Carathéodory theory of ODEs

Dalibor Pražák, fall 2024

0. Absolutely continuous functions

Here and below I, J are intervals (of arbitrary type).

Definition 1. *Function $x(t) : I \to \mathbb{R}^n$ is called absolutely continuous, denoted by $x(t) \in AC(I)$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary disjoint intervals $(a_i, b_i) \subset I$ it holds:*

$$\sum_i |a_i - b_i| < \delta \qquad \Longrightarrow \qquad \sum_i |x(a_i) - x(b_i)| < \varepsilon$$

Function $x(t) : I \to \mathbb{R}^n$ is called locally absolutely continuous, denoted by $x(t) \in AC_{\text{loc}}(I)$, if $x(t) \in AC(J)$ for any $J \subset I$ compact.

Proposition 1. *Let $x(t) \in AC(I)$. Then $x'(t)$ exists and is finite almost everywhere (a.e.) in I. Moreover, $x'(t) \in L^1(I)$ and $x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(s)\, ds$ for all t_1, $t_2 \in I$.*

Proposition 2. *Let $h(t) \in L^1(I)$ and $t_0 \in I$ is fixed. Then the function $x(t) := \int_{t_0}^{t} h(s)\, ds$ belongs to $AC(I)$. Moreover, $x'(t) = h(t)$ for a.e. $t \in I$.*

1. Carathéodory solutions

Here and below $\Omega \subset \mathbb{R}^{n+1}$ is an open set of points $(t, x) \in \mathbb{R} \times \mathbb{R}^n$; $U = U(x_0, \delta)$ is an open ball in $\mathbb{R}^n$, $Q(t_0, x_0; \delta, \Delta) := U(x_0, \delta) \times (t_0 - \delta, t_0 + \delta)$ is a cylinder in $\mathbb{R}^{n+1}$.
For a given function $x(t) : I \to \mathbb{R}^n$ we denote the graph $x = \{(t, x(t)); t \in I\} \subset \mathbb{R}^{n+1}$.

Definition 2. *Function $f = f(t, x) : \Omega \to \mathbb{R}^n$ is said to satisfy Carathéodory conditions, denoted by $f \in \mathrm{CAR}(\Omega)$, if for any $(t_0, x_0) \in \Omega$ there exists $Q(t_0, x_0; \delta, \Delta) \subset \Omega$ and a function $m(t) \in L^1(U(t_0, \delta))$ such that:*

(i) for any $x \in U(x_0, \Delta)$ fixed is the function $f(\cdot, x)$ measurable in $U(t_0, \delta)$

(ii) for almost every $t \in U(t_0, \delta)$ fixed is the function $f(t, \cdot)$ continuous in $U(x_0, \Delta)$

(iii) $|f(t, x)| \le m(t)$ for almost every t for all x in $Q(t_0, x_0; \delta, \Delta)$

Definition 3. *Let $f \in \mathrm{CAR}(\Omega)$. The function $x(t) : I \to \mathbb{R}^n$ is called a solution to*

$$x' = f(t, x) \tag{1}$$

in the sense of Carathéodory (or AC solution), if $\operatorname{graph} x \subset \Omega$, *$x(t) \in AC_{\text{loc}}(I)$ and $x'(t) = f(t, x(t))$ for almost every $t \in I$.*

Lemma 3. *Let $f \in \mathrm{CAR}(\Omega)$, and let $x(t) : I \to \mathbb{R}^n$ be a continuous function such that* $\operatorname{graph} x \subset \Omega$. *The the function $t \mapsto f(t, x(t))$ belongs to $L^1_{\text{loc}}(I)$.*

Proof. WLOG we assume that $\operatorname{graph} x \subset \overline{Q}(t_0, x_0; \delta, \Delta)$, the cylinder from Definition 2. Hence $m(t)$ is an integrable majorant. Let us prove that $x(t)$ is measurable. By uniform continuity, there exist piecewise continuous $x_n(t)$ such that $x_n(t) \to x(t)$. Now $x_n(t)$ are measurable and converge to $x(t)$ a.e. by Carathéodory conditions (i), (ii). □

Lemma 4. *Let $f \in \mathrm{CAR}(\Omega)$, and let $x(t) : I \to \mathbb{R}^n$ be a continuous function such that* $\operatorname{graph} x \subset \Omega$*. Then $x(t)$ is a solution to (1) in the sense of Carathéodory, if and only if*

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s))\, ds \tag{2}$$

for all t_1, $t_2 \in I$.

