NMMA440 - PART TWO

"ODEs with DELAY"

ODEs

- ordinary: $x' = f(x)$ $\left[x'(t) = f(x(t)) \right]$
- delayed: $x'(t) = f(x(t), x(t-r))$

$$\int_0^r k(s)\, x(t-s)\, ds$$

etc....

Remark: ODEs in Banach space

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1 - Examples

① logistic eq. (Verhulst)

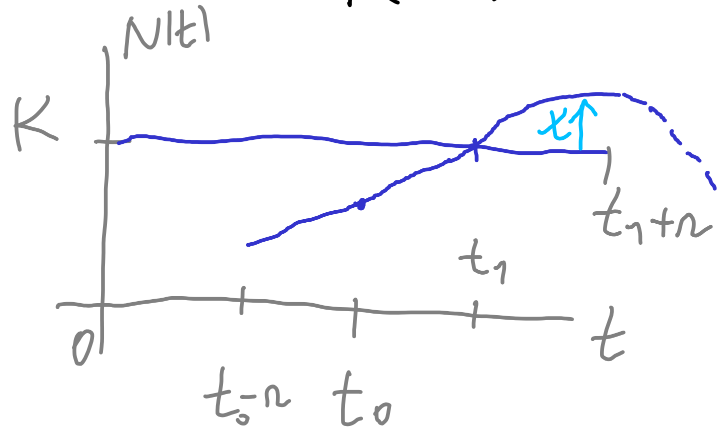
$$N'(t) = a\left(1 - \frac{N(t)}{K}\right) N(t)$$

$N(t)$ .... population

$a > 0$ ... growth rate

$K > 0$ ... capacity of environment

with delay: $N'(t) = a\left(1 - \frac{N(t-r)}{K}\right)N(t)$



Is $N(t) \equiv K$ stable? set: $N(t) = K\left(1 + x\left(\frac{t}{r}\right)\right)$, $\frac{t}{r} = s$

$x = x(s)$

$\Rightarrow$ $x'(s) = -c\, x(s-1)\,[1 + x(s)]$

$c = a \cdot r$

$\Updownarrow$

Is $x(t) \equiv 0$ stable? Not in general.

Pf. linearize $\Rightarrow$ $x'(t) = -c\, x(t-1)$
(justify later)

$\exists$ unbd. solns HW1

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② delayed SIR model

$S' = -\beta S I$

$I' = \beta S I - \alpha I$

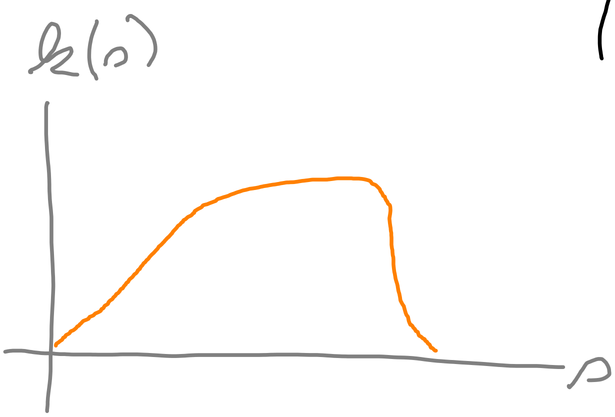
$S(t)$ ... susceptible

$I(t)$ ... infected

with delay:

(i) $-\beta S(t) I(t-\tau)$

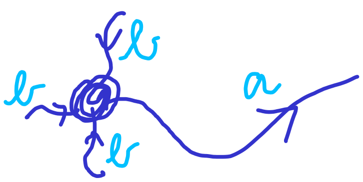
(ii) $-\beta S(t) \int_0^{\tau} k(s) I(t-s) \, ds$



③ neural field: (contin.)

$$\boxed{\partial_t u = a \cdot u + b \cdot \partial_{xx} u} \quad (1)$$

$u = u(t,x)$ ... electric pot.



delayed variant:

$$(2) \quad \boxed{\partial_t u(t,x) = a\, u(t-\tau, x) + b\, \partial_{xx} u(t-\tau, x)}$$

Claim: (2) has richer dynamics than (1)

HW1

4 Prime number theorem (1800 ~ 1900)

Set: $\pi(x) = \#\{p \leq x \,;\, p \text{ prime}\}$

Then: $\pi(x) \sim \dfrac{x}{\log(x)}, \quad x \to +\infty$

More precisely: $\displaystyle\lim_{x \to +\infty} \frac{\pi(x)}{x/\log(x)} = 1 \quad (*)$

"Proof" (Cherwell, de Visme 1950)

assume: $\exists\, y(x) \in C^1$ s.t. $y(x) \sim \pi(x)$ $(x \to +\infty)$

aim: $y'(x) \sim \dfrac{1}{\log x}$ $\quad x \to +\infty$ $\quad$ (l'Hospital $\Rightarrow (*)$)

behavior of $y'(x)$ ... ?

