1) $\left.\begin{aligned} x' &= -12x - 16y \\ y' &= 9x + 12y \end{aligned}\right\} \iff X' = AX$

where $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} -12 & -16 \\ 9 & 12 \end{pmatrix}$

Note: $A^2 = 0$, hence $e^{tA} = I + tA$

$$= \begin{pmatrix} 1-12t & -16t \\ 9t & 1+12t \end{pmatrix}$$

2) $\left.\begin{aligned} x' &= -y \\ y' &= x \end{aligned}\right\} \Rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Note: $A^2 = -I,\ A^3 = -A,\ A^4 = I, \ldots$

$$\Rightarrow \boxed{\begin{aligned} A^{2k+1} &= (-1)^k A \\ A^{2k} &= (-1)^k I \end{aligned}}$$

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \underbrace{\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1}}_{(\sin t) A} + \underbrace{\sum_{k=0}^{\infty} \frac{t^{2k}}{2k!} A^{2k}}_{(\cos t) I}$$

3) $A = \begin{pmatrix} 0, 1, 2 \\ 0, 0, 3 \\ 0, 0, 0 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0, 0, 3 \\ 0, 0, 0 \\ 0, 0, 0 \end{pmatrix}$

$$A^3 = 0$$

$$\Rightarrow e^{tA} = I + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1, t, 2t + \frac{3}{2}t^2 \\ 0, 1, 3t \\ 0, 0, 1 \end{pmatrix}$$

*4) $\lambda \in \sigma(A) \Leftrightarrow \exists v \neq 0$ s.t. $Av = \lambda v$

hence also: $A^2 v = \lambda^2 v$

$A^3 v = \lambda^3 v$

but: $A^3 + A^2 = 0$

$(\lambda^3 + \lambda^2) v = 0$

$\Rightarrow \lambda^3 + \lambda^2 = 0$ (since $v \neq 0$)

$\Rightarrow \lambda^2(\lambda + 1) = 0$, i.e.

$$\boxed{\lambda \in \sigma(A) \Leftrightarrow \lambda = 0 \text{ or } \lambda = -1}$$

Note: $A^3 = -A^2, \ A^4 = +A^2, \ A^5 = -A^2$

$$\Rightarrow \quad e^{tA} = I + tA + \underbrace{\sum_{r=2}^{\infty} \frac{t^r}{r!}(-1)^r A^2}_{=\, e^{-t} - (1-t)}$$