Proof. In view of Lemma 3, the right-hand side of (2) is well defined for all t_1, $t_2 \in I$. Both implications then readily follow from Propositions 1 and 2. □

Corollary 5. *Note that the so-called descriptive definition of Lebesgue integral is a special case: if $h(t) \in L^1(a,b)$, then $\int_a^b h(t)\, dt = H(b) - H(a)$, where $H(t) \in AC([a,b])$ is (arbitrary) function for which $H'(t) = h(t)$ a.e.*

Theorem 6 (Peano)**.** *Let $f \in \mathrm{CAR}(\Omega)$ and $(t_0, x_0) \in \Omega$ are given. Then there exists $x(t)$ a solution to (1), defined on some $I = U(t_0, \delta)$, such that $x(t_0) = x_0$.*

Proof. Assume $Q(t_0, x_0; \delta, \Delta)$ and $m(t)$ are as in Definition 2. Denote

$$X = \{x(t) \in C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n);\ x(t_0) = x_0,\ \operatorname{graph} x \subset \overline{Q}(t_0, x_0; \delta, \Delta)\}$$

Cleary X is a non-empty, convex and closed subset of the Banach space $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$. Let us define operator $\mathcal{T} : x \mapsto \hat{x}$ as

$$\hat{x}(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \qquad t \in [t_0 - \delta, t_0 + \delta] \tag{3}$$

We need to verify that $\mathcal{T}(X) \subset X$. The only non-obvious part here is the condition concerning the graph $\hat{x}$. For this is enough to take $\delta > 0$ small enough such that $\int_{t_0-\delta}^{t_0+\delta} m(t)\, dt \le \Delta$. Functions from $\mathcal{T}(X)$ are equibounded; thanks to the estimate $|\hat{x}(t_1) - \hat{x}(t_2)| \le \int_{t_1}^{t_2} m(t)\, ds$, they are equicontinuous, as well. Hence by Arzelo-Ascoli's theorem, $\mathcal{T}(X)$ is relatively compact in X. Finally, Schauder's theorem implies existence of a fixed-point. In view of Lemma 4, this is the solution we look for. □

2. Generalized Picard theorem

Theorem 7 (Generalized Banach contraction theorem.)**.** *Let Λ, X be metric spaces, where X is complete and non-empty. Let $\Phi(\lambda, x) : \Lambda \times X \to X$ is continuous w.r. to λ for each x fixed. Let further (the key assumption of uniform contraction) there exists $\kappa \in (0,1)$ such that*

$$\|\Phi(\lambda, x) - \Phi(\lambda, y)\|_X \le \kappa \|x - y\|_X \qquad \text{for all } \lambda \in \Lambda,\ x,\ y \in X. \tag{4}$$

Then:

(i) for any $\lambda \in \Lambda$ there is exactly one $x(\lambda) \in X$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$

(ii) the mapping $\lambda \mapsto x(\lambda)$ is continuous $\Lambda \to X$

(iii) $\|y - x(\lambda)\|_X \le (1 - \kappa)^{-1} \|y - \Phi(\lambda, y)\|_X$ for all $\lambda \in \Lambda$, $y \in X$

Proof. (i) Define functions $x_n : \Lambda \to X$ as $x_0(\lambda) \equiv y$, $x_{n+1}(\lambda) = \Phi(\lambda, x_n(\lambda))$, where $y \in X$ is arbitrary, fixed. From (4) we obtain by induction

$$\|x_n(\lambda) - x_{n-1}(\lambda)\|_X \le \kappa^{n-1}\|x_1(\lambda) - x_0(\lambda)\|_X = \kappa^{n-1}\|\Phi(\lambda, y) - y\|_X, \qquad n \ge 1$$

Hence, for any $m > n$

$$\begin{aligned} \|x_m(\lambda) - x_n(\lambda)\|_X &\le \sum_{j=n+1}^{m} \|x_j(\lambda) - x_{j-1}(\lambda)\|_X \le \sum_{j=n+1}^{\infty} \kappa^{j-1}\|\Phi(\lambda, y) - y\|_X \\ &= \frac{\kappa^n}{1-\kappa}\|\Phi(\lambda, y) - y\|_X. \end{aligned} \tag{5}$$

It follows that $x_n(\lambda)$ is a Cauchy sequence, for any λ fixed. Denote $x(\lambda)$ its limit. It is easy to see that $x(\lambda)$ satisfies the equation in (i). Uniqueness is a consequence of (4). In particular, we note that $x(\lambda)$ is independent of the initial choice of $y \in X$ in the sequence $x_n(\lambda)$.
(iii) Take $n = 0$ and $m \to \infty$ in (5).
(ii) Use (iii) with $y = x(\lambda_0)$ and $\lambda = \lambda_n$. We obtain

$$\|x(\lambda_0) - x(\lambda_n)\|_X \le \frac{1}{1-\kappa}\|x(\lambda_0) - \Phi(\lambda_n, x(\lambda_0))\|_X = \frac{1}{1-\kappa}\|\Phi(\lambda_0, x(\lambda_0)) - \Phi(\lambda_n, x(\lambda_0))\|_X.$$

Now $\lambda_n \to \lambda_0$ implies $x(\lambda_n) \to x(\lambda_0)$ as Φ is continuous w.r. to the first argument. □

Theorem 8 (Generalized Picard theorem)**.** *Let $I \subset \mathbb{R}$ be a bounded interval, let Π be a metric space. Assume that $f = f(t, x, p) : I \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ satisfies:*

1. *$f(\cdot, \cdot, p) \in \mathrm{CAR}(I \times \mathbb{R}^n)$ for each $p \in \Pi$ fixed*
2. *there exists $m \in L^1(I)$ such that $|f(t, x, p) - f(t, y, p)| \le m(t)|x - y|$ for a.e. $t \in I$ for all x, $y \in \mathbb{R}^n$, $p \in \Pi$*
3. *the mapping $p \mapsto \int_{t_0}^t f(s, x(s), p)\, ds$ is continuous from Π to $C(I)$, for arbitrary fixed $t_0 \in I$ and $x \in C(I)$*

Then for any given $x_0 \in \mathbb{R}^n$, $t_0 \in I$ and $p_0 \in \Pi$ there exists a unique $x \in AC(I)$, which solves $x' = f(t, x, p_0)$, $x(t_0) = x_0$ in the sense of Carathéodory. This solution depends continuously on x_0 and p_0. More precisely: if $x_{0n} \to x_0$ and $p_{0n} \to p_0$, then $x_n \rightrightarrows x$ in I, where x_n and x respectively are the solutions corresponding to x_{0n}, p_{0n} and x_0, p_0, respectively.

