}}(x-1)^3$

$f'''(x) = \frac{-2}{(1+x^2)^2} + (-2x)\cdot\frac{-2(2x)}{(1+x^2)^3}$

1.3 $(e^x)^{(k)} = e^x$; zj. $T_5^{e^x,2} = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 +$

$+ \frac{e^2}{6}(x-2)^3 + \frac{e^2}{120}(x-2)^4$

2.1 $\sin x \cdot \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left(2x - \frac{1}{6}(2x)^3 + o\left((2x)^4\right)\right)$

$= x - \frac{2}{3}x^3 + o(x^4)$

2.4 $\sqrt[6]{\cos x} = \sqrt[6]{1 \underbrace{- \frac{x^2}{2} + \frac{x^4}{120} + o(x^5)}_{= y}}$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{\cancel{120}\ 24} + o(x^5)$

$\sqrt[6]{1+y} = (1+y)^{\frac{1}{6}} = 1 + \frac{1}{6}y + \underbrace{\frac{1}{6}\cdot\left(-\frac{5}{6}\right)\cdot\frac{1}{2}}_{\frac{-5}{72}} y^2 + o(y^2)$

$= 1 + \frac{1}{6}\left(-\frac{x^2}{2} + \frac{x^4}{\cancel{120}\ 24} + o(x^5)\right) + \left(\frac{-5}{72}\right)\left(-\frac{x^2}{2} + \frac{x^4}{120} + o(x^5)\right)^2 + o\left(-\frac{x^2}{2}\right)^2$

$= 1 - \frac{1}{12}x^2 + \underbrace{\left(\frac{1}{6\cdot\cancel{120}\ 24} + \left(-\frac{5}{72}\right)\cdot\frac{1}{4}\right)}_{-\frac{1}{96}} x^4 + o(x^4)$

ale fehlich $o(x^5)$

3.1. $\sin(x+x^2) = x + x^2 - \frac{1}{6}x^3 + o(x^3)$

$e^{\sin x} = 1 + x + \frac{1}{2}x^2 + o(x^2)$

3.2 $x-1 = y$; y: $a \to 0$.

$f(x) = f(y+1) = \sin((y+1)^2 - 1) \cdot \left(\sqrt[5]{1+y} - \sqrt[5]{(1+y)^2}\right)$

$= \sin(y^2+2y) \cdot \left((1+y)^{1/5} - (1+y)^{2/5}\right)$

1. člen: $\sin(y^2+2y) = 2y + y^2 - \frac{4}{3}y^3 - 2y^4 + o(y^4)$

2. člen: $(1+y)^{1/5} = 1 + \frac{1}{5}y - \frac{2}{25}y^2 + \frac{6}{125}y^3 - \frac{21}{625}y^4 + o(y^4)$

$(1+y)^{2/5} = 1 + \frac{2}{5}y - \frac{3}{25}y^2 + \frac{8}{125}y^3 - \frac{26}{625}y^4 + o(y^4)$

$-\frac{1}{5}y + \frac{1}{25}y^2 - \frac{2}{125}y^3 + \frac{1}{125}y^4 + o(y^4)$

$-\frac{2}{5}y^2 - \frac{3}{25}y^3 + \frac{103}{375}y^4 + \frac{26}{75}y^5 + o(y^5)$

3.3 $\operatorname{tg}(\sin x) = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{107}{5040}x^7 + o(x^8)$

$\sin(\operatorname{tg} x) = x + \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{55}{1008}x^7 + o(x^8)$

3.4. $(x^2+1)^x = \exp\left(x \ln(1+x^2)\right)$

$= \exp\left(x \cdot \left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^7)\right)\right)$

$= 1 + x^3 - \frac{1}{2}x^5 + \frac{1}{2}x^6 + \frac{1}{3}x^7 + o(x^7)$

4.1 $e^x \cdot \sin x = x + x^2 + \frac{1}{3}x^3 + o(x^3)$

4.2 $(x^2+1)^x = 1 + x^3 + o(x^3)$

4.3 $\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \cdot \sin x}$; $\sin x - x = -\frac{1}{6}x^3 + o(x^3)$

4.4 $\cos(\sin x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^5)$

$\sin(\cos(\sin x) - 1) = -\frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^5)$.

4.6 $\sqrt[6]{x^6+x^5} - \sqrt[6]{x^6-x^5} = x\left(\sqrt[6]{1+\frac{1}{x}} - \sqrt[6]{1-\frac{1}{x}}\right) = f\left(\frac{1}{x}\right)$

$$f(y) = \frac{1}{y}\left(\underbrace{(1+y)^{1/6} - (1-y)^{1/6}}_{\frac{1}{3}y + o(y)}\right) \ldots$$

$(y \to 0)$

4.7 $x = \frac{1}{y}$: $\left(\frac{1}{y^3} - \frac{1}{y^2} + \frac{1}{2y}\right)e^y - \frac{1}{y^3}(1+y^6)^{1/2}$

$(y \to 0)$

$$= \frac{1}{y^3}\left(\underbrace{\left(1 - y + \frac{y^2}{2}\right)e^y}_{1+\frac{1}{6}y^3+o(y^3)} - \underbrace{(1+y^6)^{1/2}}_{1+o(y^3)}\right)$$

4.8 $x = 1+y$: $\frac{y^2}{\sin y - \ln(1+y)\cdot(1+y)^{1/2}} + \frac{8}{y} = \frac{y^3 + 8\left(-\frac{1}{8}y^3 - \frac{1}{24}y^4 + o(y^4)\right)}{-\frac{1}{8}y^4 + o(y^4)}$

$(y \to 0)$

$\sin y - \ln(1+y)\cdot(1+y)^{1/2} = -\frac{1}{8}y^3 + \frac{1}{24}y^4 + o(y^4)$

4.9 $$\cos^{\sin x} = \exp(\sin x \cdot \underbrace{\ln(\cos x)}_{\ln\left(1+\left(-\frac{x^2}{2}\ldots\right)\right)}) = 1 - \frac{1}{2}x^3 + o(x^5)$$

$\cos x^x = 1 - \frac{1}{2}x^3 - \frac{1}{12}x^5 + o(x^5)$

$x - \sin x = \frac{1}{6}x^3 + o(x^3)$