A1. [6b] Vypočítejte limitu

$$\lim_{x\to+\infty}\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)^{\frac{1}{1-\cos\frac{1}{x}}}$$

A2. [5b] Vypočítejte limitu

$$\lim_{x\to+\infty}\frac{\ln(x^2+1)+x+\sin x}{1-\ln(x^x)}$$

B1. [6b] Vypočítejte limitu

$$\lim_{x\to0+}\left(1+\sin\sqrt{x}\right)^{\frac{1}{1-\sqrt[3]{1+x}}}$$

B2. [5b] Vypočítejte limitu

$$\lim_{x\to+\infty}\frac{\ln(e^{(x^2)}+1)-x}{\ln(e^x-1)-x^3+\cos x}$$

(A1) $\lim\limits_{x\to+\infty} \left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)^{\frac{1}{1-\cos\frac{1}{x}}}$.

$f(t) = \exp h(t)$; $h(t) = \frac{1}{1-\cos\frac{1}{x}} \cdot \ln\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)$

úprava odmocniny: $\sqrt[3]{a}-\sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}$

$\Rightarrow \sqrt[3]{x^2}-\sqrt[3]{x^2+1} = \frac{-1}{\sqrt[3]{x^4}+\sqrt[3]{x^2(x^2+1)}+\sqrt[3]{(x^2+1)^2}} \to 0$

neboť jmen. čitatel $\to +\infty$:

např. 1. člen: $\left.\begin{array}{l} x^4 \to +\infty \,;\; x\to+\infty \\ \sqrt[3]{y}\to+\infty ,\; y\to+\infty \end{array}\right\}$ VoLSF (b).

$h(t) = \frac{\frac{1}{x^2}}{1-\cos\frac{1}{x}} \cdot \frac{\ln\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)}{\sqrt[3]{x^2}-\sqrt[3]{x^2+1}} \cdot \frac{-x^2}{\sqrt[3]{x^4}+\sqrt[3]{x^2(x^2+1)}+\sqrt[3]{(x^2+1)^2}}$

$= P_1(t)\cdot P_2(t)\cdot P_3(t)$

$P_1(t)\to 2$; neboť $\frac{1-\cos y}{y^2}\to\frac{1}{2}$, $y\to 0$

$\left.\begin{array}{l} \frac{1}{x}\to 0 \,;\; x\to+\infty \\ \neq 0 \text{ na } P(+\infty) \end{array}\right\}$. VoLSF (a)

$P_2(x) \to 1$, mert $\frac{\ln(1+y)}{y} \to 1$, $y \to 0$

„$V_0 LSF(b)$" $\sqrt[3]{x^2} - \sqrt[3]{x^2+1} \to 0$; $x \to +\infty$

$\neq 0$ mo $P(+\infty)$

$$P_3(x) = \underbrace{\frac{-x^2}{\sqrt[3]{x^4}}} \cdot \underbrace{\frac{1}{1+\sqrt[3]{1+\frac{1}{x^2}}+\sqrt[3]{\left(1+\frac{1}{x^2}\right)^2}}} \to -\infty \cdot \frac{1}{3} = -\infty.$$

$-x^{2/3}$ $\to \frac{1}{3}$; $V_0 AL$;

$\downarrow$

$-\infty$ $\quad 1+\frac{1}{x^2} \to 1$; $x \to +\infty$

$\sqrt[3]{y}$ mojiv $y_0 = 1$

$V_0 LSF(a)$.

---

cellem $\quad h(x) = P_1(x) \cdot P_2(x) \cdot P_3(x) \to 2 \cdot 1 \cdot (-\infty) = -\infty$

tehy $\boxed{f(x) \to 0, \; x \to +\infty}$

A2 $\displaystyle \lim_{x\to+\infty} \frac{\ln(x^2+1)+x+\sin x}{1-\ln(x^x)}$

$\ln(x^2+1) = \ln x^2\left(1+\frac{1}{x^2}\right) = 2\ln x + \ln\left(1+\frac{1}{x^2}\right)$

$\ln(x^x) = x\cdot\ln x$

---

vytkneme „vedoucí člen" = nahoře x

dole $x\cdot\ln x$

$$f(x) = \frac{1+2\frac{\ln x}{x}+\frac{1}{x}\ln\left(1+\frac{1}{x^2}\right)+\frac{\sin x}{x}}{-1+\frac{1}{x\cdot\ln x}}\cdot\frac{\cancel{x}}{\cancel{x}\cdot\ln x}$$

$$\to \frac{1+0+0\cdot0+0}{-1+\frac{1}{+\infty}}\cdot\frac{1}{+\infty} = 0.$$

---

podrobně: $\frac{\ln x}{x}\to 0$ (základní limita)

