NMMA440 – PART TWO

"ODEs with DELAY"

ODEs
- ordinary: $x' = f(x)$ $\quad \left[x'(t) = f(x(t)) \right]$
- delayed: $x'(t) = f(x(t), x(t-r))$

$$\int_0^r k(s)\, x(t-s)\, ds$$

etc....

Remark: ODEs in Banach space

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1 - Examples

① logistic eq. (Verhulst)

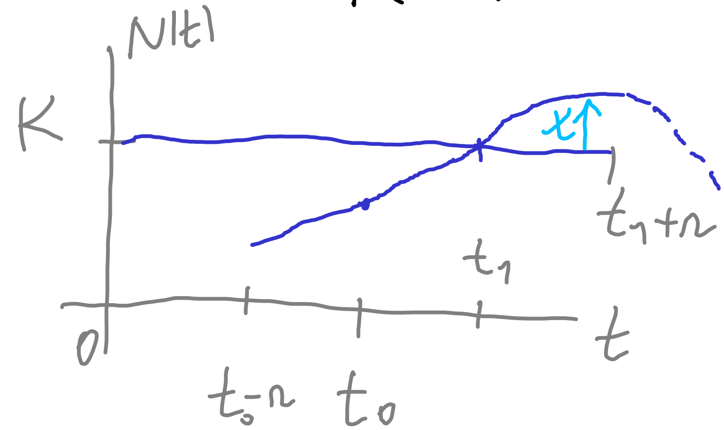
$$N'(t) = a\left(1 - \frac{N(t)}{K}\right) N(t)$$

$N(t)$ .... population

$a > 0$ ... growth rate

$K > 0$ ... capacity of environment

with delay: $N'(t) = a\left(1 - \frac{N(t-r)}{K}\right)N(t)$



Is $N(t) \equiv K$ stable? set: $N(t) = K\left(1 + x\left(\frac{t}{r}\right)\right)$, $\frac{t}{r} = s$

$x = x(s)$

$\Updownarrow$

$$\Rightarrow \quad x'(s) = -c\,x(s-1)\left[1 + x(s)\right]$$

$c = a \cdot r$

Is $x(t) \equiv 0$ stable? Not in general.

Pf. linearize (justify later) $\Rightarrow$ $x'(t) = -c\,x(t-1)$

$\exists$ unbd. solns HW1

② delayed SIR model

$$S' = -\beta S I$$
$$I' = \beta S I - \alpha I$$

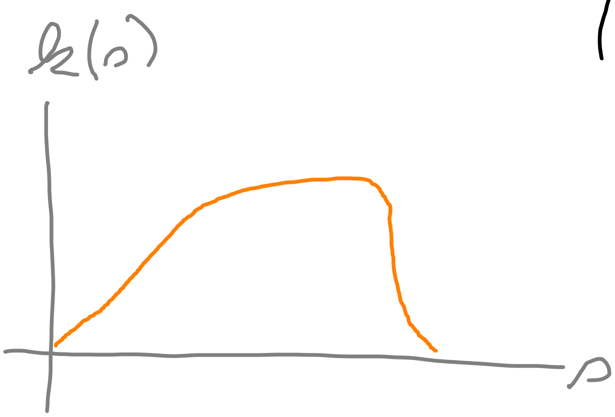
$S(t)$ … susceptible

$I(t)$ … infected

with delay:

(i) $-\beta S(t) I(t-r)$

(ii) $-\beta S(t) \int_0^r k(s) I(t-s)\, ds$



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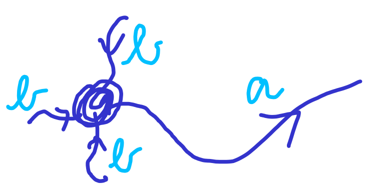
③ neural field: $\boxed{\partial_t u = a \cdot u + b \cdot \partial_{xx} u}$ (1)

(contin.)

$u = u(t,x)$ … electric pot.



delayed variant:

(2) $\boxed{\partial_t u(t,x) = a\, u(t-r,x) + b\, \partial_{xx} u(t-r,x)}$

Claim: (2) has richer dynamics than (1)

HW 1

④ Prime number theorem (1800 ~ 1900)

Set: $\pi(x) = \#\{p \le x ;\ p \text{ prime}\}$

Then: $\pi(x) \sim \dfrac{x}{\log(x)}$, $x \to +\infty$

More precisely: $\lim\limits_{x \to +\infty} \dfrac{\pi(x)}{x/\log(x)} = 1 \quad (*)$

"Proof" (Cherwell, de Visme 1950)

assume: $\exists\, y(x) \in C^1$ s.t. $y(x) \sim \pi(x)$ $(x \to +\infty)$

aim, $y'(x) \sim \dfrac{1}{\log x}$ $\quad x \to +\infty$ $\quad$ (l'Hospital $\Rightarrow (*)$)

behavior of $y'(x)$ ... ?

$$\underbrace{\pi(m) - \pi(n)}_{\#\{p \in (n,m] ;\ p \text{ prime}\}} \simeq y(m) - y(n) = \int_n^m \underline{\underline{y'(x)}}\, dx$$

density of primes (= probability of being prime) (close to x)

$$\mathrm{Prob}\{2 \mid x\} = \frac{1}{2}$$

$$\mathrm{Prob}\{3 \mid x\} = \frac{1}{3}$$

$$\vdots$$

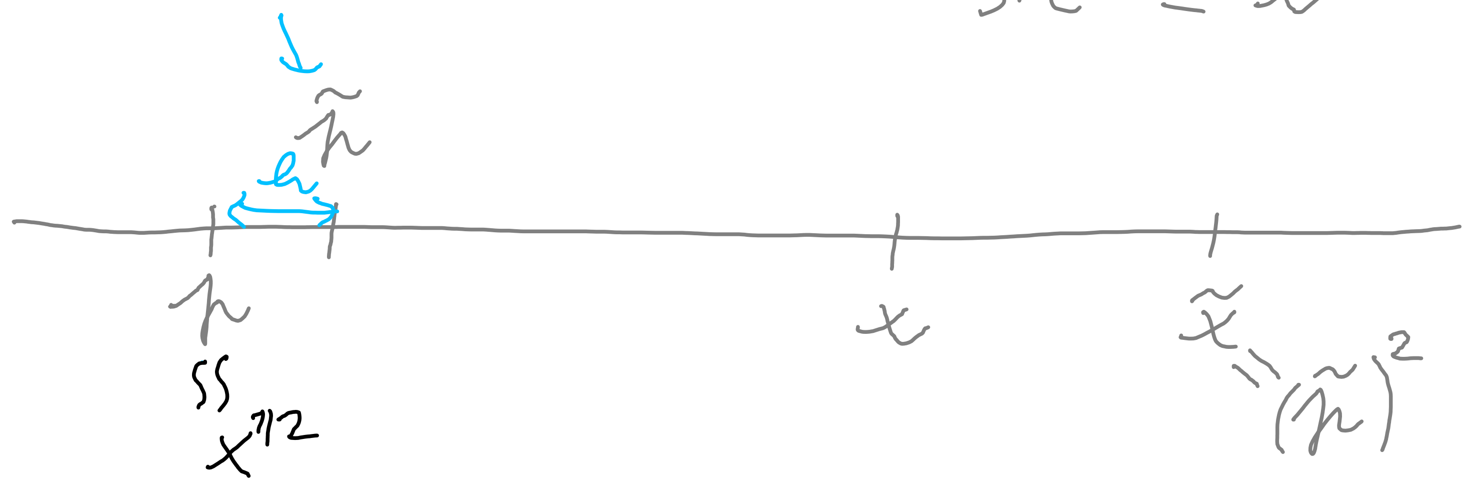
$$\mathrm{Prob}\{p \mid x\} = \frac{1}{p}$$

