Lemma 2 One can write $T(t) = N(t) + K(t)$,

where:

1. $N(t)N(s) = N(t+s)$

   $N(t) = 0, \quad t \geq r$

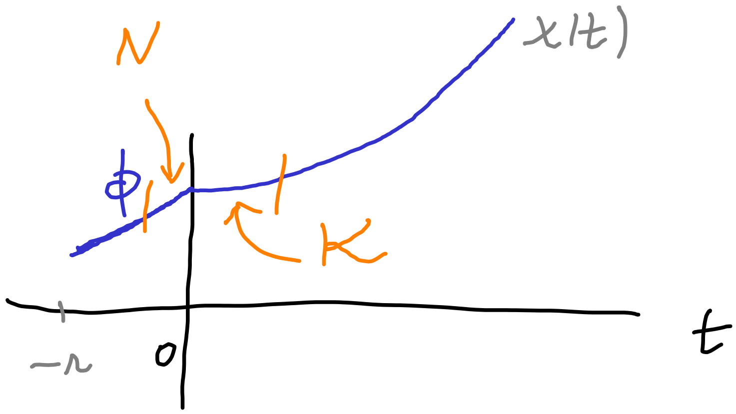
2. $K(t)$ is compact

Proof. Set:

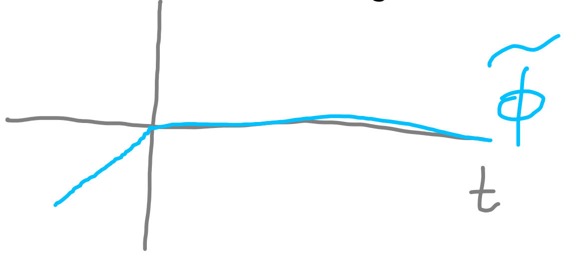
$$[N(t)\phi](\theta) = \begin{cases} \overbrace{\phi(t+\theta) - \phi(0)}^{\tilde{\phi}} & (t+\theta < 0) \\ 0 & (t+\theta \geq 0) \end{cases}$$

$\phi \in \mathcal{C}$ $\qquad \theta \in [-r, 0)$

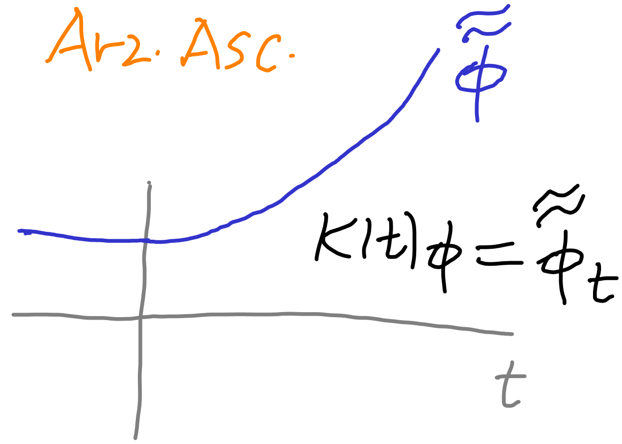
$$[K(t)\phi](\theta) = \begin{cases} \phi(0) & (t+\theta < 0) \\ \underbrace{\phi(0) + \int_0^t L(T(s)\phi)\,ds}_{\tilde{\tilde{\phi}}} & (t+\theta \geq 0) \end{cases}$$



$N(t) = \tilde{\phi}_t$



Arz. Asc.



Remarks.

- $T(t)$ compact for $t \geq r$ ($\Leftarrow L2$)
- $T(t)\phi \in \mathcal{D}(A)$, for $t \geq r$ ($\Leftarrow L1$)
- $T(t)$ differentiable for $t \geq r$ ($\Rightarrow$ norm continuous)

(3.3) $\Leftrightarrow$ $\frac{d}{dt} x_t = A x_t$, $t \geq r$

- $T(t)$ not analytic ($T(t)\phi \notin \mathcal{D}(A)$, $t < r$ in general)

Theorem 7.

Let $(A, \mathcal{D}(A))$ be the generator of $T(t)$, let $\operatorname{Re}\lambda < \beta$ for $\forall \lambda \in \sigma(A)$, for $\beta \in \mathbb{R}$. Then: $\exists M \geq 1$ s.t. $\|T(t)\|_{\mathcal{L}(\mathcal{C})} \leq M e^{\beta t}$ $\forall t \geq 0$.