Proof. For the sake of simplicity, let $I = [0, T]$ and $t_0 = 0$. We will apply Theorem 7 with $\Lambda = \mathbb{R}^n \times \Pi$, $X = C([0, T], \mathbb{R}^n)$, where Φ is the mapping

$$(x_0, p_0, x(\cdot)) \mapsto x_0 + \int_0^t f(s, x(s), p_0)\, ds.$$

By the third assumption, Φ is continuous w.r. to (x_0, p_0) for any $x(\cdot)$ fixed. The key assumption (uniform contraction) will be verified for a special (yet equivalent) norm $\|x\|_X = \sup_{t\in[0,T]} |x(t)|e^{-Lt}$, where $L > 0$ will be specified later. Set $\hat{x} = \Phi(x_0, p_0, x)$, $\hat{y} = \Phi(x_0, p_0, y)$. Then

$$\begin{aligned} |\hat{x}(t) - \hat{y}(t)| &= \Big|\int_0^t f(s, x(s), p_0) - f(s, y(s), p_0)\, ds\Big| \le \int_0^t m(s)|x(s) - y(s)|\, ds \\ &\le \|x - y\|_X \int_0^t m(s)e^{Ls}\, ds \qquad \text{for any } t \in I\,. \end{aligned}$$

□

Hence $\|\hat{x}-\hat{y}\|_X \le \kappa \|x-y\|_X$, where

$$\kappa = \sup_{t\in I} \int_0^t m(s) e^{-L(t-s)}\, ds\,.$$

Let us write[1] $m(s) = m_1(s) + m_2(s)$, $m_1(s) = m(s)\chi_{\{m>M\}}(s)$, $m_2(s) = m(s)\chi_{\{m\le M\}}(s)$. We can choose $M > 0$ large enough so that $\int_I m_1 < 1/4$. Then

$$\int_0^t m_1(s) e^{-L(t-s)}\, ds \le \int_I m_1(s)\, ds < \frac{1}{4}.$$

On the other hand,

$$\int_0^t m_2(s) e^{-L(t-s)}\, ds \le M \int_I e^{-L(t-s)}\, ds < M \int_0^\infty e^{-Ls'}\, ds' = \frac{M}{L} < \frac{1}{4},$$

since we finally take $L > 4M$. Hence $\kappa < 1/2$, which finishes the proof.

3. Maximal solution

Definition 4. *A solution $x(t) : I \to \mathbb{R}^n$ of* (1) *will be called maximal in Ω, if there exists no proper extension (i.e. defined on some strictly larger $\hat{I} \supset I$).*
It is right-maximal or left-maximal, if it cannot be extended after the endpoint of I or before the initial point of I, respectively. Clearly, it is maximal if and only if it is both left- and right-maximal.

If $f \in \mathrm{CAR}(\Omega)$ and Ω is open, then solution $x(t) : (a,b) \to \mathbb{R}^n$ is not right-maximal, if and only if (i) $b < \infty$, (ii) there exists $\lim_{t\to b-} x(t) = x_0 \in \mathbb{R}^n$ and (iii) $(b, x_0) \in \Omega$. These conditions are clearly necessary. Sufficiency follows from the local existence (Theorem 6) and the fact that solutions can be glued together in a continuous manner (Lemma 4).
Note that a maximal solution is always defined on an *open* interval, as long as $\Omega \subset \mathbb{R}^{n+1}$ is open.

Theorem 9. *Each solution has at least one maximal extension.*

Proof. Let $(x, (a,b))$ is a given solution. We construct a sequence of right extensions as follows. Set $(x_0, (a, b_0)) = (x, (a,b))$. As $(x_{n+1}, (a, b_{n+1})$ we take any extension of $(x_n, (a, b_n))$ with $b_{n+1} > (b_n + \beta_n)/2$, where β_n is the supremum of all the right points of possible extensions. In case that $\beta_n = +\infty$, we take $b_{n+1} > b_n + 1$.
We claim that the limit solution $(x, (a, \beta))$, where $\beta = \lim_n b_n = \sup_n b_n$, is right-maximal. Assume not: then $\beta < +\infty$ and there is a non-trivial extension $(\tilde{x}, (a, \beta+\delta))$. Observe that for any n, this is also an extension to $(x_n, (a, b_n))$ and thus $\beta_n \ge \beta + \delta$. However, $b_n \to \beta$. For n large enough, this contradicts the conditions for the choice of b_{n+1}. □

Remark. *The problem of finding maximal solution is to choose some continuation in possibly uncoutably many points of non-uniqueness. Usually, this is overcome by Zorn's lemma, i.e. the axiom of choice (AC). Previous proof is a bit more complicated, but it only uses a countable version of AC.*
If the solutions are unique, no choice has to be done at all, since all extensions are equal on the common interval of definition.