$\left(x \in \mathbb{N} \text{ fixed} \right)$

$$\mathrm{Prob}\{x \text{ prime}\} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{p}\right)$$

$p \ldots$ largest prime s.t. $\leq x^{1/2}$

following prime



(1) $\tilde{p} = p + h \approx p + \dfrac{1}{y'(x^{1/2})}$ average gap

(2) $y'(\tilde{x}) = \left(1 - \dfrac{1}{x^{1/2}}\right) y'(x)$

Taylor: $\tilde{x} = (x^{1/2} + h)^2, \quad h \approx \dfrac{1}{y'(x^{1/2})}$

$$\Rightarrow \quad y'(\tilde{x}) = y'(x) + 2h\, x^{7/2}\, y''(x)$$

finally :

$$\boxed{2x\,\frac{y''(x)}{y'(x)} + y'(x^{7/2}) = 0}$$

set : $\log x = 2^t \quad \left(x = e^{(2^t)}\right)$

$$r(x)\log x = 1 + u(t)$$

(where $r(x) = y'(x)$)

$$\Rightarrow \quad u'(t) = -\alpha\, u(t-1)\left[1 + u(t)\right] \quad (**)$$

where $\alpha = \log 2 \in (0,1)$

aim: $r(x) \sim \dfrac{1}{\log(x)}$ $\iff$ $u(t) \to 0,\ t \to +\infty$

$x \to +\infty$

we will prove

(by linearization)

2 - Basic theory

Remark ordinary ODE (non-delay)

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

initial cond.

$f(\cdot,\cdot)$ ... cont. $\overset{\text{Peano}}{\Longrightarrow}$ local $\exists$ $(t \geq t_0$ & $t \leq t_0)$

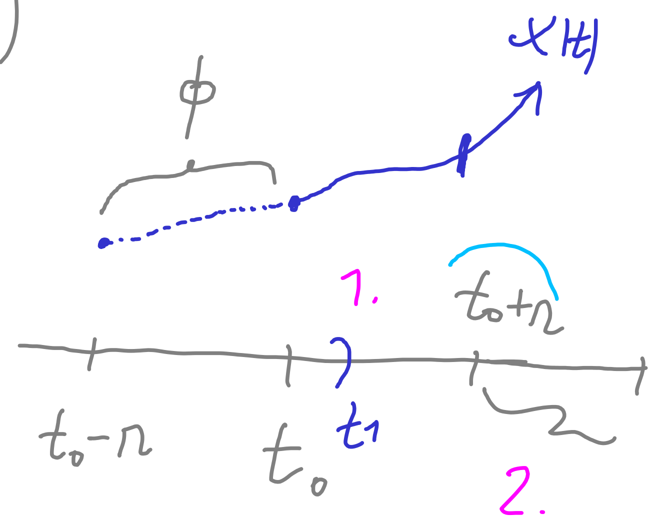
moreover Lipsch. (local) $\overset{}{\underset{\text{Picard}}{\Longrightarrow}}$ also uniqueness (global)

Delayed ODE (sample problem)

$$x'(t) = f(t, x(t), x(t-\underline{r}))$$

(1) $$\left.x\right|_{[t_0 - r,\, t_0]} = \phi$$

initial condition (history)



Method of steps. (for (1))

1. $t \in [t_0, t_0 + r]$: (1) $\Leftrightarrow$ $$\begin{cases} x'(t) = f(t, x(t), \phi(t-r)) \\ x(t_0) = \phi(t_0) \end{cases}$$

usual ODE theory

$\Rightarrow x^{(1)}(t)\,,\ t \in [t_0, t_1)\,,$

If $t_1 = r$: 2. $t \in [t_0 + r, t_0 + 2r]$

(1) $\iff$ $$x'(t) = f(t, x(t), x^{(1)}(t-r))$$
$$x(t_0 + r) = x^{(1)}(t_0 + r)$$

etc...

Generalizations ??

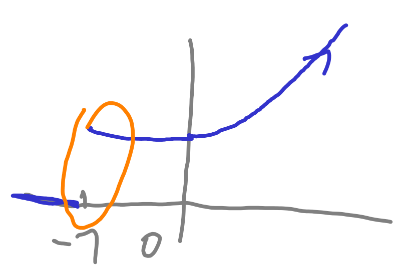
(1) $$x'(t) = f(t, x(t), x(t-r))$$

$$x(t-r_1),\ x(t-r_2)$$
$$x(t-\rho(t))$$ .... OK

not possible: $$x(t/2) \quad (t \geq 0)$$
$$\int_0^r k(s)\, x(t-s)\, ds$$

Remarks (1) backward in time? not in general

$$x'(t) = x(t-1)$$
$$x_{[-1,0]} = 1$$



$\Rightarrow$ $x(t) = x'(t+1)$

② delay can improve things....

$$x'(t) = f(t, x(t-r))$$

+ i.c.

$f(\cdot,\cdot)$ cont. $\Rightarrow \exists!$ sol. $\forall t \geq t_0$

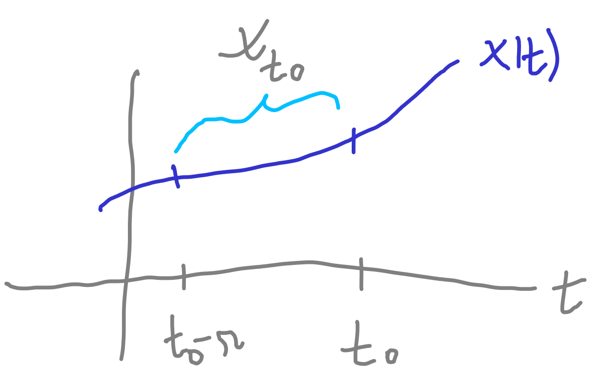
(compare usual ODE)

Notation: $C := C([-r,0]; \mathbb{R}^n)$

$\phi \in C \ldots \|\phi\| = \sup_{s \in [-r,0]} |\phi(s)|$

$x(t) : [t_0 - r, t_1] \to \mathbb{R}^n$; $t_0 < t_1$

denote: $x_t \in C$ by $x_t(s) = x(t+s)$, $s \in [-r,0]$



Observe: $x(t) \in C([t_0 - r, t_1]; \mathbb{R}^n)$

$\Rightarrow t \longmapsto x_t$

$[t_0, t_1] \to C$ is continuous

Proof: by uniform cont. of $x(t)$

Def. [delay diff. eq.]

