**Theorem 1,2** : local $\exists$ uniqueness for DDEs
(similar to ODE)

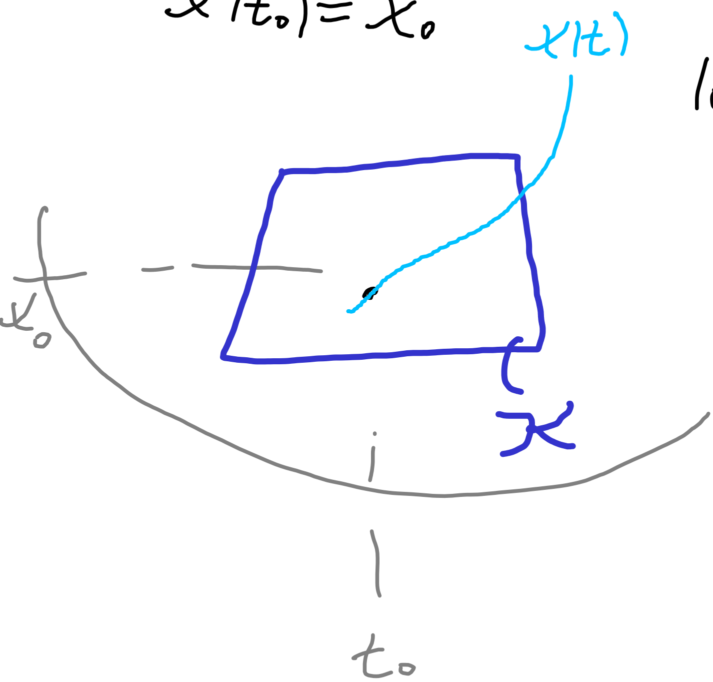
**Recall:** ordinary ODE :

$$x'(t) = f(t, x(t))$$
$$x(t_0) = x_0$$

maximal solution
(non-continuable)

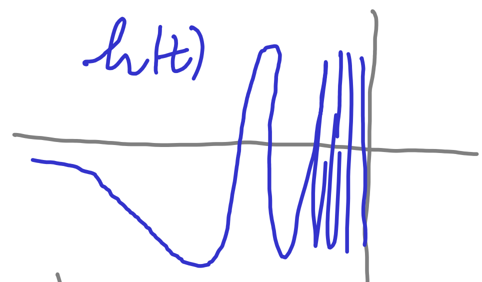
$\Downarrow$

leaves any closed, bdd. set
($\Leftrightarrow$ compact)




$\Omega \subset \mathbb{R}^{n+1}$

**Example** : $h(t) = \sin(1/t)$
(Yorke)
$(t < 0)$




$\nexists \lim_{t \to 0-} h(t)$

**Set:** $F : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$

where : $F(t, \varphi) = \begin{cases} h'(t)\left(1 + \dfrac{\|h_t - \varphi\|}{t}\right) & \text{if } \|h_t - \varphi\| < -t \\ 0 & \text{elsewhere} \end{cases}$

$\mathcal{C} = C([-r, 0], \mathbb{R})$

claim: $h(t)$ solves $x'(t) = F(t, x_t)$
$t < 0$ ($\Rightarrow$ cannot be continued to $t=0$)

but: $(t, x_t) \in \underbrace{[-1,0] \times \mathcal{C}_{[-2,2]}}$ ← closed, bdd (not compact)

F cont., but not bdd

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Notation. $\mathcal{C} = C([-r,0]; \mathbb{R}^n)$

$A \subset \mathbb{R}^n$ : $\mathcal{C}_A = \{ \varphi \in \mathcal{C} \, ; \, \varphi(t) \in A, \, \forall t \in [-r,0] \}$

Def. Let $\mathcal{D} \subset \mathbb{R} \times \mathcal{C}$ be open. Then $F : \mathcal{D} \to \mathbb{R}^n$ is called quasi-bounded, if it is bounded on ~~the~~ any subset

$$[t_0, t_1] \times \mathcal{C}_B \subset \mathcal{D}$$

where $B \subset \mathbb{R}^n$ is closed and bounded.

Observe: $f(t,x,y) : (a,b) \times \Omega \times \Omega \to \mathbb{R}^n$ continuous, $(a,b) \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open

$\Rightarrow$ $F(t,\psi) := f(t, \psi(0), \underline{\psi(-r)})$

or $\int_0^r k(s)\psi(-s)\,ds$

is quasi-bounded

on $\mathcal{D} = (a,b) \times \mathcal{C}_\Omega$

$\Updownarrow$

$x'(t) = f(t, x(t), \underline{x(t-r)})$

or $\int_0^r k(s)\,x(t-s)\,ds$

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**Theorem 3.** Assume: $\Omega \subset \mathbb{R}^n$ open

$F : [t_0, \beta) \times \mathcal{C}_\Omega \to \mathbb{R}^n$ cont.

quasi-bounded

$\phi \in \mathcal{C}_\Omega$ given

Then: $\exists\ x(t) : [t_0, t_1) \to \mathbb{R}^n$ a maximal solution to $(t_1 \le \beta)$

$x'(t) = F(t, x_t)$, $x_{t_0} = \phi$

Such that: if $t_1 < \beta$, then for arbitrary

$B \subset \Omega$ closed, bounded

$\exists\ t \in (t_0, t_1)$ s.t. $x(t) \notin B$.