[1] χ_A is characteristic function of the set A.

Theorem 10 (On leaving the compact)**.** *Assume $f \in \mathrm{CAR}(\Omega)$, $\Omega \subset \mathbb{R}^{n+1}$ is open, and (x, I) is a maximal solution to* (1) *in Ω. Let $K \subset \Omega$ be a compact set such that $(t_0, x(t_0)) \in K$ for some $t_0 \in I$. Then there exists $t_1 > t_0$ in I such that $(t_1, x(t_1)) \notin K$. Similarly, there exists $t_2 < t_0$ in I such that $(t_2, x(t_2)) \notin K$.*

Proof. Let $I = (a, b)$. Assume that the graph of the restriction $\tilde{x} = x_{[t_0,b)}$ is contained in K. The function $x(t)$ is locally AC, hence $\tilde{x}(t)$ is globally AC on $[t_0, b)$. Consequently, there is a finite limit $x_0 = \tilde{x}(b-)$. Clearly $(b, x_0) \in \Omega$ and according to the remark after Definition 4, we can extend $\tilde{x}$ beyond the point b, which contradicts the right-maximality of (x, I). □

4. Uniqueness

Lemma 11 (Gronwall)**.** *Assume $u \in C(I)$, $\rho \in L^1(I)$ are nonnegative and $t_0 \in I$, $c \geq 0$ such that*

$$u(t) \leq c + \Big| \int_{t_0}^{t} \rho(s) u(s)\, ds \Big| \qquad \text{for all } t \in I. \tag{6}$$

Then

$$u(t) \leq c \exp\Big(\Big| \int_{t_0}^{t} \rho(s)\, ds \Big| \Big) \qquad \text{for all } t \in I.$$

Proof. WLOG we only consider $t \in I \cap [t_0, \infty)$, which means that integrals are nonnegative and we can omit the absolute values. Set $\Phi(t)$ equal to the right-hand side of (6). Then

$$\Phi'(t) = \rho(t) u(t) \leq \rho(t) \Phi(t) \qquad \text{for a.e. } t$$

By a standard procedure (integrating factor, yet in the class of AC functions) we get $\Phi(t) \leq \Phi(0) \exp(\int_{t_0}^{t} \rho(s)\, ds)$, for all $t \in I \cap [t_0, \infty)$. Noting that $\Phi(0) = c$ and $u(t) \leq \Phi(t)$ finishes the proof. □

Lemma 12. *Assume $v \in AC(I)$, $\rho \in L^1(I)$, $\rho \geq 0$ satisfy*

$$\big| v'(t) \big| \leq \rho(t) |v(t)| \qquad \text{for a.e. } t \in I. \tag{7}$$

Then

$$|v(t)| \leq |v(t_0)| \exp\Big(\Big| \int_{t_0}^{t} \rho(s)\, ds \Big| \Big) \qquad \text{for all } t_0,\ t \in I. \tag{8}$$

Proof. Let us fix $t_0 \in I$. Then (see Proposition 1)

$$\begin{aligned} |v(t)| &\leq |v(t_0)| + |v(t) - v(t_0)| = |v(t_0)| + \Big| \int_{t_0}^{t} v'(s)\, ds \Big| \\ &\leq |v(t_0)| + \Big| \int_{t_0}^{t} \rho(s) |v(s)|\, ds \Big| \qquad \text{for all } t \in I. \end{aligned}$$

We now apply Lemma 11 with $u(t) = |v(t)|$, $c = |v(t_0)|$. □

Definition 5. *We say that the equation* (1) *has in Ω the property of* local uniqueness*, if for any two solutions (x, I), (y, J) in Ω, satisfying $x(t_0) = y(t_0)$ for some $t_0 \in I \cap J$, there exists $\delta > 0$ such that $x = y$ on $I \cap J \cap U(t_0, \delta)$.*
The equation has the property of global uniqueness*, if $x(t_0) = y(t_0)$ for some $t_0 \in I \cap J$ implies $x = y$ everywhere in $I \cap J$.*

Obviously, global uniqueness implies local uniqueness; however, both notions are equivalent by the following argument: the set $R = \{t \in I \cap J;\ x(t) = y(t)\}$ is both closed[2] (by continuity of solutions) and open[2] (thanks to local uniqueness). Hence $R \neq \emptyset$ implies $R = I \cap J$ as the intersection of two intervals is a connected set.

