A1/ $\displaystyle\lim_{x\to 0+}\left(1+\sqrt[3]{\operatorname{tg} x}\right)^{\frac{1}{\sqrt{1+\sqrt{x}}-1}}$

$f(x)=\exp h(x)$; $\quad h(x)=\dfrac{1}{\sqrt{1+\sqrt{x}}-1}\ln\left(1+\sqrt[3]{\operatorname{tg} x}\right).$

úprava odmocniny: $\sqrt{1+\sqrt{x}}-1=\dfrac{\sqrt{x}}{\sqrt{1+\sqrt{x}}+1}$

$$h(x)=\frac{\ln\left(1+\sqrt[3]{\operatorname{tg} x}\right)}{\sqrt[3]{\operatorname{tg} x}}\cdot\left(\sqrt{1+\sqrt{x}}+1\right)\cdot\frac{\sqrt[3]{\operatorname{tg} x}}{\sqrt{x}}=P_1\cdot P_2\cdot P_3$$

$\left.\begin{array}{l}\dfrac{\ln(1+y)}{y}\to 0,\ y\to 0 \\ \quad\text{(zákl. limita)} \\ \sqrt[3]{\operatorname{tg} x}\to 0;\ x\to 0+ \\ \quad \neq 0;\ x\in P_+(0,\delta) \\ \quad\text{(funkce } \sqrt[3]{y},\ \operatorname{tg} x \text{ rostoucí,} \\ \quad\quad \text{spojité v okolí } 0\text{)}\end{array}\right\}\ \overset{\text{VoLSF(b)}}{\Longrightarrow}\ P_1\to 1;\ x\to 0+.$

$\sqrt{1+\sqrt{x}}+1\to 2;\ x\to 0+$ dle VoLSF(a)

neboť $\sqrt{y}$ je spojitá v $y_0=1$,

spojitá zprava v $y_0=0$.

---

zbývá vyšetřit

$$P_3=\frac{\sqrt[3]{\operatorname{tg} x}}{\sqrt[3]{x}}\cdot\frac{\sqrt[3]{x}}{\sqrt{x}}=\sqrt[3]{\frac{\operatorname{tg} x}{x}}\cdot x^{-\frac{1}{6}};$$

$$\frac{\operatorname{tg} x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \to 1 \cdot 1 \ ; x \to 0+$$

(základní limita & spojitost cos v $y_0 = 0$)

$$x^{-\frac{1}{6}} = \exp\Big(-\frac{1}{6}\underbrace{\ln x}_{\to -\infty}\Big) \to +\infty \ ; x \to 0+$$

---

celkem tedy: $P_3 \longrightarrow 1 \cdot (+\infty) = +\infty$

$h(t) \to +\infty$

$f(t) \to +\infty$

A2/ $\lim\limits_{x \to +\infty} \dfrac{x^2 + \sin x^2 + \ln\left(e^x\sqrt{x+1}\right)}{1 - \ln\left(e^{2x}+1\right)}$

úprava logaritmů:

$$\ln\left(e^x\sqrt{x+1}\right) = \ln e^x + \ln\sqrt{x+1} = x + \tfrac{1}{2}\ln(x+1)$$

$$\ln\left(e^{2x}+1\right) = \ln e^{2x}\left(1+e^{-2x}\right) = 2x + \ln\left(1+e^{-2x}\right)$$

$$f(x) = \frac{x^2 + \sin x^2 + x + \frac{1}{2}\ln(x+1)}{1 - 2x + \ln\left(1+e^{-2x}\right)}$$

$$= \frac{x^2\left(1 + \frac{1}{x^2}\sin x^2 + \frac{1}{x} + \frac{1}{2x^2}\ln(x+1)\right)}{x\left(\frac{1}{x} - 2 + \frac{1}{x}\ln\left(1+e^{-2x}\right)\right)} = x \cdot R(x).$$

ukážeme, že $R(x) \to -\frac{1}{2}$; tedy $f(x) \to -\infty$.