$$\underbrace{\pi(m) - \pi(n)}_{\#\{p \in (n,m] \,;\, p \text{ prime}\}} \simeq y(m) - y(n) = \int_n^m \underline{\underline{y'(x)}}\,dx$$

density of primes (= probability of being prime) (close to x)

$$\text{Prob}\{2 \mid x\} = \frac{1}{2}$$

$$\left(x \in \mathbb{N} \text{ fixed} \right)$$

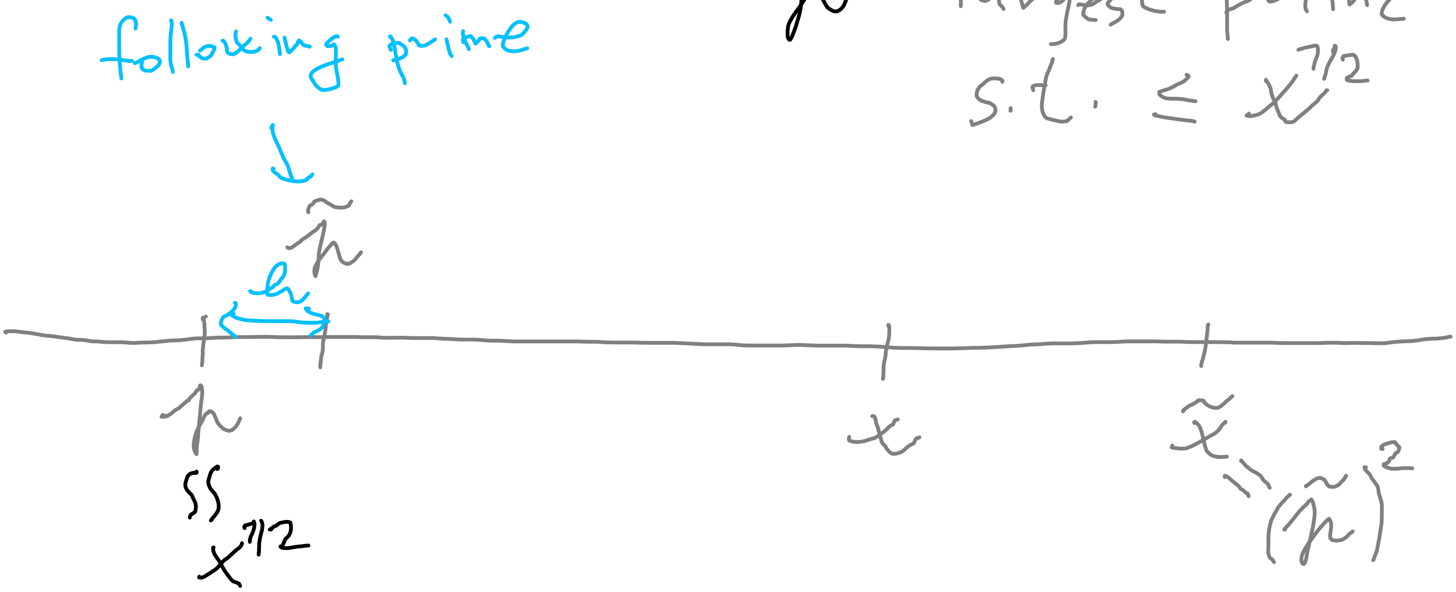
$$\text{Prob}\{3 \mid x\} = \frac{1}{3}$$

$$\vdots$$

$$\text{Prob}\{p \mid x\} = \frac{1}{p}$$

$$\text{Prob}\{x \text{ prime}\} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{p}\right)$$

p ... largest prime s.t. $\leq x^{1/2}$



$$(1) \quad \tilde{p} = p + h \approx p + \underbrace{\frac{1}{y'(x^{1/2})}}_{\text{average gap}}$$

$$(2) \quad y'(\tilde{x}) = \left(1 - \frac{1}{x^{1/2}}\right) y'(x)$$

Taylor: $\tilde{x} = (x^{1/2} + h)^2$, $h \approx \dfrac{1}{y'(x^{1/2})}$

$$\Rightarrow \quad y'(\tilde{x}) = y'(x) + 2h\, x^{1/2}\, y''(x)$$

finally:

$$\boxed{2x\,\frac{y''(x)}{y'(x)} + y'(x^{1/2}) = 0}$$

set:

$$\log x = 2^t \qquad \left(x = e^{(2^t)}\right)$$

$$r(x)\log x = 1 + u(t)$$

(where $r(x) = y'(x)$)

$$\Rightarrow \quad u'(t) = -\alpha\, u(t-1)\left[1 + u(t)\right] \qquad (**)$$

where $\alpha = \log 2 \in (0,1)$

aim: $r(x) \sim \dfrac{1}{\log(x)}$, $x \to +\infty$ $\iff$ $u(t) \to 0$, $t \to +\infty$

we will prove (by linearization)