Definition 6. *Function $f(t,x) : \Omega \to \mathbb{R}^n$ is locally Lipschitz continuous with respect to x in the generalized sense, provided that for any $(t_0, x_0) \in \Omega$ there exists a cylinder $Q(t_0, x_0; \delta, \Delta) \subset \Omega$ and a function $l(t) \in L^1(U(t_0, \delta))$ such that $|f(t,x) - f(t,y)| \le l(t)|x - y|$ for almost all t, for all x, y in $Q(t_0, x_0; \delta, \Delta)$.*

Theorem 13. *Let $f \in \mathrm{CAR}(\Omega)$ be locally Lipschitz continuous with respect to x in the generalized sense. Then the equation has the property of local (and hence global) uniqueness in Ω.*

Proof. Let (x, I), (y, J) be solutions in Ω, and let $x(t_0) = y(t_0) =: x_0$ for some $t_0 \in I \cap J$. Let δ, Δ and $l(t)$ be as in Definition 6. WLOG $\delta > 0$ is small so that after the possible restriction to $\hat{I} = I \cap J \cap U(t_0, \delta)$, the graphs of x and y stay in $Q(t_0, x_0; \delta, \Delta)$.
Set $v(t) = x(t) - y(t)$. Then $|v'(t)| \le l(t)|v(t)|$ for a.e. $t \in \hat{I}$. Recall that $v(t_0) = 0$. By Lemma 12, we thus obtain $v(t) = 0$ in $\hat{I}$. □

Definition 7. *Nondecreasing, continuous function $\omega : [0, \infty) \to [0, \infty)$ will be called generalized modulus of continuity of the function $f = f(t,x)$ with respect to x in Ω, provided that for any $(t_0, x_0) \in \Omega$ there exists a cylinder $Q(t_0, x_0; \delta, \Delta) \subset \Omega$ and a function $k(t) \in L^1(U(t_0, \delta))$ such that $|f(t,x) - f(t,y)| \le k(t)\omega(|x - y|)$ for almost all t for all x, y v $Q(t_0, x_0; \delta, \Delta)$.*

Theorem 14 (Osgood)**.** *Let the function $f = f(t,x)$ has a generalized modulus of continuity ω with respect to x such that*

$$\int_0^\eta \frac{du}{\omega(u)} = \infty \tag{9}$$

for any $\eta > 0$. Then the equation $x' = f(t,x)$ has the property of local uniqueness.

Proof. Let x, y be solutions on $[t_0, t_0 + \delta]$ such that $x(t_0) = y(t_0)$ and $x(t_0 + \delta) \neq y(t_0 + \delta)$. Set $u(t) = |x(t) - y(t)|$. This is an AC function and $u'(t) \le |x'(t) - y'(t)| \le k(t)\omega(u(t))$ a.e. For $\varepsilon > 0$ arbitrary we have

$$\int\limits_0^{u(t_0+\delta)} \frac{dy}{\omega(y) + \varepsilon} = \int\limits_{t_0}^{t_0+\delta} \frac{u'(t)\,dt}{\omega(u(t)) + \varepsilon} \le \int\limits_{t_0}^{t_0+\delta} k(t)\,dt \tag{10}$$

The first two integrals are equal, as they are increments of a C^1 and AC functions $G(y) = \int dy/(\omega(y) + \varepsilon)$ and $G(u(t))$ respectively, on corresponding intervals, cf. Corollary 5.
Consider now $\varepsilon \to 0+$. Since $u(t_0 + \delta) > 0$, the left-hand side goes to $+\infty$ thanks to (9) and Levi's theorem. But the right-hand side is a fixed finite number - a contradiction. □

Remark. *This is an obvious generalization of the classical uniqueness result, based on Lipschitz continuity of $f(t,x)$ w.r.t. to x (just set $l(t) = L$ and $\omega(u) = u$). Interestingly, the condition (9) is optimal, as the following shows.*

[2] Relative to $I \cap J$.

Proposition 15. *Let $\omega : [0, \eta] \to [0, \infty)$ be nondecreasing continuous function such that $\omega(0) = 0$ and*

$$\int_0^\eta \frac{du}{\omega(u)} < \infty \tag{11}$$

Then there exists a nontrivial solution to $x' = \omega(x)$, $x(0) = 0$.

Proof. Set $G(y) := \int_0^y du/\omega(u)$ for $y \in [0, \eta]$. Clearly $\omega(u) > 0$ for $u > 0$, hence $G(y)$ is strictly increasing. So is the function $x := G_{-1}$, defined on $[0, G(\eta)]$, and $x'(t) = 1/G'(x(t)) = \omega(x(t))$ for $t > 0$. □

5. Continuity of the solution map

Assume that $f \in \mathrm{CAR}(\Omega)$ and the equation (1) has the property of local (and hence global) uniqueness in Ω. We define the *solution map* φ via $\varphi(t, t_0, x_0) = x(t)$, where $x(\cdot)$ is the *maximal* solution to (1), subject to the initial condition $x(t_0) = t_0$.
Clearly φ is well-defind on a certain subset of $\mathbb{R} \times \mathbb{R} \times \Omega$ and $\varphi(t_0, t_0, x_0) = x_0$ for all $(t_0, x_0) \in \Omega$.
In various arguments, it is important to guarantee that the (maximal) solution is defined at least on a certain interval. This is the content of the following lemma.