Pf. $\exists$ maximal solution $\Leftarrow$ Zorn's lemma

(non-continuable beyond some $t_1 \le \beta$)

2nd part: ?? $x(t) \in B, \ \forall t \in [t_0, t_1)$

$B \subset \Omega$ given closed, bdd

WLOG: $x(t) \in B \ \forall t \in (t_0 - r, t_1)$

$\Leftrightarrow x_t \in C_B, \ t \in [t_0, t_1)$

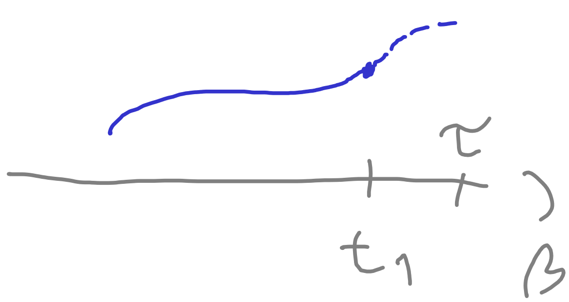
but (F quasi-bdd!!) $\Rightarrow |F(t, \psi)| \le M$

$\forall (t, \psi) \in [t_0, t_1] \times C_B$

$\Rightarrow |x'(t)| \le M, \ t \in [t_0, t_1)$

note: $t_1 < +\infty$

$\Rightarrow \exists \lim_{t \to t_1^-} x(t) =: x_1$




Apply Thm 1

$\Rightarrow$ extension of the sol. to $[t_1, \tau)$, $\tau > t_1$ $\lightning$

**Corollary.** $F : [t_0, \beta) \times C \to \mathbb{R}^n$ cont.

(*) $|F(t, \psi)| \le a(t) + b(t) \|\psi\| \quad \forall t \in [t_0, \beta), \ \psi \in C$

with $a(t), b(t)$ cont. on $[t_0, \beta)$

$\Rightarrow \exists$ solution on $[t_0, \beta)$
(global)

**Pf.** (*) $\Rightarrow$ $F$ quasi-bounded

Thm 3 $\Rightarrow \exists \ x(t) : [t_0, t_1) \to \mathbb{R}^n$ maximal

$t_1 \le \beta$

?? $t_1 < \beta$ : Thm 3 $\Rightarrow$ $x(t)$ leaves any $B \subset \mathbb{R}^n$ closed, bdd

$x'(t) = F(t, x_t)$

$t_1 < +\infty$

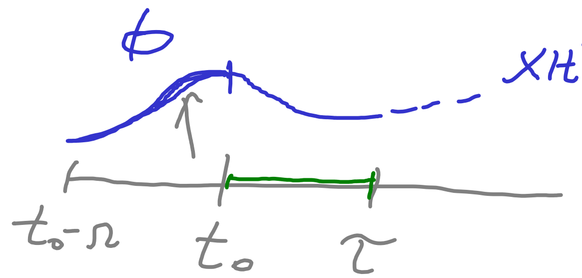
$$|x(t)| \le |x(t_0)| + \int_{t_0}^{t} |F(s, x_s)| \, ds \qquad t \in [t_0, t_1) \subset [t_0, t_1]$$

$$\overset{(*)}{\le} \underbrace{a(s)}_{\le C_1} + \underbrace{b(s)}_{} \|x_s\|$$

$$|x(t)| \le |x(t_0)| + \int_{t_0}^{t} C_1 + C_1 \|x_s\| \, ds \qquad t \in [t_0, t_1)$$

$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{t} \underbrace{\|x_s\|}_{\le \sigma(s) + \|\phi\|} \, ds \qquad = \sup \text{ on } [\cancel{t}\,s - r, \cancel{t}\,s]$$

Set: $\sigma(\tau) := \sup_{t \in [t_0, \tau]} |x(t)| \qquad \tau \in [t_0, t_1)$




$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{t} \sigma(s) \, ds$$

$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{\tau} \sigma(s) \, ds \qquad \forall t \in [t_0, \tau]$$

sup

$$\Rightarrow \sigma(\tau) \le \tilde{C}_0 + C_1 \int_{t_0}^{\tau} \sigma(s) \, ds$$

Gronwall: $\sigma(\tau) \le \tilde{C}_0 \cdot \exp(C_1(\tau - t_0)) =: C_3$

$\forall \tau \in [t_0, t_1)$

$\Rightarrow x(t) \in B, \quad \forall t \in [t_0, t_1)$

where $B = \{x \in \mathbb{R}^n ; |x| \le C_3\}$

↯ ↯