3 - Linear equation

Definition. By linear DDE we mean

$$(3.1) \qquad x'(t) = L(t, x_t) + h(t)$$

where: $L = L(t, \varphi) : [t_0, \beta) \times \mathcal{C} \to \mathbb{R}^n$ cont.

bdd, linear w.r.t. φ (for any t fixed)

$h(t) : [t_0, \beta) \to \mathbb{R}^n$ cont.

Moreover: $|L(t, \varphi)| \leq m(t) \|\varphi\| \quad \forall t \in (t_0, \beta)$

where $m(t)$ is continuous

Special cases: linear homogeneous DDE

$$(3.2) \qquad x'(t) = L(t, x_t)$$

and: linear, homogeneous & autonomous DDE (with const. coeff.s)

$$(3.3) \qquad x'(t) = L(x_t)$$

where $L : \mathcal{C} \to \mathbb{R}^n$

bounded, linear

Theorem 4. Given $\phi \in \mathcal{C} \Rightarrow \exists!\, x(t)$ global solution to (3.1) on (t_0, β) such that $x_{t_0} = \phi$

Proof. Set $F(t,\psi) = L(t,\psi) + h(t)$

$$\Rightarrow |F(t,\psi)| \leq m(t)\|\psi\| + |h(t)|$$

Corollary (after Thm 3) $\Rightarrow$ global existence

uniqueness? $|F(t,\psi) - F(t,\tilde{\psi})| =$

$$= |F(t,\psi-\tilde{\psi})| \leq m(t)\,\|\psi-\tilde{\psi}\|$$

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Theorem 2 $\Rightarrow$ $F(t,\psi)$ locally Lipsch. w.r.to ψ

Recall: $x'(t) = A(t)x(t) + h(t)$

fundamental matrix: $U'(t) = A(t)U(t)$

$U(t)$ regular $(\forall t)$

variation of constants:

$$x(t) = U(t)U^{-1}(t_0)x_0 + \int_{t_0}^{t} U(t)U^{-1}(s)h(s)\,ds$$

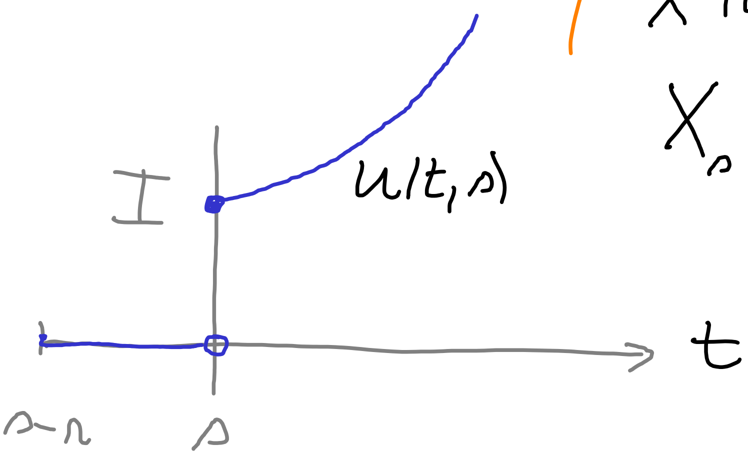
$(x(t_0) = x_0)$

Definition. Fundamental matrix $U=U(t,s)\in\mathbb{R}^{n\times n}$ (for (3.2))

is defined as
$$U(t,s)=\begin{cases} I+\int_s^t L(\theta, U_\theta(\cdot,s))\,d\theta & (t\geq s)\\ 0 & (s-r\leq t<s)\end{cases}$$

Ekvivalently: for s fixed, $t\mapsto U(t,s)$ solves

$$\begin{cases} X'(t)=L(t,X_t)\\ X_s=\begin{cases} 0 & s-r\leq t<s\\ I & t=s\end{cases}\end{cases}$$



well defined?

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Theorem 4
(some modification)

Theorem 5. The function
$$x(t)=\int_{t_0}^{t} U(t,s)h(s)\,ds$$
solves (3.1) with $x_{t_0}=0$.