Lemma 16. *Assume that $Q = Q(t_0, x_0; \delta, \Delta)$ and $m(t)$ are as in Definition 2. Moreover let*

$$\int_{t_0-\delta}^{t_0+\delta} m(t)\, dt < \Delta/3 \tag{12}$$

Let x be a solution, defined at least on $U(t_0, \delta)$ such that $x(t_0) = x_0$. Let (y, J) be a maximal *solution, satisfying $|y(t') - x(t')| < \Delta/3$ for some $t' \in (t_0 - \delta, t_0 + \delta)$. Then J contains the interval $[t_0 - \delta, t_0 + \delta]$, and $|y(t) - x_0| < \Delta$ for all $t \in [t_0 - \delta, t_0 + \delta]$.*

Proof. Let us first show that $|y(t) - x_0| < \Delta$ for all $t \in J \cap [t_0 - \delta, t_0 + \delta]$. For contradiction, let $t'' > t'$ be smallest time such that $|y(t'') - x_0| = \Delta$. Hence $|y(t) - x_0| < \Delta$ for all t between t', t'', and so

$$\begin{aligned} |y(t'') - x_0| &\le |y(t'') - y(t')| + |y(t') - x(t')| + |x(t') - x_0| \\ &= \Big|\int_{t'}^{t''} y'(t)\, dt\Big| + |y(t') - x(t')| + \Big|\int_{t_0}^{t'} x'(t)\, dt\Big| \\ &< 2\int_{t_0-\delta}^{t_0+\delta} m(t)\, dt + \Delta/3 < \Delta \end{aligned}$$

– a contradiction. However, by Theorem 10 y has to leave the compact $\overline{Q}$ at certain times both larger and smaller than t'. In view of the above, this is only possible if $J \supset [t_0 - \delta, t_0 + \delta]$, strictly.

□

Lemma 17. *Let $f \in \mathrm{CAR}(\Omega)$. Then the equation* (1) *has the property of local uniqueness in Ω, if and only if the solutions are locally continuously dependent on the initial condition.*

Proof. By local continuous dependence on the initial condition we mean the following: for any solution (x, I) and $t_0 \in I$ there exists $U(t_0, \delta) \subset I$ such that if x_n are solutions on $U(t_0, \delta)$ and $x_n(t') \to x(t')$ for at least one $t' \in U(t_0, \delta)$ fixed, then $x_n \rightrightarrows x$ on $U(t_0, \delta)$.
Ad $\Leftarrow$: let $y(t)$ be arbitrary solution with $x(t_0) = y(t_0)$. By Lemma 16, we can assume that $y(t)$ is defined at least on $U(t_0, \delta)$. Taking now $t' = x_0$ and $x_n = y$ for all n, the conclusion follows trivially.
Ad $\Rightarrow$: again by Lemma 16, we can assume that $\operatorname{graph} x_n \subset Q(t_0, x_0; \delta, \Delta)$. Repeating the argument of Theorem 6, the sequence is relatively compact in $C([t_0 - \delta, t_0 + \delta])$. If $x_n \not\rightrightarrows x$, we can find a subsequence such that $x_{\tilde{n}} \rightrightarrows \tilde{x} \neq x$. On the other hand, as $x_n(t') \to x(t')$ for some t', we have $\tilde{x}(t') = x(t')$. This contradicts the assumption of uniqueness. □

Theorem 18. *Let $f \in \mathrm{CAR}(\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is open. Let the equation* (1) *has the property of local uniqueness in Ω. Then the solution map is continuous and its domain of definition is an open subset of $\mathbb{R} \times \mathbb{R} \times \Omega$.*
Moreover: the map $(t_0, x_0) \mapsto I$ which assigns to the initial condition the interval of existence of the corresponding maximal solution is lower semicontinuous.

Proof. Let $(t_1, t_0, x_0) \in \mathcal{D}(\varphi)$, let $x(t) = \varphi(t, t_0, x_0)$ is the corresponding maximal solution, defined on the interval (a, b). Let $t_1 > t_0$ be arbitrary fixed such that $[t_0, t_1] \subset I$.
The set $\{(t, x(t));\ t \in [t_0, t_1]\}$ is compact, and so can be covered by a finite number of cylinders (see Definition 2) $Q_k = Q_k(\tau_k, x(\tau_k); \delta_k, \Delta_k)$, $0 \le k \le N$. WLOG $\tau_0 = t_0$ and $\tau_N = t_1$,

$$Q_{k-1} \cap Q_k \neq \emptyset\,, \tag{13}$$

and finally

$$\int_{I_k} m_k(\tau)\, d\tau < \Delta_k/3 \tag{14}$$

where we set $I_k = [\tau_k - \delta_k, \tau_k + \delta_k]$. Let for simplicity of notation write $y(t) = \varphi(t, t_0', x_0')$, albeit $y(t)$ still depends also on t_0' and x_0'. Then

$$|y(t_0') - x(t_0')| \le |x_0' - x_0| + |x(t_0) - x(t_0')| < \Delta_0/3$$

whenever (t_0', x_0') is close enough to (t_0, x_0). By Lemma 16, y is defined at least on I_0 and $\operatorname{graph}(y|_{I_0}) \subset Q_0$. Thanks to $|y'(t)| \le m_0(t)$ we estimate

$$|y(t_0) - x(t_0)| \le |y(t_0) - y(t_0')| + |x_0' - x(t_0)| \le |\int_{t_0}^{t_0'} m_0(s)\, ds| + |x_0' - x_0|\,.$$

