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1 Introduction. Examples.

We will study the so-called *delayed differential equations*, or (DDE), typically of the form $x'(t) = f(x(t), x(t-r))$ or $x'(t) = f(x(t), \int_0^r k(s)x(t-s)\,ds)$. It is useful to compare/contrast them with "ordinary" ODEs $x'(t) = f(x(t))$.

Examples. ① Logistic (Verhults) equation with delay

$$N'(t) = a\left(1 - \frac{N(t-r)}{K}\right)N(t) \tag{1.1}$$

where $N(t)$ is a population, $a > 0$ the maximal growth rate, and $K > 0$ the carrying capacity of environment. By setting $N(t) = K(1 + x(t/r))$ and $s = t/r$, we obtain

$$x'(s) = -cx(s-1)\big(1 + x(s)\big) \tag{1.2}$$

where $c = ar$. We observe that if $c \geq \pi/2$, $x \equiv 0$ is not stable (in contrast with the non-delayed case).
② Delayed SIR model, where the transmission term $\beta S(t)I(t)$ is replaced by $\beta S(t)I(t-r)$ or $\beta S(t)\int_0^r k(s)I(t-s)\,ds$.
③ Neural field model

$$\partial_t u(t,x) = au(t-r,x) + b\partial_{xx}u(t-r,x) \tag{1.3}$$

The presence of delay is biologically well motivated, and indeed, it brings a rich dynamics (wavetrains).
④ "Proof" of the Prime number theorem. By a "very heuristic" argument, the prime distribution function

$$y(x) = \#\big\{p \leq x; p \text{ prime }\big\} \tag{1.4}$$

satisfies the DDE

$$2x\frac{y''(x)}{y'(x)} + y'(x^{1/2}) = 0 \tag{1.5}$$

On setting $y'(x)\log x = 1 + u(t)$ and $\log x = 2^t$, one deduces

$$u'(t) = -\alpha u(t-1)\big(1 + u(t)\big) \tag{1.6}$$

where $\alpha = \log 2$. From here it can be (rigorously) deduced that $u(t) \to 0$, $t \to \infty$. Hence $y'(x) \sim 1/\log x$ and thus, finally, $y(x) \sim x/\log x$, for $x \to \infty$.

2 Basic theory.

Remarks. ① Special type of problem

$$x'(t) = f(t, x(t), x(t-r))$$

can be solved, for a given "initial history" $x|_{[t_0-r,t_0]} = \phi$ by the *method of steps*: $t \in [t_0, t_0+r,$ then $t \in [t_0+r, t_0+2r]$, etc. But this method does not work with delays like $x(t/2)$ or $\int_0^r k(s)x(t-s)\,ds$.
② In general, one cannot expect solution to exist backward in time.
③ On the other hand, delay can also improve things: one has global (forward) existence and uniqueness for $x'(t) = f(t, x(t-r))$, assuming only *continuity* of f. (Compare with the ODE situation.)

Notation. We denote $\mathcal{C} = C([-r, 0; \mathbb{R}^n)$ with the usual supremum norm. For given $x(t) : [t_0 - r, t_1] \to \mathbb{R}^n$ and $t \in [t_0, t_1]$, we define $x_t \in \mathcal{C}$ by $x_t(s) = x(t+s)$, $x \in [-r, 0]$.
Observe that continuity of $x(t)$ implies continuity of the map $t \mapsto x_t$.

Definition. By a delay differential equation (DDE) we mean

$$x'(t) = F(t, x_t) \tag{2.1}$$

usually with an initial condition

$$x_{t_0} = \phi \tag{2.2}$$

We will always assume that $F = F(t, \psi) : \mathcal{D} \to \mathbb{R}^n$ is continuous, $\mathcal{D} \subset \mathbb{R} \times \mathcal{C}$ is open, and $(t_0, \phi) \in \mathcal{D}$.
By solution on $[t_0, t_1)$ we understand a function $x(t) : [t_0 - r, t_1) \to \mathbb{R}^n$ satisfying (2.2), such that (2.1) holds for all $t \in [t_0, t_1)$, with right-sided derivative at $t = t_0$.