$\frac{1}{x}\cdot\ln\left(1+\frac{1}{x^2}\right)\to 0\cdot 0$; VoAL,

spojitost $\ln y$ v $y_0=1$

$\sin x\cdot\frac{1}{x}\to 0$: $|\sin x|\leq 1$

$\frac{1}{x}\to 0$

$\frac{1}{x\cdot\ln x}\to\frac{1}{\infty\cdot\infty}=0.$

(B1) $\lim\limits_{x\to 0+} \left(1+\sin\sqrt{x}\right)^{\frac{1}{1-\sqrt[3]{1+x}}}$,

$$f(x) = \exp h(x); \quad h(x) = \frac{1}{1-\sqrt[3]{1+x}} \cdot \ln\left(1+\sin\sqrt{x}\right)$$

úprava odmocniny: $$\frac{1}{\sqrt[3]{a}-\sqrt[3]{b}} = \frac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{a-b}$$

$$\frac{1}{1-\sqrt[3]{1+x}} = \frac{1+\sqrt[3]{1+x}+\sqrt[3]{(1+x)^2}}{-x}$$

$$h(x) = \frac{\ln(1+\sin\sqrt{x})}{\sin\sqrt{x}} \cdot \frac{\sin\sqrt{x}}{\sqrt{x}} \cdot \frac{1+\sqrt[3]{1+x}+\sqrt[3]{(1+x)^2}}{-\sqrt{x}},$$

$$= R_1(x)\cdot R_2(x)\cdot R_3(x)$$

$$\left.\begin{array}{l} \frac{\sin y}{y} \to 1,\ y\to 0 \\ \sqrt{x}\to 0,\ x\to 0+ \\ \quad >0\ \forall x\in P_+(0,\delta) \end{array}\right\} \text{VoLSF(G)} \Rightarrow R_2(x)\to 1$$

$$\left.\begin{array}{l} \frac{\ln(1+y)}{y} \to 1,\ y\to 0 \\ \sin\sqrt{x}\to 0;\ x\to 0+ \\ \quad \neq 0\ \forall x\in P_+(0,\delta) \end{array}\right\} \text{VoLSF(G)} \Rightarrow R_1(x)\to 1$$

$$R_3(x) = \left(1+\sqrt[3]{1+x}+\sqrt[3]{(1+x)^2}\right)\cdot\frac{-1}{\sqrt{x}}$$

$$\left.\begin{array}{l}1.\ \text{část} \to 3;\ \text{VoAL},\quad 1+x\to 1;\ x\to 0+\\ \qquad\qquad \sqrt[3]{y}\ \text{spojitá v}\ y_0=1\end{array}\right\}\ \text{VoLSF(a)}$$

$$\left.\begin{array}{l}2.\ \text{část}:\ \sqrt{x}\to 0;\ x\to 0+\\ \qquad\qquad >0\ \text{na}\ P_+(0,\delta)\end{array}\right\}\Longrightarrow \frac{1}{0+}\ \text{typ}\qquad -\frac{1}{\sqrt{x}}\to-\infty$$

---

celkem: $h(x) = R_1(x)\cdot R_2(x)\cdot R_3(x) \longrightarrow 1\cdot 1\cdot(-\infty) = -\infty$

tedy $\boxed{f(x)\to 0,\ x\to 0+}$

(B2) $$\lim_{x\to+\infty} \frac{\ln\left(e^{x^2}+1\right)-x}{\ln\left(e^{x}-1\right)-x^3+\cos x}$$

$$\ln\left(e^{x^2}+1\right)=\ln e^{x^2}\left(1+e^{-x^2}\right)=x^2+\ln\left(1+e^{-x^2}\right);$$

$$\ln\left(e^{x}-1\right)=x+\ln\left(1-e^{-x}\right);$$

---

vytkneme vedoucí člen: čitatel x^2

jmenovatel x^3

$$f(x)=\frac{x^2}{x^3}\cdot\frac{1+\frac{1}{x^2}\ln\left(1+e^{-x^2}\right)-\frac{1}{x}}{-1+\frac{1}{x^2}+\frac{1}{x^3}\ln\left(1-e^{-x}\right)+\frac{\cos x}{x^3}}\to 0\cdot\frac{1}{-1}=0$$

---

podrobně: $\frac{x^2}{x^3}=\frac{1}{x}\to\frac{1}{+\infty}=0$ V. o AL

$\frac{1}{x^2}\ln\left(1+e^{-x^2}\right)\to 0\cdot\ln 1=0$

V. o AL; $e^{-x^2}\to 0$; neboť $-x^2\to-\infty$

spojitost $\ln y$ v $y_0=1$

podobně $\frac{1}{x^3}\ln\left(1-e^{-x}\right)$

konečně: $\left.\begin{array}{l}\cos x \text{ omezené}\\ \frac{1}{x^3}\to 0\end{array}\right\}\Rightarrow \frac{\cos x}{x^3}\to 0$