If $(t_0', x_0') \to (t_0, x_0)$, the right-hand side tends to zero and by Lemma 17 it follows that even $y \rightrightarrows x$ in I_0. In particular $y(t'') \to x(t'')$ at some $t'' \in I_1$, cf. (13) above.
By repeating this argument for $k = 2, \dots, N$, we eventually get that $y \rightrightarrows x$ on $[t_0 - \delta_0, t_1 + \delta_N]$. From here it clearly follows that

$$\varphi(t_1', t_0', x_0') - \varphi(t_1, t_0, x_0) = y(t_1') - x(t_1') + x(t_1') - x(t_1) \to 0$$

for $(t_1', t_0', x_0') \to (t_1, t_0, x_0)$, i.e. the continuity of the map φ. Moreover, we see that $[t_0 - \delta_0, t_1 + \Delta_N] \times I_0 \times U(x(t_0), \Delta') \subset \mathcal{D}(\varphi)$ for suitable $\Delta' > 0$, i.e. the domain of definition $\mathcal{D}(\varphi)$ is open.
By lower-semicontinuity of intervals we mean: if the (maximal) solution is defined on some open I and $K \subset I$ is compact, then any close enough (maximal) solution is defined on some open $J \supset K$. And this also follows from the above argument. □

6. Differentiability of the solution map

In this section we will assume that $f(t,x)$ is locally Lipschitz (with respect to all its variables) on some open set $\Omega \subset \mathbb{R}^{n+1}$. By previous section, the solution map $\varphi = \varphi(t,t_0,x_0)$ is continuous on its domain of definition $\mathcal{D}(\varphi)$. We claim that it is actually locally Lipschitz with respect all the variables. Concerning x_0, this follows from the proof of Theorem 13. Concerning t and t_0, this is a consequence of (local) boundedness of f, which implies that solutions have (locally) bounded derivatives $x'(t)$.
For simplicity of notation, all expressions of the form $\varphi(t,t_0,x_0)$ are implicitly restricted to $(t,t_0,x_0) \in \mathcal{D}(\varphi)$, i.e. whenever they make sense.

Lemma 19. *Let $N \subset \Omega$ be set of measure zero, let t_0 be fixed. Then for a.e. x fixed, $(t, \varphi(t,t_0,x)) \notin N$ for a.e. t.*

Proof. If we show that

$$M = \{(t,x) \in \Omega;\ (t, \varphi(t,t_0,x) \in N\}$$

has measure zero, the conclusion is immediate by Fubini's theorem, since a.a. x-cuts are null sets of t.
However, $\varphi(t,t_0,x) = y \iff x = \varphi(t_0,t,y)$, and so

$$M = \{(t, \varphi(t_0,t,y));\ (t,y) \in N\}$$

By the Lipschitz continuity of φ, the set M is a Lipschitz image of a null set N; hence also a null set. ☐

Theorem 20. *Let $t_0 \in \mathbb{R}$ be fixed. Then for a.e. $x_0 \in \mathbb{R}^n$ such that $(t_0,x_0) \in \Omega$, the function $u(t) = \frac{\partial \varphi}{\partial w}(t,t_0,x_0)$ is defined for a.e. t and any $w \in \mathbb{R}^n$. Moreover, $u(t)$ is a AC solution to the first variation equation*

$$u' = \nabla_x f(t, \tilde{x}(t))u, \qquad u(t_0) = w \tag{15}$$

where $\tilde{x}(t) := \varphi(t,t_0,x_0)$.

Proof. Let $t_0 \in \mathbb{R}$ be fixed. Let $N \subset \Omega$ be the set of all points where either $\varphi(\cdot,t_0,\cdot)$ or f are not differentiable. By Rademacher's theorem, N is a null set. From Lemma 19, one deduces that for a.e. x_0 fixed, $u(t) := \frac{\partial \varphi}{\partial w}(t,t_0,x_0)$ is defined for a.e. t. Similarly, the matrix function $A(t) := \nabla_x f(t, \tilde{x}(t))$ is defined for a.e. t.
Set $y(t) := \varphi(t,t_0,x_0+hw)$, for $w \in \mathbb{R}^d$ fixed and $h \neq 0$ small, real number. We write

$$\frac{y(t)-x(t)}{h} = \frac{y(t_0)-x(t_0)}{h} + \int_{t_0}^{t} \frac{f(s,y(s))-f(s,x(s))}{h}\, ds \tag{16}$$

Letting $h \to 0$, the first term goes to $u(t)$. The second term is just w. In the last term, the integrand goes to $A(s)u(s)$, for a.e. s, and is bounded (by Lipschitz continuity of f and φ). We thus conclude

$$u(t) = w + \int_{t_0}^{t} A(s)u(s)\, ds \tag{17}$$

again for a.e. t. Thus $u(t)$ can be represented by an AC function, which in turn is a (unique) solution to (15), cf. Lemma 4. ☐

Corollary 21. *If $f = f(t,x)$ is C^1, then $\frac{\partial\varphi}{\partial w}(t,t_0,x_0)$ is defined everywhere, and it can computed as (now a classical) solution to* (15)*.*

Proof. Denote $\tilde{u}(t)$ the solution to (15). We know by previous theorem that $\tilde{u}(t) = \frac{\partial\varphi}{\partial w}(t,t_0,x_0)$ for almost all the arguments. However, $\tilde{x}(t)$ and hence $A(t)$ depends on these arguments continuously; hence by Theorem 8, also $\tilde{u}(t)$ is continuous w.r. to its arguments. The conclusion now follows by next lemma. □