Remark. One has the usual equivalent integral form

$$x(t) = \phi(t_0) + \int_{t_0}^{t} F(s, x_s)\,ds, \qquad t \in [t_0, t_1) \tag{2.3}$$

Theorem 1. Problem (2.1), (2.2) is locally (forward) solvable. More precisely, there exist a solution $x(t)$ on $[t_0, t_1)$ for some $t_1 > t_0$.

Theorem 2. If $F(t, \psi)$ is locally Lipschitz w.r. to ψ, then one has (global, forward) uniqueness. This means, if $x(t)$ and $\tilde{x}(t)$ are solutions on $[t_0, t_1)$ and $[t_0, \tilde{t}_1)$, respectively, then $x(t) = \tilde{x}(t)$ for all $t_0 \le t < \min\{t_1, \tilde{t}_1\}$.

Remark. In the ODE theory, we know that any maximal (i.e. non-continuable) solution leaves any closed, bounded set. This is not the case for DDEs:

Example. [Yorke]. Let $h(t) = \sin(1/t)$, for $t < 0$, and define $F : (-\infty, 0] \times \mathcal{C} \to \mathbb{R}$ by

$$F(t, \psi) = \begin{cases} h'(t)\left[1 + \frac{\|h_t - \psi\|}{t}\right], & \|h_t - \psi\| < -t \\ 0, & \text{elsewhere} \end{cases}$$

One verifies that F is continuous (even locally Lipschitz). Now $h(t)$ solves the DDE $x'(t) = F(t, x_t)$, for $t \in (-1, 0)$. But it cannot be continued to $t = 0$, although it stays in a closed, bounded set $(t, x(t)) \in [-1, 0] \times [-1, 1]$.
This counterexample employs the fact that closed, bounded set in $\mathcal{C}$ need not be compact, hence continuous function(al) is not bounded here. Fortunately enough, for typical DDEs, one has additional better property.

Definition. For $A \subset \mathbb{R}^n$, denote

$$\mathcal{C}_A = \big\{ \psi \in \mathcal{C};\ \psi(s) \in A \quad \forall s \in [-r, 0] \big\}$$

Let $\mathcal{D} \subset \mathbb{R} \times \mathcal{C}$ be open. Then $F : \mathcal{D} \to \mathbb{R}^n$ is called *quasi-bounded* if F is bounded on any subset $[t_0, t_1] \times \mathcal{C}_B \subset \mathcal{D}$, where $B \subset \mathbb{R}^n$ is closed, bounded.

Remark. Assume $\Omega \subset \mathbb{R}^n$ is open, and $f = f(t, x, y) : (a, b) \times \Omega \times \Omega \to \mathbb{R}^n$ is continuous. Then $F(t, \psi) = f(t, \psi(0), \psi(-r))$ or $F(t, \psi(0), \int_0^r k(s)\psi(-s)\, ds)$ is quasi-bounded.
This indeed corresponds to typical examples of DDE of the form

$$\begin{aligned} x'(t) &= f(t, x(t), x(t - r)) \\ x'(t) &= f(t, x(t), \int_0^r k(s) x(t - s)\, ds) \end{aligned}$$

Theorem 3. Let $\Omega \subset \mathbb{R}^n$ be open, let $F(t, \psi) : [t_0, \beta) \times \mathcal{C}_\Omega \to \mathbb{R}^n$ be continuous, quasi-bounded. Let $\phi \in \mathcal{C}_\Omega$ be given.
Then there exist $x(t)$ a maximal (non-continuable) solution to (2.1), (2.2) on $[t_0, t_1)$, for some $t_1 \le \beta$. Moreover, if $t_1 < \beta$, then for arbitrary bounded, closed $B \subset \Omega$ there is $t \in (t_0, t_1)$ such that $x(t) \notin B$.

Corollary. Let $F(t, \psi) : [t_0, \beta) \times \mathcal{C} \to \mathbb{R}^n$ be continuous, and one has the estimate

$$|F(t, \psi)| \le a(t) + b(t)\|\psi\| \tag{2.4}$$

where $a(t)$, $b(t)$ are continuous on $[t_0, \beta)$. Then (2.1), (2.2) is globally solvable, i.e., (any) maximal solution is defined up to $t < \beta$.