Proof. (sketch)

$$r'(t) = \frac{d}{dt}\int_{t_0}^{t} u(t,s)\,h(s)\,ds$$

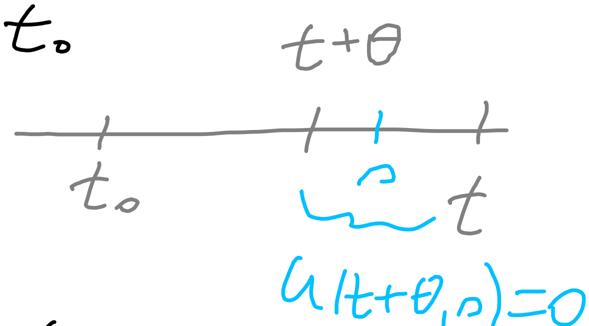
$$= \underbrace{u(t,t)}_{I}\,h(t) + \int_{t_0}^{t} \underbrace{\partial_t u(t,s)}_{\overset{\shortparallel}{L(t,u_t(\cdot,s))}}\,h(s)\,ds \qquad (**)$$

but: $L(t, r_t) = \;??$

observe: $r_t(\theta) = r(t+\theta) = \int_{t_0}^{\cancel{t+\theta}} u(t+\theta,s)\,h(s)\,ds$ *)

$\theta \in [-r,0]$

*)



$$\Rightarrow\quad r_t = \int_{t_0}^{t} u_t(\cdot,s)\,h(s)\,ds$$

$$L(t,r_t) = L\left(t, \int_{t_0}^{t} u_t(\cdot,s)\,h(s)\,ds\right)$$

$$= \int_{t_0}^{t} L\left(t, u_t(\cdot,s)\,h(s)\,\cancel{ds}\right)ds = (**)$$

init. cond.

$r(t_0+\theta)=0 \quad \forall\,\theta\in[-r,0]$ $\quad\Rightarrow\quad$ (3.1) holds

Remarks (and notation)

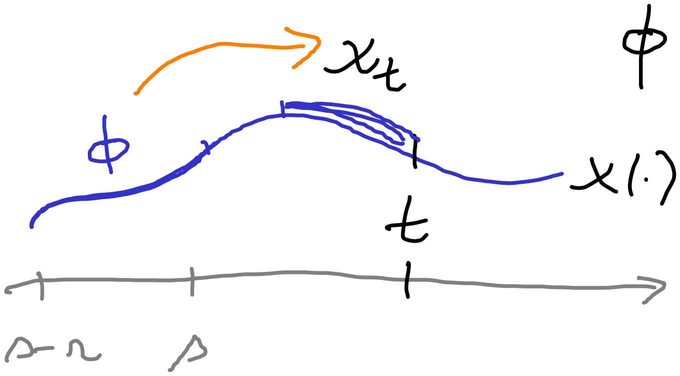
1) FA ⇒ Riesz Thm .... $L : \mathcal{C} \to \mathbb{R}^1$ lin, bdd $\quad \mathcal{C} = C([-r,0], \mathbb{R}^1)$

$$L(\psi) = \int_{-r}^{0} \psi(\theta)\, d\underbrace{\mu}_{\text{representing measure}}(\theta)$$

e.g. $$x'(t) = \underbrace{a x(t) + b x(t-r) + \int_{-r}^{0} g(\theta) x(t+\theta)\, d\theta}_{L(x_t)}$$

$$\Rightarrow \mu = a \cdot \delta_0 + b\, \delta_{-r} + g(s)\, d\underbrace{\lambda}_{\text{Leb. meas.}}$$

2) define: $T(t,s) : \mathcal{C} \to \mathcal{C}$ $\quad t \geq s$



$\phi \longmapsto x_t$ ... where $x(\cdot)$ solves (3.2) w.i.c. $x_s = \phi$

general solution to (3.1)

$$\xi = \begin{cases} 0, & \theta \in [-r, 0) \\ I, & \theta = 0 \end{cases}$$

$$x_t = T(t,t_0)\phi + \int_{t_0}^{t} \underbrace{T(t,s)\xi}_{\cancel{u(t,s)} = U_t(\cdot, s)} h(s)\, ds \in \mathcal{C}$$

$(x(t) = x_t(0))$