$$\left.\begin{aligned} x'(t) &= F(t, x_t) \\ x_{t_0} &= \phi \end{aligned}\right\} \text{(DDE)}$$

where: $t_0 \in \mathbb{R}$, $\phi \in C$

$F = F(t, x) : J \times \mathcal{D} \to \mathbb{R}^n$ ... continuous

$J \subset \mathbb{R}^n$ int., $\mathcal{D} \subset C$ (open) s.t. $t_0 \in J$, $\phi \in \mathcal{D}$

Solution: $x(t) : I \to \mathbb{R}^n$ ... continuous

where $I = [t_0 - r, t_1)$, $t_1 > t_0$

s.t. $\forall t \in [t_0, t_1)$:

$(t, x_t) \in J \times \mathcal{D}$

$(\text{DDE})_1$ holds (one-sided der. $t = t_0$)

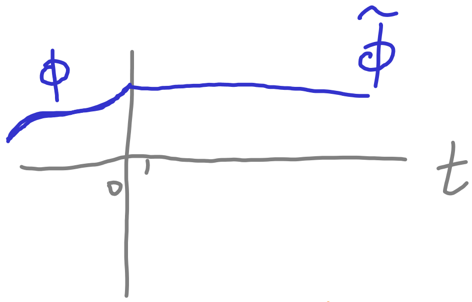
and $x|_{[t_0 - r, t_0]} = \phi$ $\left(\Leftrightarrow (\text{DDE})_2\right)$

Remark: $(\text{DDE})_1 \Leftrightarrow x(t) = \phi(t_0) + \int_{t_0}^{t} F(s, x_s)\, ds$ $\forall t \in [t_0, t_1)$

Pf: continuity of the integrand

Theorem 1 (DDE) is locally solvable. (t > t_0 only)

Pf. WLOG: $t_0 = 0$, denote: $\tilde{\Phi}(t) = \begin{cases} \phi(t); & t \in [-r, 0] \\ \phi(0); & t > 0 \end{cases}$



set: $x(t) = \tilde{\phi}(t) + \underbrace{y(t)}_{\text{new unknown.}}$

$$\text{(DDE)} \iff \left[\begin{aligned} & y(t) = \int_0^t F(s, \tilde{\phi}_s + y_s)\, ds \\ & y_0 = 0 \end{aligned}\right.$$

recall: Schauder's theorem:

X ... Banach space

$K \subset X$ closed, convex, bdd.

$\mathcal{F}: K \to K$ cont. s.t. $\overline{\mathcal{F}(K)}$ compact

$\Rightarrow \exists\, y \in K$ s.t. $y = \mathcal{F}(y)$

application: $X = C([-r, \tau]; \mathbb{R}^n)$, $\tau > 0$

$$K = \{ y \in X,\ y_0 = 0,\ \|y_t\| \le \rho \quad \forall t \in [0, \tau] \}$$

$\mathcal{F} : y \longmapsto \tilde{y}$

where $\tilde{y}(t) = \begin{cases} 0, & t \in [-r, 0] \\ \int_0^t F(s, \tilde{\phi}_s + y_s)\, ds & t \in [0, \tau] \end{cases}$

note: F cont. (at $(0, \phi)$) $\Rightarrow$ $\exists\, M > 0$, $\exists\, \tau, \rho > 0$ small

s.t. $|F(s, \tilde{\phi}_s + y_s)| \leq M$

$\forall s \in [0, \tau], \forall y \in K$

let also: $M\tau \leq \rho$

$\Rightarrow \tilde{y} \in K$, i.e. $\mathcal{F}(K) \subset K$

Pf. $|\tilde{y}(t)| \leq \int_0^t |F(\ldots)|\, ds \leq t \cdot M$

$\leq \tau M \leq \rho$

it remains: 1) $\mathcal{F}$ cont. $\Leftarrow$ Lebesgue

2) $\mathcal{F}(K)$ rel. compact $\Leftarrow$ Arzela-Asc.

Theorem 2 Assume $F = F(t,x)$ loc. Lipschitz w.r.t. x.
Then (DDE) is (globally) unique.

Pf. $x^{(1)}(t),\ t \in [t_0 - r, t_1)$

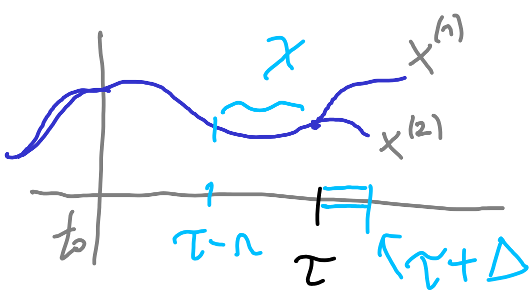
$x^{(2)}(t),\ t \in [t_0 - r, t_2)$ s.t. $x^{(1)}_{t_0} = x^{(2)}_{t_0} = \phi$

$\overset{?}{\Longrightarrow}$ $x^{(1)} \equiv x^{(2)}(t),\ \forall t \in [t_0, \tilde{t})$

where $\tilde{t} = \min\{t_1, t_2\}$.

?? $\exists t \in (t_0, \tilde{t})$ s.t. $x^{(1)}(t) \neq x^{(2)}(t)$

take $\tau = \inf\{t \in (t_0, \tilde{t});\ x^{(1)}(t) \neq x^{(2)}(t)\}$



$\exists L > 0$

$x^{(1)}_\tau = x^{(2)}_\tau$ "

$\exists \mathcal{U}$ neigh. of (τ, x)

s.t. $|F(t,x) - F(t,\tilde{x})| \le L\|x - \tilde{x}\|$

$\forall (t,x), (t,\tilde{x}) \in \mathcal{U}$

fix $\Delta > 0$ small. s.t. $(t, x^{(1)}_t), (t, x^{(2)}_t) \in \mathcal{U}$

$\forall t \in [\tau, \tau + \Delta]$

$L \cdot \Delta < 1$

integral form:

$$x^{(i)}(t) = \cancel{x^{(i)}(\tau)} + \int_{\cancel{0}\ \tau}^{t} F(s, x_s^{(i)})\, ds$$

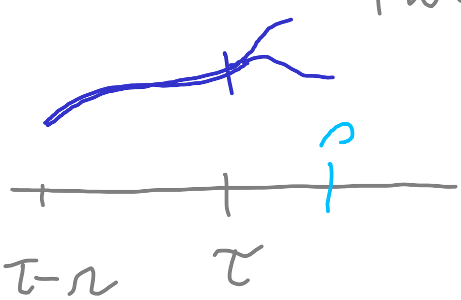
$(i=1,2)$ $\quad t \in [\tau, \tau+\Delta]$

$$|x^{(1)}(t) - x^{(2)}(t)| \le \int_{\cancel{0}\ \tau}^{t} \underbrace{|F(s, x_s^{(1)}) - F(s, x_s^{(2)})|}_{\le L \underbrace{\|x_s^{(1)} - x_s^{(2)}\|}} ds$$

but:

(WLOG: $\Delta \le r$)

$$\|x_s^{(1)} - x_s^{(2)}\| = \underbrace{\sup_{\sigma \in [-r,0]} |x^{(1)}(s+\sigma) - x^{(2)}(s+\sigma)|}$$



$$= \sup_{\sigma \in \cancel{[0,s]}\ [\tau, \tau+s]} |x^{(1)}(\sigma) - x^{(2)}(\sigma)|$$

$$\le S := \sup_{\sigma \in [\tau, \tau+\Delta]} |x^{(1)}(\sigma) - x^{(2)}(\sigma)|$$

$$\Rightarrow \quad |x^{(1)}(t) - x^{(2)}(t)| \le S \cdot L \cdot \Delta \quad \Big/ \sup_{t \in [\tau, \tau+\Delta]}$$

$$S \le S \cdot L\Delta$$

$$S(1 - \underbrace{L\Delta}_{<1}) \le 0 \quad \Rightarrow \quad \boxed{S = 0}$$

Theorem 1.2: local $\exists$ uniqueness for DDEs (similar to ODE)

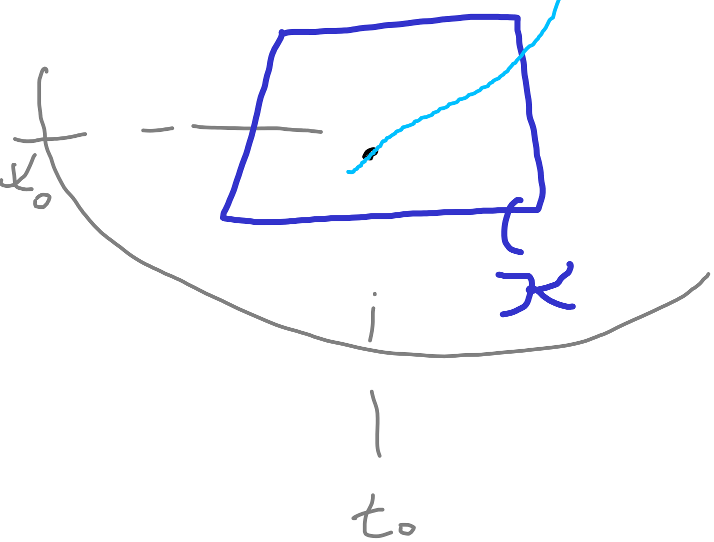
Recall: ordinary ODE:

$$x'(t) = f(t, x(t))$$
$$x(t_0) = x_0$$

maximal solution (non-continuable)

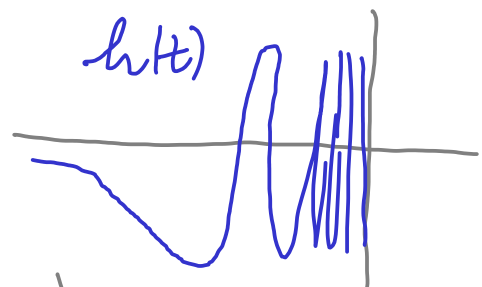
$\Downarrow$

leaves any closed, bdd. set ($\Leftrightarrow$ compact)



Example: $h(t) = \sin(1/t)$ $(t < 0)$

(Yorke)



$\nexists \lim_{t \to 0-} h(t)$

Set: $F : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$

where: $F(t, \varphi) = \begin{cases} h'(t)\left(1 + \dfrac{\|h_t - \varphi\|}{t}\right) & \text{if } \|h_t - \varphi\| < -t \\ 0 & \text{elsewhere} \end{cases}$

$\mathcal{C} = C([-r, 0], \mathbb{R})$

claim: $h(t)$ solves $x'(t) = F(t, x_t)$

$t < 0$ ($\Rightarrow$ cannot be continued to $t=0$)

but: $(t, x_t) \in \underbrace{[-1,0] \times \underbrace{C_{[-2,2]}}}$ ← closed, bdd (not compact)

F cont., but not bdd

Notation. $C = C([-r,0]; \mathbb{R}^n)$

$A \subset \mathbb{R}^n$: $C_A = \{\psi \in C ; \psi(t) \in A, \forall t \in [-r,0]\}$

Def. Let $\mathcal{D} \subset \mathbb{R} \times C$ be open. Then $F : \mathcal{D} \to \mathbb{R}^n$ is called quasi-bounded, if it is bounded on ~~the~~ any subset $[t_0, t_1] \times C_B \subset \mathcal{D}$ where $B \subset \mathbb{R}^n$ is closed and bounded.

Observe: $f(t,x,y) : (a,b) \times \Omega \times \Omega \to \mathbb{R}^n$ continuous, $(a,b) \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open

$\Rightarrow \quad F(t,\psi) := f(t, \psi(0), \psi(-r))$

or $\int_0^r k(s)\psi(-s)\,ds$

is quasi-bounded

on $\mathcal{D} = (a,b) \times \mathcal{C}_\Omega$

$\Updownarrow$

$x'(t) = f(t, x(t), x(t-r))$

or $\int_0^r k(s)x(t-s)\,ds$

Theorem 3. Assume: $\Omega \subset \mathbb{R}^n$ open

$F : [t_0, \beta) \times \mathcal{C}_\Omega \to \mathbb{R}^n$ cont.

quasi-bounded

$\phi \in \mathcal{C}_\Omega$ given

Then: $\exists\, x(t) : [t_0, t_1) \to \mathbb{R}^n$ a maximal solution to $(t_1 \le \beta)$

$x'(t) = F(t, x_t), \quad x_{t_0} = \phi$

Such that: if $t_1 < \beta$, then for arbitrary

$B \subset \Omega$ closed, bounded

$\exists\, t \in (t_0, t_1)$ s.t. $x(t) \notin B$.

Pf. $\exists$ maximal solution $\Leftarrow$ Zorn's lemma

(non-continuable beyond some $t_1 \leq \beta$)

2nd part: ?? $x(t) \in B, \ \forall t \in [t_0, t_1)$

$B \subset \Omega$ given closed, bdd

WLOG: $x(t) \in B \ \forall t \in (t_0 - r, t_1)$

$\Longleftrightarrow x_t \in \mathcal{C}_B, \ t \in [t_0, t_1)$

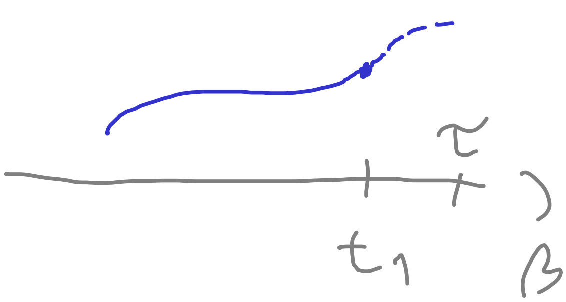
but (F quasi-bdd!!) $\Rightarrow |F(t, \psi)| \leq M$

$\forall (t, \psi) \in [t_0, t_1] \times \mathcal{C}_B$

$\Rightarrow |x'(t)| \leq M, \ t \in (t_0, t_1)$

note: $t_1 < +\infty$

$\Rightarrow \exists \lim_{t \to t_1^-} x(t) =: x_1$



Apply Thm 1

$\Rightarrow$ extension of the sol. to $[t_1, \tau)$, $\tau > t_1$ $\lightning$

Corollary. $F : [t_0, \beta) \times \mathcal{C} \to \mathbb{R}^n$ cont.