3 Linear equation.

By linear DDE we understand

$$x'(t) = L(t, x_t) + h(t) \tag{3.1}$$

where we assume $L(t, \psi) : [t_0, \beta) \times \mathcal{C}$ is continous and moreover, $L(t, \cdot) : \mathcal{C} \to \mathbb{R}^n$ is linear, bounded operator, for any $t \in [t_0, \beta)$ fixed. We will also require

$$|F(t, \psi)| \le m(t)\|\psi\|$$

for all $t \in [t_0, \beta)$, where $m(t)$ is some continuous function. Finally, $h(t) : [t_0, \beta) \to \mathbb{R}^n$ is also continuous. Important special cases are linear homogeneous problem

$$x'(t) = L(t, x_t) \tag{3.2}$$

and linear, homogeneous and autonomous problem, i.e.

$$x'(t) = L(x_t) \tag{3.3}$$

where $L : \mathcal{C} \to \mathbb{R}^n$ is a bounded, linear operator.

Remark. By the Riesz representation theorem, we know that any bounded, linear $L : C([-r, 0], \mathbb{R}) \to \mathbb{R}$ can be written in the form

$$L(\psi) = \int_{-r}^{0} \psi(\theta)\, d\mu(\theta) \tag{3.4}$$

where μ is a unique Borel measure. More generally, a bounded, linear operator $L : \mathcal{C} \to \mathbb{R}^n$ can be written as

$$L(\psi) = \int_{-r}^{0} d\mu(\theta)\psi(\theta)$$

where μ is a matrix-valued measure.

Theorem 4. For any $\phi \in \mathcal{C}$, there is a unique solution to (3.1) on $[t_0, \beta)$, satisfying $x_{t_0} = \phi$.

Definition. For any $s \in [t_0, \beta)$ and $t \geq s - r$, we define the fundamental matrix $U(t, s) \in \mathbb{R}^{n \times n}$ by the relation

$$U(t, s) = \begin{cases} I + \int_s^t L(\theta, U_t(\cdot, s))\, ds & t \geq s \\ 0 \quad s - r \leq t < s \end{cases} \tag{3.5}$$

Equivalently: for $s \in [t_0, \beta)$ fixed, the function $t \mapsto U(t, s)$ solves

$$X'(t) = L(t, X_t) \tag{3.6}$$

on $[s, \infty)$ with the initial condition $X_s = \xi$, where

$$\xi(\theta) = \begin{cases} 0 & s - r \leq \theta < s \\ I & \theta = s \end{cases} \tag{3.7}$$

Note that $U(t, s)$ is indeed well-defined, thanks to the Theorem 4 (in fact, a slight extension of thereof).

Theorem 5. [Variation of constants.] The function

$$z(t) = \int_{t_0}^{t} U(t, s)h(s)\, ds, \qquad t \in [t_0, \beta) \tag{3.8}$$

solves (3.1), together with the zero initial condition, i.e. $z_{t_0} = 0 \in \mathcal{C}$.

Definition. For any $s \in [t_0, \beta)$ and $t \geq s$, we define $T(t,s) : \mathcal{C} \to \mathcal{C}$ the solution operator to the problem (3.2) by $T(t,s)\phi = x_t$, where $x(\cdot)$ is the (unique) solution with the initial condition $x_s = \phi$.

In view of Theorem 5, general solution to (3.1) can be written as

$$x_t = T(t,t_0)\phi + \int_{t_0}^{t} T(t,s)\xi h(s)\, ds \tag{3.9}$$

Note that this is equality in $\mathcal{C}$. One can obtain the solution by $x(t) = x_t(0)$. Observe also that $T(t,s)\xi = U_t(\cdot, s) \in C([-r,0], \mathbb{R}^{n\times n})$.

4 Linear autonomous equation

In this section, we study the linear, autonomous (and homogeneous) equation, i.e.

$$x'(t) = L(x_t) \tag{4.1}$$

We define the solution operator $T(t) : \mathcal{C} \to \mathcal{C}$ by $T(t)\phi := x_t$, with $x(t)$ being the (unique) solution subject to the initial condition $x_0 = \phi$.

Remark. ① We already defined the solution operator $T(t,s)$ for a more general equation (3.2). However, as (4.1) is autonomous, one has $T(t,s) = T(t-s, 0)$ for any $t \geq s \geq 0$. Hence a one-parameter solution operator is enough here.

