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Theorem P.1 [Solution of ODE with separated variables.] Consider the equation

$$x' = h(x)g(t)$$

Let I, J be open intervals such that $g(t)$ is continuous on I, $h(x)$ is continuous and non-zero on J. Let

$$H(x) = \int \frac{dx}{h(x)}, \quad x \in J, \qquad G(t) = \int g(t)\,dt, \quad t \in I$$

Let constant c and an open interval $I_c \subset I$ be chosen such that $G(t) + c \in H(J)$ for all $t \in I_c$. Then

$$x(t) = H^{-1}(c + G(t)), \qquad t \in I_c$$

solves the equation. Moreover, all solutions $x(t) : I \to J$ are of this form.

Theorem P.2 [Solution of linear 1st order ODE.] Consider the equation

$$x' + a(t)x = b(t)\,.$$

Let I be an open interval such that $a(t)$, $b(t)$ are continuous on I. Let $A(t) = \int a(t)$ on I. Then

$$x(t) = \exp(-A(t)) \left[c + \int b(t)\exp(A(t))\,dt\right], \qquad t \in I$$

solves the equation. Moreover, there are no other solutions on I.

Theorem P.3 [Peano existence theorem.] Consider the equation $X' = F(t, X)$. Let the right hand side F be continuous. Then through every point passes at least one solution.

Theorem P.4 [Picard existence and uniqueness theorem.] Consider the equation $X' = F(t, X)$. Let the right-hand side F be C^1. Then through every point passes exactly one solution.

Theorem P.5 [Linearized stability theorem.]
Consider system of equations $X' = F(X)$. Let X_0 be stationary point (i.e. $F(X_0) = 0$), let F be C^1 close to X_0. Denote $A = \nabla F(X_0)$ the so called linearization matrix, and $\sigma(A)$ its spectrum. Then it holds:

1. If $\forall \lambda \in \sigma(A)$ one has $\operatorname{Re}\lambda < 0$, then X_0 is asymptotically stable.
2. If $\exists \lambda \in \sigma(A)$ such that $\operatorname{Re}\lambda > 0$, then X_0 is unstable.

Theorem P.6 [Stable / unstable direction theorem.]
Let X_0, A and $\sigma(A)$ be as in the previous theorem. Let $\lambda \in \sigma(A)$ be a simple, real and non-zero eigenvalue. Let v be the corresponding eigenvector. Then it holds:

1. If $\lambda < 0$, there exists a pair of solutions close to X_0, behaving like $X_0 \pm e^{\lambda t}v$, for $t \to +\infty$ ("stable directions").
2. If $\lambda > 0$, there exists a pair of solutions close to X_0, behaving like $X_0 \pm e^{\lambda t}v$, for $t \to -\infty$ ("unstable directions").

Theorem P.7 [Characterization of the first integral.] Consider autonomous system $X' = F(X)$, for $X \in \Omega$. Let $V = V(X) : \Omega \to \mathbb{R}$ be a C^1 function. Then V is a first integral (in Ω) iff and only if there holds:

1. $\nabla V(X) \not\equiv 0$, i.e. $\frac{\partial V}{\partial x_i}(X)$ do not all vanish in Ω

2. $\nabla V(X) \cdot F(X) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(X) F_i(X) = 0$ everywhere in Ω

Theorem P.8 [Linear homogeneous ODE with constant coefficients.] Consider the system

$$X' = AX \tag{L-1}$$

Furthermore, consider n-th order equation

$$a_0 x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1} x' + a_n x = 0 \tag{L-2}$$

For both problems, solutions form n-dimensional space. For (L-1), solution is uniquely determined by initial condition of the form $X(t_0) = C \in \mathbb{R}^n$. For (L-2), the initial condition has the form

$$x(t_0) = c_1,\ x'(t_0) = c_2,\ \ldots,\ x^{(n-1)}(t_0) = c_n$$