(*) $|F(t,\psi)| \leq a(t) + b(t)\|\psi\| \quad \forall t \in [t_0,\beta)$, $\psi \in \mathcal{C}$

with $a(t), b(t)$ cont. on $[t_0, \beta)$

$\Rightarrow \exists$ solution on $[t_0, \beta)$ (global)

Pf. (*) $\Rightarrow$ F quasi-bounded

Thm 3 $\Rightarrow$ $\exists\, x(t) : [t_0, t_1) \to \mathbb{R}^n$ maximal

$t_1 \leq \beta$

?? $t_1 < \beta$: Thm 3 $\Rightarrow$ $x(t)$ leaves any $B \subset \mathbb{R}^n$ closed, bdd

$x'(t) = F(t, x_t)$

$t_1 < +\infty$

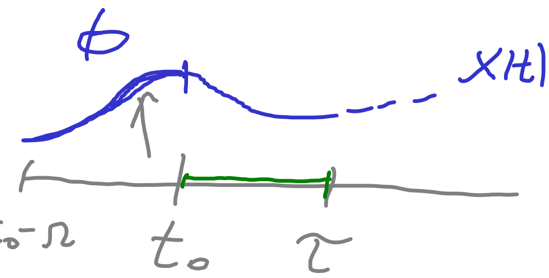
$$|x(t)| \leq |x(t_0)| + \int_{t_0}^{t} |F(s, x_s)|\, ds \qquad t \in [t_0, t_1) \subset [t_0, t_1]$$

$$\overset{(*)}{\leq} \underbrace{a(s)}_{\leq c_1} + \underbrace{b(s)}_{\leq c_1}\|x_s\|$$

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^{t} c_1 + c_1 \|x_s\|\, ds \qquad t \in [t_0, t_1)$$

$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{t} \underbrace{\|x_s\|}_{\le \sigma(s) + \|\phi\|} \, ds \qquad = \sup \text{ on } [\cancel{t}\,s - r, \cancel{t}\,s]$$

set: $$\sigma(\tau) := \sup_{t \in [t_0, \tau]} |x(t)| \qquad \tau \in [t_0, t_1)$$



$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{t} \sigma(s) \, ds$$

$$|x(t)| \le \tilde{C}_0 + C_1 \int_{t_0}^{\tau} \sigma(s) \, ds \qquad \forall t \in [t_0, \tau]$$

sup

$$\Rightarrow \sigma(\tau) \le \tilde{C}_0 + C_1 \int_{t_0}^{\tau} \sigma(s) \, ds$$

Gronwall: $$\sigma(\tau) \le \tilde{C}_0 \cdot \exp\left(C_1(\tau - t_0)\right) =: C_3$$

$$\forall \tau \in [t_0, t_1)$$

$$\Rightarrow x(t) \in B, \quad \forall t \in [t_0, t_1)$$

where $B = \{x \in \mathbb{R}^n ; |x| \le C_3\}$

$\lightning\lightning$

3 - Linear equation

Definition. By linear DDE we mean

$$(3.1) \qquad x'(t) = L(t, x_t) + h(t)$$

where : $L = L(t, \varphi) : [t_0, \beta) \times \mathcal{C} \to \mathbb{R}^n$ cont.

bdd, linear w.r.t. φ

(for any t fixed)

$h(t) : [t_0, \beta) \to \mathbb{R}^n$ cont.

Moreover : $|L(t, \varphi)| \le m(t) \|\varphi\| \quad \forall t \in (t_0, \beta)$

where $m(t)$ is continuous

Special cases: linear homogeneous DDE

$$(3.2) \qquad x'(t) = L(t, x_t)$$

and : linear, homogeneous & autonomous DDE

(with const. coeff.s)

$$(3.3) \qquad x'(t) = L(x_t)$$

where $L : \mathcal{C} \to \mathbb{R}^n$

bounded, linear

Theorem 4. Given $\phi \in \mathcal{C} \Rightarrow \exists !\, x(t)$ global solution to (3.1) on (t_0, β) such that $x_{t_0} = \phi$

Proof. Set $F(t,\psi) = L(t,\psi) + h(t)$

$$\Rightarrow |F(t,\psi)| \leq m(t)\|\psi\| + |h(t)|$$

Corollary (after Thm 3) $\Rightarrow$ global existence

uniqueness? $|F(t,\psi) - F(t,\tilde{\psi})| =$

$\Uparrow$ $= |F(t,\psi-\tilde{\psi})| \leq m(t)\|\psi - \tilde{\psi}\|$

Theorem 2 $\Rightarrow F(t,\psi)$ locally Lipsch. w.r.to ψ

Recall: $x'(t) = A(t)x(t) + h(t)$

fundamental matrix: $U'(t) = A(t)U(t)$

$U(t)$ regular $(\forall t)$

variation of constants:

$$x(t) = U(t)U^{-1}(t_0)x_0 + \int_{t_0}^{t} U(t)U^{-1}(s)h(s)\,ds$$

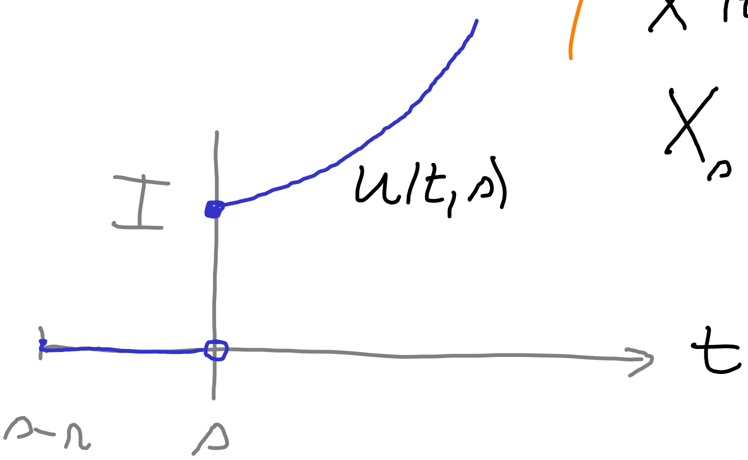
$(x(t_0) = x_0)$

Definition. Fundamental matrix $U = U(t,s) \in \mathbb{R}^{m \times m}$ (for (3.2))

is defined as

$$U(t,s) = \begin{cases} I + \int_s^t L(\theta, U_\theta(\cdot, s))\, d\theta & (t \geq s) \\ 0 & (s-r \leq t < s) \end{cases}$$

Ekvivalently: for s fixed, $t \mapsto U(t,s)$ solves



$$\begin{cases} X'(t) = L(t, X_t) \\ X_s = \begin{cases} 0 & s-r \leq t < s \\ I & t = s \end{cases} \end{cases}$$

well defined?