② Obviously, $T(t)$ are a c_0-semigroup on $\mathcal{C}$. We will investigate the spectrum of its generator, and the consequences to the stability (cf. part I of the lecture) of solutions.

Lemma 1. Generator of $T(t)$ is $(A, \mathcal{D}(A))$, where

$$\mathcal{D}(A) = \left\{\phi \in C^1([-r,0];\mathbb{R}^n);\ \phi'(0) = L(\phi)\right\}$$

and $A : \mathcal{D}(A) \to \mathcal{C}$ is defined by

$$[A(\phi)](\phi) = \begin{cases} \phi'(\theta), & \theta \in [-r, 0) \\ L(\phi), & \theta = 0 \end{cases}$$

Remark. Recall that given operator $(A, \mathcal{D}(A))$, we define the resolvent set

$$\rho(A) = \left\{\lambda \in \mathbb{C};\ \exists(\lambda I - A)^{-1}\text{continuous}\right\}$$

and the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$. One can split

$$\sigma(A) = P\sigma(A) \cup C\sigma(A) \cup R\sigma(A)$$

where $P\sigma(A)$ is the point spectrum (i.e., $(\lambda I - A)$ is not injective $\iff$ there is a non-zero eigenvector); $C\sigma(A)$ is the continuous spectrum (inverse exists but is not bounded) and finally $R\sigma(A)$ is the residual spectrum (i.e. $(\lambda I - A)$ is not onto).

Theorem 6. Let $(A, \mathcal{D}(A))$ be the generator of $T(t)$, let μ be the measure representing L. Then

1. $\sigma(A) = P\sigma(A)$
2. $\lambda \in \sigma(A)$ iff $\det \Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \lambda I - \int_{-r}^{0} e^{\lambda\theta} \, d\mu(\theta) \tag{4.2}$$

Remark. Formally, $\Delta(\lambda)$ arises by plugging $x(t) = e^{\lambda t} I$ into (4.1), exactly as in the ODE situation.

Lemma 2. The solution operator can be written as

$$T(t) = N(t) + K(t)$$

where

1. $N(t)$ satisfies: $N(t)N(s) = N(t+s)$ and $N(t) = 0$ for $t \geq r$.
2. $K(t)$ is compact

Remark. It follows that $T(t)$ is compact for $t \geq r$. Also, by Lemma 1, for $t \geq r$ one has $T(t)\phi \in \mathcal{D}(A)$, i.e., it is a so-called differentiable semigroup. On the other hand, the semigroup is not analytic, since $T(t)\phi \notin \mathcal{D}(A)$ for $t \in (0, r)$ in general.

Theorem 7. Let $(A, \mathcal{D}(A))$ be the generator of $T(t)$, let $\operatorname{Re}\lambda < \beta$ for all $\lambda \in \sigma(A)$. Then there is $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{L}(\mathcal{C})} \leq M e^{\beta t}, \qquad \forall t \geq 0$$

In other words, $\omega(T) \leq s(A)$; cf. Chapter 6 of part I.

Corollary. [Linearized stability.] Consider the problem

$$x'(t) = \int_{-r}^{0} x(t+\theta) \, d\mu(\theta) + g(x_t)$$

μ is a Borel measure and $g : \mathcal{C} \to \mathbb{R}^n$ is continuous, $g(0) = 0$, such that

$$\frac{g(\psi)}{\|\psi\|} \to 0, \qquad \text{for } \|\psi\| \to 0$$

i.e. g is a lower order term. Assume there is $\beta > 0$ such that $\operatorname{Re}\lambda < -\beta$ for all λ satisfying the characteristic equation $\det \Delta(\lambda) = 0$, with $\Delta(\lambda)$ given by (4.2).
Then the zero solution is asymptotically stable.

Question. Is DDE a finite or infinite-dimensional problem? Consider $x'(t) = x(t-1)$. Then the characteristic equation $\lambda - e^{-\lambda}$ has infinitely many roots. Hence, there are infinitely many linearly independent solutions.
On the other hand, there are only finitely many roots with $\operatorname{Re}\lambda \geq 0$, each having finite multiplicity. Consequently, one can write

$$\mathcal{C} = X_0 \oplus Y$$

such that X_0 is finite-dimensional (effectively, the DDE reduces to an ODE here), and $T(t)$ decays exponentially on Y. Hence the problem is asymptotically finite-dimensional.