---

dle VoAL stačí ukázat, že:

$\frac{1}{x^2}\sin x^2 \to 0$: dle Věty 2.8; $\frac{1}{x^2} \to 0$

$\sin x^2$ omezená fce

$$\frac{1}{2x^2}\ln(x+1) = \frac{x+1}{2x^2} \cdot \frac{\ln(x+1)}{x+1} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2}\right)\frac{\ln(x+1)}{x+1}$$

$$\to 0 \cdot 0;$$

dle VoAL a základní limity

$$\frac{\ln y}{y} \to 0;\ y \to +\infty.$$

B1/ $\lim\limits_{x\to-\infty} \left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right)^{\frac{1}{e^{1/x}-1}}$

$$f(t)=e^{h(t)} \; ; \; h(t)=\frac{1}{e^{1/x}-1}\cdot \ln\left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right).$$

úprava odmocniny: $\sqrt{x^2+1}-\sqrt{x^2+2}=\dfrac{-1}{\sqrt{x^2+1}+\sqrt{x^2+2}} \xrightarrow[x\to-\infty]{} 0,$

tedy píšeme:

$$h(t)=\frac{\ln\left(1+\sqrt{x^2+1}-\sqrt{x^2+2}\right)}{\sqrt{x^2+1}-\sqrt{x^2+2}}\cdot\frac{1/x}{e^{1/x}-1}\cdot\frac{-x}{\sqrt{x^2+1}+\sqrt{x^2+2}}$$

$$= P_1\cdot P_2\cdot P_3$$

$\left.\begin{array}{l} \dfrac{\ln(1+y)}{y}\to 1 \; ; \; y\to 0 \\ \sqrt{x^2+1}-\sqrt{x^2+2}\to 0 ; x\to-\infty \\ \quad\text{a jiné } \neq 0 \text{ pro } x\in P(-\infty) \end{array}\right\} \xrightarrow{\text{VoLSF(b)}} P_1\to 1$

$\left.\begin{array}{l} \dfrac{y}{e^y-1}=\dfrac{1}{\frac{e^y-1}{y}}\to 1 \; ; \; y\to 0 \\ \quad\text{(základní limita)} \\ \dfrac{1}{x}\to 0 \; ; \; x\to-\infty \\ \quad \neq 0 \; ; \; x\in P(-\infty) \end{array}\right\} \xrightarrow{\text{VoLSF(b)}} P_2\to 1$

---

zbývá spočítat P_3.

uvažme, že $\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = |x|\cdot\sqrt{1+\frac{1}{x^2}} = -x\sqrt{1+\frac{1}{x^2}}$,

$x \in P(-\infty,\delta)$

$$P_3 = \frac{-x}{-x\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}\right)} = \frac{1}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1+\frac{2}{x^2}}} \to \frac{1}{2};$$

neboť $\frac{1}{x^2} \to 0$ dle VoAL

$\sqrt{y}$ spojitá v $y_0 = 1$.

---

celkem tedy $h(x) = P_1\cdot P_2\cdot P_3 \to 1\cdot 1\cdot \frac{1}{2} = \frac{1}{2}$

$$f(x) \to e^{1/2} = \sqrt{e}\ ;\ x \to -\infty.$$

B21

$$\lim_{x\to+\infty} \frac{\ln(x+1) + \sqrt[3]{x} + \ln\left(1+e^{x/6}\right)}{2+\cos\sqrt{x}+\sqrt{x}};$$

úprava logaritmů:

$$\ln(x+1) = \ln x\left(1+\tfrac{1}{x}\right) = \ln x + \ln\left(1+\tfrac{1}{x}\right)$$

$$\ln\left(1+e^{x/6}\right) = \ln e^{x/6}\left(1+e^{-x/6}\right) = \tfrac{1}{6}x + \ln\left(1+e^{-x/6}\right)$$

---

vytknutí vedoucího členu: nahoře x, dole $\sqrt{x}$:

$$f(x) = \sqrt{x}\cdot R(x) \to (+\infty)\cdot\frac{1}{6} = +\infty;$$

kde

$$R(x) = \frac{\frac{1}{x}\ln x + \frac{1}{x}\ln\left(1+\frac{1}{x}\right) + \frac{\sqrt[3]{x}}{x} + \frac{1}{6} + \frac{1}{x}\ln\left(1+e^{-x/6}\right)}{\frac{2}{\sqrt{x}} + \frac{\cos\sqrt{x}}{\sqrt{x}} + 1};$$

stačí ukázat, že vše kromě $1/6$ a 1 jde do 0.

$\frac{1}{x}\ln x \to 0$; $x\to+\infty$ (základní limita)

$\frac{1}{x}\ln\left(1+\frac{1}{x}\right) \to 0\cdot\ln 1 = 0$; dle VoAL a spojitosti $\ln y$ v bodě $y_0=1$

$\frac{1}{x}\ln\left(1+e^{-x/6}\right) \to 0$ analogicky

$\frac{2}{\sqrt{x}} \to \frac{2}{+\infty} = 0$ dle VoAL

$\frac{\cos\sqrt{x}}{\sqrt{x}} \to 0$ dle V. 2.4; neboť $\frac{1}{\sqrt{x}} \to 0$

$\cos\sqrt{x}$ je omezený.