$\Uparrow$

Theorem 4

(some modification)

Theorem 5. The function

$$x(t) = \int_{t_0}^{t} U(t,s) h(s)\, ds \quad \text{solves (3.1)}$$

with $x_{t_0} = 0$.

Proof. (sketch)

$$r'(t) = \frac{d}{dt}\int_{t_0}^{t} u(t,s)\,h(s)\,ds$$

$$= \underbrace{u(t,t)}_{I}\,\underline{\underline{h(t)}} + \int_{t_0}^{t} \underbrace{\partial_t u(t,s)\,h(s)\,ds}_{= L(t,\,u_t(\cdot,s))} \quad (**)$$

but: $L(t, r_t) = \;??$

observe: $r_t(\theta) = r(t+\theta) = \int_{t_0}^{\cancel{t+\theta}\ *)} u(t+\theta, s)\,h(s)\,ds$

$\theta \in [-r, 0]$

$*)$

$$\Rightarrow \quad r_t = \int_{t_0}^{t} u_t(\cdot, s)\,h(s)\,ds$$

$$L(t, r_t) = L\left(t, \int_{t_0}^{t} u_t(\cdot,s)\,h(s)\,ds\right)$$

$$= \int_{t_0}^{t} L(t, u_t(\cdot, s])\,h(s)\,\cancel{ds}\,)\;ds = (**)$$

init. cond.

$r(t_0+\theta) = 0 \quad \forall\, \theta \in [-r, 0]$ $\quad\Rightarrow\quad$ (3.1) holds

Remarks (and notation)

1) FA $\Rightarrow$ Riesz Thm .... $L : \mathcal{C} \to \mathbb{R}^1$ lin, bdd, $\quad \mathcal{C} = C([-r,0],\mathbb{R}^1)$

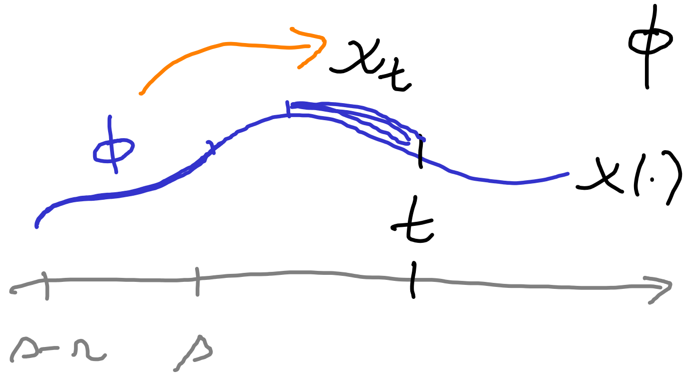
$$L(\psi) = \int_{-r}^{0} \psi(\theta)\, d\underbrace{\mu}_{\text{representing measure}}(\theta)$$

e.g. $$x'(t) = \underbrace{a x(t) + b x(t-r) + \int_{-r}^{0} g(\theta) x(t+\theta)\, d\theta}_{L(x_t)}$$

$$\Rightarrow \mu = a\cdot\delta_0 + b\,\delta_{-r} + g(s)\, d\underbrace{\lambda}_{\text{Leb. meas.}}$$

2) define: $\{T(t,s) : \mathcal{C} \to \mathcal{C}$ $\quad t \geq s$

$\phi \longmapsto x_t$ ... where $x(\cdot)$ solves (3.2) w.i.c. $x_s = \phi$



general solution to (3.1)

$$\xi = \begin{cases} 0, & \theta \in [-r,0) \\ I, & \theta = 0 \end{cases}$$

$$x_t = T(t,t_0)\phi + \int_{t_0}^{t} \underbrace{T(t,s)\xi}_{\cancel{u(t,s)} = u_t(\cdot,s)} h(s)\, ds \in \mathcal{C}$$

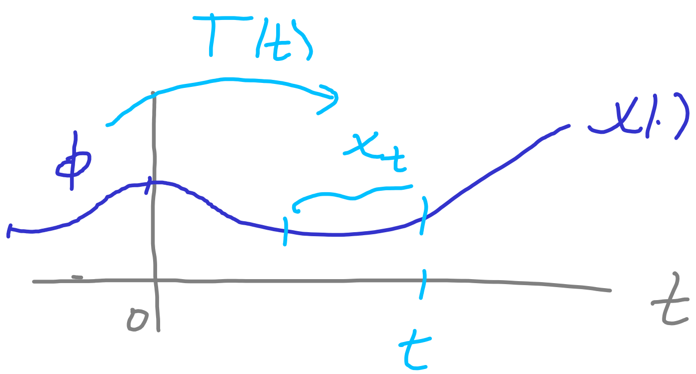
$(x(t) = x_t(0))$

4 - Linear autonomous eq'n

(3.3) $x'(t) = L(x_t)$ $\qquad L : \mathcal{C} \to \mathbb{R}^m$ (bdd, linear)

Def. $T(t) : \mathcal{C} \to \mathcal{C}$ $\qquad$ solution operator

$(t \geq 0)$ $\qquad \phi \mapsto x_t$ .. $\quad x(t)$ ... solution of (3.3) s.t. $x_0 = \phi$



Remark: $T(t,s) = T(t-s, \cancel{0})$ $\quad \forall t \geq s$

autonomous eq.

Observe: $T(t)$ .... C_0-semigroup (on $\mathcal{C}$)

- $T(0) = I$
- $T(t)T(s) = T(t+s)$
- $T(h)\phi \to \phi$ $\quad \forall \phi \in \mathcal{C}$ fixed
  $h \to 0+$ $\qquad (\Leftarrow t \mapsto x_t$ is cont.$)$

... part I of lecture ....

(T. Bárta)

- generator $A = ?$
- $\sigma(A) = ?$

$\Updownarrow$

- growth bound

$\|T(t)\| \leq M \cdot e^{\omega t}$

$\Rightarrow$ stability

Lemma 1 Generator of $T(t)$ is $(A, \mathcal{D}(A))$

where $\mathcal{D}(\tilde{A}) = \{ \phi \in C^1([-r,0]; \mathbb{R}^n)$ s.t.
$\phi'(0) = L\phi \}$

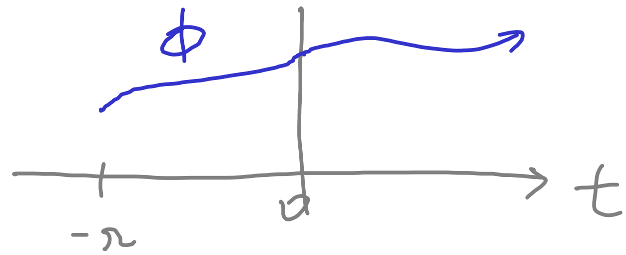
$$[\tilde{A}\phi](\theta) = \begin{cases} \phi'(\theta), & \theta \in [-r,0) \\ L(\phi) & \theta = 0 \end{cases}$$

Pf. let $(A, \mathcal{D}(A))$ be generator of $T(t)$

by def. : $\phi \in \mathcal{D}(A) \Leftrightarrow \exists \overbrace{\lim_{h \to 0+} \frac{T(h)\phi - \phi}{h} = \psi}^{} \in \mathcal{C}$