Lemma 22. *Let $F(x)$ be locally Lipschitz on open set $Q \subset \mathbb{R}^m$. Let there exist a continuous $G(x)$ such that $\nabla F(x) = G(x)$ for a.e. x. Then $F(x)$ is C^1 and $\nabla F(x) = G(x)$ for all x.*

Proof. Without loss of generality Q is a bounded cube, and $G(x)$ is uniformly continuous on Q. For $m = 1$ one has (cf. Proposition 1) $F(x_1) - F(x_2) = \int_{x_1}^{x_2} G(x)dx$ for all x_1, $x_2 \in Q$. Hence $F'(x) = G(x)$ everywhere and $F'(x)$ is continuous.
For $m \geq 2$ general we will show that

$$\frac{\partial F}{\partial x_i}(x) = G_i(x) \tag{18}$$

everywhere, hence also the continuity of ∇F. WLOG let $i = 1$. Writing $x = (x_1, y) \in \mathbb{R}\times\mathbb{R}^{m-1}$, by Fubini's theorem (18) holds for almost every x_1, unless $y \in N$, where $N \subset \mathbb{R}^{m-1}$ is a null set. By the already proven case $m = 1$, (18) holds for all x_1 if $y \notin N$. For $y_0 \in N$, we can take $y_n \notin N$ such that $y_n \to y_0$. Now $\frac{\partial F}{\partial x_1}(\cdot, y_n) = G_1(\cdot, y_n)$ and the right-hand side goes uniformly to $G_1(\cdot, y_0)$, thanks to uniform continuity of G. Hence (18) holds even with $y = y_0$ and the proof is finished. □

7. Linear equation

By a linear ODE we mean

$$x' = A(t)x + b(t) \tag{19}$$

where $A(t) : I \to \mathbb{R}^{n\times n}$ a $b(t) : I \to \mathbb{R}^n$. We assume that $A(t)$ and $b(t)$ are locally integrable. Note that the right-hand side of (19) satisfies Carathéodory conditions: it is continuous w.r. to x whenever $\|A(t)\| + |b(t)| < \infty$. In fact, it is Lipschitz continuous w.r. to x in the generalized sense (cf. Definition 6) - we can take $\ell(t) = \|A(t)\|$.
This implies local existence and uniqueness of solutions. But an important property of *linear* equations is *global* existence of solutions.

Theorem 23. *Let $A(t) \in L^1_{\mathrm{loc}}(I)$, $b(t) \in L^1_{\mathrm{loc}}(I)$, where $I \subset \mathbb{R}$ is an open interval. Then for any initial condition $x(t_0) = x_0$, where $t_0 \in I$, there exists a unique solution to* (19)*, defined on I.*

Proof. Denote $\Omega = I \times \mathbb{R}^n$. By Theorems 9 and 13 there exists exactly one maximal solution, defined on some open interval $J \subset I$. Assume $[\alpha,\beta] \subset I$ is arbitrary compact interval such that $t_0 \in [\alpha,\beta]$. Let us further set $m(t) = \|A(t)\|$, $c = |x_0| + \int_\alpha^\beta |b(t)|\,dt$, $C = c\exp\big(\int_\alpha^\beta m(s)\,ds\big)$. One has

$$|x(t)| \leq c + \int_{t_0}^t m(t)|x(t)|\,dt \tag{20}$$

Hence, by Lemma 12, $|x(t)| \le c \exp \left| \int_{t_0}^{t} m(s)\, ds \right| \le C$ for all $t \in [\alpha, \beta]$. On the other hand, by Theorem 10 there exist t_1, $t_2 \in J$ such that $t_2 < t_0 < t_1$ and $(t_1, x(t_1))$ a $(t_2, x(t_2))$ do not lie in the compact set $K = [\alpha, \beta] \times \overline{U}(x_0, C)$.
Now, as $|x(t)| \le C$ for all $\alpha \le t \le \beta$, one necessarily has $t_1 > \beta$, $t_2 < \alpha$, hence $J \supset [\alpha, \beta]$. Since $[\alpha, \beta] \subset I$ was arbitrary, we obtain that $J = I$. □

Remark. *Analogously, we can prove* global existence *for the general nonlinear problem* (1), *provided that* $f \in \mathrm{CAR}(I \times \mathbb{R}^n)$ *and one has the estimate* $|f(t,x)| \le a(t)|x| + b(t)$ *for some* $a(t)$, $b(t) \in L^1_{\mathrm{loc}}(I)$.

Theorem 24. *Let* $b(t) \in L^1_{\mathrm{loc}}(I)$, $A \in \mathbb{R}^{n \times n}$ *is a constant matrix. Then the solution to*

$$x' = Ax + b(t) \tag{21}$$

can be written as

$$x(t) = e^{(t-t_0)A} x(t_0) + \int_{t_0}^{t} e^{(t-s)A} b(s)\, ds \tag{22}$$

for arbitrary t_0, $t \in I$.

Proof. The equivalence of (21) a (22) is done by a routine computation, but in the class of AC solutions, cf. Propositions 1 and 2. We also note that the integrand on the right-hand side of (22) is L^1_{loc}. □