Remark. In other words: $\boxed{\omega(T) \leq s(A)}$

Proof.

$$\omega(T) \leq \frac{1}{t} \ln\left(\underline{\underline{r}}(T(t))\right)$$

$(\Leftarrow \text{Prop. 1})$

$$\sigma(T(t)) \setminus \{0\} \subseteq e^{t\sigma(A)}$$

$(\Leftarrow \text{Theorem 5})$

since $T(t)$ is norm-cont. for $t \geq r$

alternatively:

(Thm. 3)

$$\underbrace{\underline{\underline{P}}\sigma(T(t)) \setminus \{0\}}_{\overset{\text{compactness}}{=}\ \sigma(T(t)) \setminus \{0\}} = e^{t \underbrace{\underline{\underline{P}}\sigma(A)}_{\overset{(\text{Thm 6})}{=}\ \sigma(A)}}$$

Corollary: $$x'(t) = \underbrace{\int_{-r}^{0} x(t+\theta)\,d\mu(\theta)}_{L(x_t)} + g(x_t)$$

where: μ ... Borel measure

$g : C \to \mathbb{R}^n$ cont., $g(0)=0$

$\dfrac{g(\psi)}{\|\psi\|} \to 0,\ \psi \to 0$ (lower order)

(some $\beta > 0$)

Assume: $\operatorname{Re}\lambda < \cancel{0}\,{-\beta}$ $\forall \lambda$ s.t. $\det \Delta(\lambda) = 0$

where $$\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{-\lambda\theta}\,d\mu(\theta)$$

Then: $x \equiv 0$ is asymptotically stable.

Proof.: $x'(t) = L(x_t) + g(x_t)$

Thm 7: $\|T(t)\| \le m\, e^{-\beta t},\quad t \ge 0$

Thm 5: (var. const.) $$x_t = T(t)\phi + \int_0^t T(t-s)\,\xi\, g(x_s)\,ds$$

$$\Rightarrow \quad \|x_t\| \leq R_1(t) + R_2(t)$$

where: $R_1(t) \leq M \cdot e^{-\beta t} \|\phi\|$

$$R_2(t) \leq \int_0^t \tilde{M} \cdot e^{-\beta(t-s)} \underbrace{|g(x_s)|}_{\leq \underline{\underline{\alpha}} \|x_s\|} ds$$

small

$$\Rightarrow \quad \|x_t\| \leq M e^{-\beta t} \left(\|\phi\| + \tilde{\alpha} \int_0^t e^{\beta s} \|x_s\| ds \right)$$

$$\Rightarrow \quad \|x_t\| \to 0, \text{ by Gronwall}$$

Application:

1) Verhulst w. delay:

$$\underline{x'(t) = -cx(t-1)} \; \overbrace{-cx(t-1)x(t)}^{g(x_t)}$$

HW 1: $\boxed{c \leq \frac{\pi}{2}} \Rightarrow \operatorname{Re} \lambda < 0$ $\qquad \|g(\psi)\| \leq \|\psi\|^2$

2) PΠT

Question: DDE $\begin{cases}\text{finite-dim} \\ \text{infinite-dim}\end{cases}$??

.....

$$\left\lfloor x'(t) = x(t-1) \right\rceil$$

1) char. eq. $\underbrace{\lambda - e^{-\lambda} = 0}_{}$ $\iff$ $\underbrace{1 - \frac{e^{-1/z}}{z}}_{f(z)} = 0$

claim: ∞-many roots!!

$\left(\lambda = \frac{1}{z} \neq 0\right)$

$\Downarrow$

$x(t) = e^{\lambda t}$

for ∞-many $\lambda \in \mathbb{C}$

observe: $f(z)$ has essential singularity at $z = 0$

Picard's Thm: $w = 0$ exception $f(z) \neq 1$

$\Downarrow$

$f(z) = 0$ ∞-often in $\dot{D}(0,\delta)$ $\forall \delta > 0$

2) by Hw3: $\sigma(A) \cap \{ \mathrm{Re}\,\lambda \geq 0 \}$ is finite (with finite multiplicity)

can be shown

$$\mathcal{C} = X_0 \oplus Y$$

X_0: finite-dimens. (ODE like)

Y: exponential decay ($\mathrm{Re}\,\sigma(A) < 0$)