$A\phi = \psi$

$C = C([-r,0], \mathbb{R}^n)$



$$\underbrace{\frac{\phi(h+\theta)-\phi(\theta)}{h} \rightrightarrows \psi(\theta)}$$ w.r. $\theta \in [-r,0]$.

for $h \to 0+$

in particular: $\underline{\phi \in \mathcal{D}(A)} \Rightarrow \exists\, \phi_+'(\theta) = \psi(\theta)$

$\forall \theta \in [-r,0]$

with ψ cont.

also note:

$\phi_+'(0) = L(\phi)$

$(\Leftarrow$ equation$)$

$\Downarrow (HW^*)^3$

$\phi \in C^1([-r,0])$

$\Rightarrow \phi \in \mathcal{D}(\tilde{A})$

$A\phi = \tilde{A}\phi$

conversely: $\mathcal{D}(\tilde{A}) \subset \mathcal{D}(A)$

$\phi \in \mathcal{D}(\tilde{A}) \Rightarrow \phi'$ cont in $[-r,0]$

— " — $t \geq 0$ $(\Leftarrow$ eq.$)$

$$\frac{1}{h}\left[\phi(h+\theta)-\phi(\theta)\right]=\phi'(\theta+\eta)\rightrightarrows\phi'(\theta)$$

Lagrang

where $|\eta|<h$

Recall. resolvent set:

$$\rho(A)=\{\lambda\in\mathbb{C};\ \exists(\lambda I-A)^{-1}\ \underline{\text{continuous}}\}$$

spectrum: $\sigma(A)=\mathbb{C}\setminus\rho(A)$

one can write:

$$\sigma(A)=\underline{\underline{P\sigma(A)}}\cup C\sigma(A)\cup R\sigma(A)$$

point sp. (not inj.)

continuous sp (inverse $\exists$ but not cont.)

residual sp. (not onto)

$\Updownarrow$

$\exists\phi\neq0,\ \phi\in\mathcal{D}(A)$ s.t. $A\phi=\lambda\phi$

(3.3) $x'(t)=L(x_t)$

Recall. $L : \underbrace{C([-r,0];\mathbb{R}^n)}_{\mathcal{C}} \to \mathbb{R}^n$ bdd., linear

$\xRightarrow{\text{Riesz Thm}}$ $\exists!$ Borel measure s.t.

$$\boxed{L(\phi) = \int_{-r}^{0} \phi(\theta)\, d\mu(\theta)} \quad \forall \phi \in \mathcal{C}$$

(more precisely: $(n>1)$ $\int_{-r}^{0} \underset{\mathbb{R}^{n\times n}}{\underset{\uparrow}{d\mu(\theta)}}\, \underset{\mathbb{R}^{n}}{\underset{\uparrow}{\phi(\theta)}}$)

Theorem 6

Let $(A, \mathcal{D}(A))$ be generator of $T(t)$, let μ be the measure reprezenting L.

Then: 1. $\sigma(A) = P\sigma(A)$

2. $\lambda \in \sigma(A) \Leftrightarrow \det \Delta(\lambda) = 0$

where $\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta}\, d\mu(\theta)$

Note $\Delta(\lambda) \leftarrow$ put $x(t) = e^{\lambda t} I$ into (3.3)

Proof. $\lambda \overset{?}{\in} \rho(A) \Leftrightarrow$ given $\psi \in \mathcal{C}$ $\exists \phi \in \mathcal{D}(A)$ s.t

$$(A - \lambda I)\phi = \psi$$

$$A\phi = \lambda\phi + \psi$$

Lemma 1 $\Rightarrow$

$$\phi'(\theta) = \lambda\phi(\theta) + \psi(\theta)$$

$\forall \theta \in [-r, \cancel{0)}\,]$

(by cont.)

solve by (V.C.)

$$\phi(\theta) = e^{\lambda\theta} b + \int_{-r}^{0} e^{\lambda(\theta - s)} \psi(s)\, ds$$

$\forall \theta \in [-r, 0]$

but: $\phi \in \mathcal{D}(A) \overset{L.1}{\Longrightarrow} \phi'(0) = L(\phi)$

$$= \int_{-r}^{0} \phi(\theta)\, d\mu(\theta)$$

also: $\phi'(0) = \lambda\phi(0) + \psi(0)$

hence, altogether:

$$\lambda b + \psi(0) = \int_{-r}^{0} \left(e^{\lambda\theta} b + \int_{0}^{\theta} e^{\lambda(\theta-s)} \psi(s)\,ds \right) d\mu(\theta)$$

(*)

$$\Delta(\lambda) b = -\psi(0) + \int_{-r}^{0} \left(\int_{0}^{\theta} e^{\lambda(\theta-s)} \psi(s)\,ds \right) d\mu(\theta)$$

observe 1. $\det \Delta(\lambda) \neq 0$

$\Rightarrow \quad b = \Delta^{-1}(\lambda) \cdot \{ -\psi(0) + \dots \}$

i.e. $\psi \xrightarrow{\text{continous}} b \rightsquigarrow \phi \overset{(*)}{\in} \mathcal{D}(A)$

$\Rightarrow \lambda \in \rho(A)$

2. $\Leftarrow$ also true

3. $\det \Delta(\lambda) = 0 \Rightarrow \exists b \neq 0$ s.t.

$\Delta(\lambda) b = 0$

i.e. (*) holds (with $\psi \equiv 0$)

but: $\Rightarrow \exists \phi \neq 0, \ \phi \in \mathcal{D}(A)$

s.t. $A\phi = \lambda\phi$

or $\lambda \in P\sigma(A)$

Lemma 2 One can write $T(t) = N(t) + K(t)$,

where:

1. $N(t)N(s) = N(t+s)$

   $N(t) = 0$, $t \geq r$

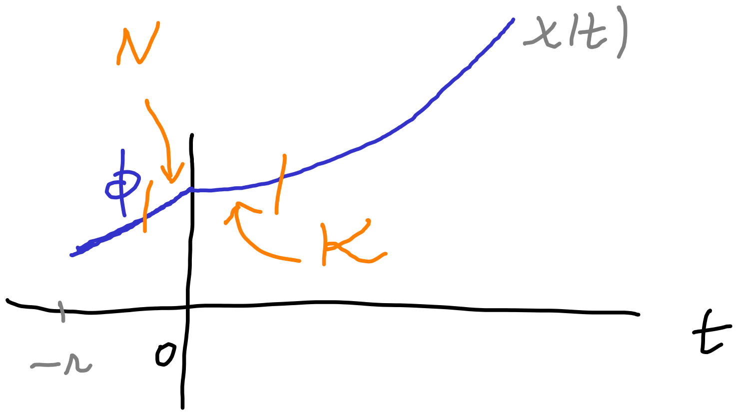
2. $K(t)$ is compact

Proof: Set:

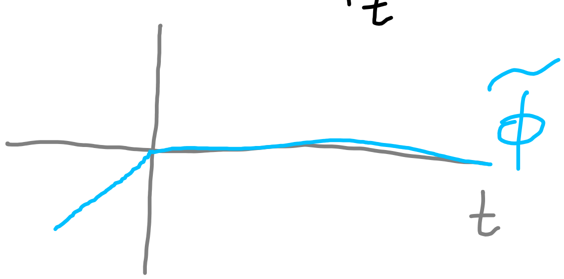
$$[N(t)\phi](\theta) = \begin{cases} \phi(t+\theta) - \phi(0) & (t+\theta < 0) \\ 0 & (t+\theta \geq 0) \end{cases}$$

$\phi \in \mathcal{C}$ $\qquad$ $\theta \in [-r, 0)$

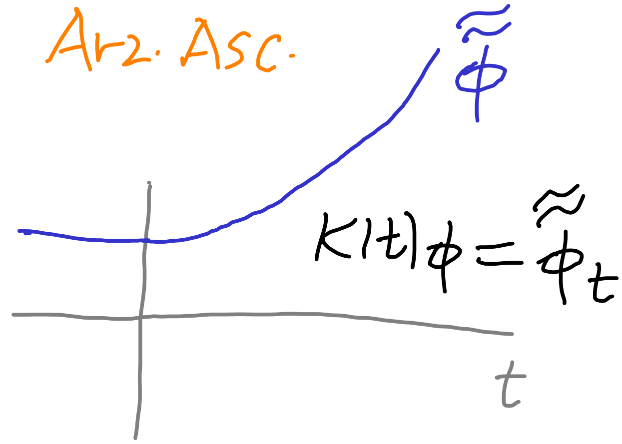
$$[K(t)\phi](\theta) = \begin{cases} \phi(0) & (t+\theta < 0) \\ \phi(0) + \int_0^t L(T(s)\phi)\, ds & (t+\theta \geq 0) \end{cases}$$



$N(t) = \tilde{\phi}_t$



Arz. Asc.



Remarks.

- $T(t)$ compact for $t \geq r$
  $\left(\Leftarrow L2 \right)$
- $T(t)\phi \in \mathcal{D}(A)$, for $t \geq r$
  $\left(\Leftarrow L1 \right)$
- $T(t)$ differentiable for $t \geq r$ $\left(\Rightarrow \text{norm continuous} \right)$

$(3.3) \Leftrightarrow \frac{d}{dt} x_t = A x_t \quad , t \geq r$

- $T(t)$ not analytic
  $\left(T(t)\phi \notin \mathcal{D}(A), \; t < r \right)$
  in general

Theorem 7.

Let $(A, \mathcal{D}(A))$ be the generator of $T(t)$, let $\operatorname{Re} \lambda < \beta$ for $\forall \lambda \in \sigma(A)$, for $\beta \in \mathbb{R}$.

Then: $\exists M \geq 1$ s.t. $\|T(t)\|_{\mathcal{L}(\mathcal{C})} \leq M e^{\beta t}$ $\forall t \geq 0$.

Remark. In other words: $\boxed{\omega(T) \leq s(A)}$

Proof. $\omega(T) \leq \frac{1}{t} \ln\left(\underline{\underline{r}}(T(t))\right)$

$(\Leftarrow \text{Prop. } 1)$

$\sigma(T(t)) \setminus \{0\} \subseteq e^{t\sigma(A)}$

$(\Leftarrow \text{Theorem } 5)$

since $T(t)$ is norm-cont. for $t \geq r$

alternatively:

$(\text{Thm. } 3)$

$$\underbrace{\underline{\underline{P}}\sigma(T(t)) \setminus \{0\}}_{\substack{\parallel \\ \sigma(T(t)) \setminus \{0\}}} = e^{t \underbrace{\underline{\underline{P}}\sigma(A)}_{\substack{\parallel \\ \sigma(A)}}}$$

compactness

$(\text{Thm } 6)$

Corollary: $$x'(t) = \underbrace{\int_{-r}^{0} x(t+\theta)\,d\mu(\theta)}_{L(x_t)} + g(x_t)$$

where: μ ... Borel measure

$g : C \to \mathbb{R}^n$ cont., $g(0)=0$

$$\frac{g(\psi)}{\|\psi\|} \to 0,\ \psi \to 0$$

(lower order)

(some $\beta > 0$)

Assume: $\operatorname{Re}\lambda < \cancel{0}\ -\beta \quad \forall \lambda$ s.t. $\det \Delta(\lambda) = 0$

where $$\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{-\lambda\theta}\, d\mu(\theta)$$

Then: $x \equiv 0$ is asymptotically stable.

Proof.: $$x'(t) = L(x_t) + g(x_t)$$

Thm 7: $$\|T(t)\| \le n \cdot e^{-\beta t}, \quad t \ge 0$$

Thm 5:
(var. const.)
$$x_t = T(t)\phi + \int_0^t T(t-s)\,\xi\, g(x_s)\,ds$$

$$\Rightarrow \quad \|x_t\| \leq R_1(t) + R_2(t)$$

where: $$R_1(t) \leq M \cdot e^{-\beta t} \|\phi\|$$

$$R_2(t) \leq \int_0^t \tilde{M} \cdot e^{-\beta(t-s)} \underbrace{|g(x_s)|}_{\leq \alpha \|x_s\|} ds$$

α small

$$\Rightarrow \quad \|x_t\| \leq M e^{-\beta t} \left(\|\phi\| + \tilde{\alpha} \int_0^t e^{\beta s} \|x_s\| ds \right)$$

$$\Rightarrow \quad \|x_t\| \to 0, \text{ by Gronwall}$$

Application:

1) Verhulst w. delay:

$$x'(t) = -c x(t-1) \underbrace{- c x(t-1) x(t)}_{= g(x_t)}$$

HW 1: $\boxed{c \leq \frac{\pi}{2}} \Rightarrow \operatorname{Re} \lambda < 0$

$\|g(\psi)\| \leq \|\psi\|^2$

2) PMT

Question: DDE $\begin{cases}\text{finite-dim} \\ \text{infinite-dim}\end{cases}$??

.....

$$x'(t) = x(t-1)$$

1) char. eq. $\underbrace{\lambda - e^{-\lambda} = 0}$ $\Longleftrightarrow$ $\underbrace{1 - \frac{e^{-\frac{1}{r}}}{r}}_{f(r)} = 0$

claim: ∞-many roots!!

$(\lambda = \frac{1}{r} \neq 0)$

$\Downarrow$

$x(t) = e^{\lambda t}$

for ∞-many $\lambda \in \mathbb{C}$

observe: $f(r)$ has essential singularity at $r=0$

Picard's Thm: $w=0$ exception $f(r) \neq 1$

$\Downarrow$

$f(r) = 0$ ∞-often in $D(0,\delta)$ $\forall \delta > 0$

2) by Thm 3: $\underbrace{\sigma(A)\cap\{\operatorname{Re}\lambda\geq 0\}}_{\text{is finite}}$ (with finite multiplicity)

$\Downarrow$ can be shown

$$\mathcal{C} = \underbrace{X_0}_{\text{finite-dimens.}} \oplus \underbrace{Y}_{\text{exponential decay}}$$

finite-dimens. (ODE like)

exponential decay ($\operatorname{Re}\sigma(A)<0$)