# Chapter XI

# Linear Second Order Equations

## 1. Preliminaries

One of the most frequently occurring types of differential equations in mathematics and the physical sciences is the linear second order differential equation of the form

$$u'' + g(t)u' + f(t)u = h(t) \tag{1.1}$$

or of the form

$$(p(t)u')' + q(t)u = h(t). \tag{1.2}$$

Unless otherwise specified, it is assumed that the functions $f(t)$, $g(t)$, $h(t)$, and $p(t) \neq 0$, $q(t)$ in these equations are continuous (real- or complex-valued) functions on some $t$-interval $J$, which can be bounded or unbounded. The reason for the assumption $p(t) \neq 0$ will soon become clear.

Of the two forms (1.1) and (1.2), the latter is the more general since (1.1) can be written as

$$(p(t)u')' + p(t)f(t)u = p(t)h(t), \tag{1.3}$$

if $p(t)$ is defined as

$$p(t) = \exp \int_a^t g(s)\, ds \tag{1.4}$$

for some $a \in J$. As a partial converse, note that if $p(t)$ is continuously differentiable then (1.2) can be written as

$$u'' + \frac{p'(t)}{p(t)} u' + \frac{q(t)}{p(t)} u = \frac{h(t)}{p(t)},$$

which is of the form (1.1).

When the function $p(t)$ is continuous but does not have a continuous derivative, (1.2) cannot be written in the form (1.1). In this case, (1.2) is to be interpreted as the first order, linear system for the binary vector

$x = (x^1, x^2) \equiv (u, p(t)u'),$

$$x^{1\prime} = \frac{x^2}{p(t)}, \qquad x^{2\prime} = -q(t)x^1 + h(t). \tag{1.5}$$

In other words, a solution $u = u(t)$ of (1.2) is a continuously differentiable function such that $p(t)u'(t)$ has a continuous derivative satisfying (1.2). When $p(t) \neq 0$, $q(t)$, $h(t)$ are continuous, the standard existence and uniqueness theorems for linear systems of § IV 1 are applicable to (1.5), hence (1.2). [We can also deal with more general (i.e., less smooth) types of solutions if it is only assumed, e.g., that $1/p(t)$, $q(t)$, $h(t)$ are locally integrable; cf. Exercise IV 1.2.]

The particular case of (1.2) where $p(t) \equiv 1$ is

$$u'' + q(t)u = h(t). \tag{1.6}$$

When $p(t) \neq 0$ is *real-valued*, (1.2) can be reduced to this form by the change of independent variables

$$ds = \frac{dt}{p(t)}, \qquad \text{i.e.,} \quad s = \int_a^t \frac{dr}{p(r)} + \text{const.} \tag{1.7}$$

for some $a \in J$. The function $s = s(t)$ has a derivative $ds/dt = 1/p(t) \neq 0$ and is therefore strictly monotone. Hence $s = s(t)$ has an inverse function $t = t(s)$ defined on some $s$-interval. In terms of the new independent variable $s$, the equation (1.2) becomes

$$\frac{d^2u}{ds^2} + p(t)q(t)u = p(t)h(t), \tag{1.8}$$

where $t$ in $p(t)q(t)$ and $p(t)h(t)$ is replaced by the function $t = t(s)$. The equation (1.8) is of the type (1.6).

If $g(t)$ has a continuous derivative, then (1.1) can be reduced to an equation of the form (1.6) also by a change of the dependent variable $u \to z$ defined by

$$u = z \exp\left(-\tfrac{1}{2}\int_a^t g(s)\, ds\right) \tag{1.9}$$

for some $a \in J$. In fact, substitution of (1.9) into (1.1) leads to the equation

$$z'' + \left[f(t) - \frac{g^2(t)}{4} - \frac{g'(t)}{2}\right]z = h(t) \exp \tfrac{1}{2}\int_a^t g(s)\, ds, \tag{1.10}$$

which is of the type (1.6).

In view of the preceding discussion, the second order equations to be considered will generally be assumed to be of the form (1.2) or (1.6). The following exercises will often be mentioned.

*Exercise* 1.1. (*a*) The simplest equations of the type considered in this chapter are

$$(1.11) \qquad u'' = 0, \qquad u'' - \sigma^2 u = 0, \qquad u'' + \sigma^2 u = 0,$$

where $\sigma \neq 0$ is a constant. Verify that the general solution of these equations is

$$(1.12) \quad u = c_1 + c_2 t, \qquad u = c_1 e^{\sigma t} + c_2 e^{-\sigma t}, \qquad u = c_1 \cos \sigma t + c_2 \sin \sigma t,$$

respectively. (*b*) Let $a, b$ be constants. Show that $u = e^{\lambda t}$ is a solution of

$$(1.13) \qquad u'' + bu' + au = 0,$$

if and only if $\lambda$ satisfies

$$(1.14) \qquad \lambda^2 + b\lambda + a = 0.$$

Actually, the substitution $u = ze^{-bt/2}$ [cf. (1.9)] reduces (1.13) to

$$z'' + \sigma^2 z = 0, \qquad \sigma^2 = a - \tfrac{1}{4}b^2.$$

Hence by (*a*) the general solution of (1.13) is

$$(1.15) \qquad u = e^{-bt/2}(c_1 + c_2 t) \qquad \text{or} \quad u = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

according as (1.14) has a double root $\lambda = \frac{1}{2}b$ or distinct roots $\lambda_1, \lambda_2 = -\frac{1}{2}b \pm (\frac{1}{4}b^2 - a)^{1/2}$. When $a$, $b$ are real and $\frac{1}{4}b^2 - a < 0$, nonreal exponents in the last part of (1.15) can be avoided by writing

$$(1.16) \qquad u = e^{-bt/2}[c_1 \cos (a - \tfrac{1}{4}b^2)^{1/2} t + c_2 \sin (a - \tfrac{1}{4}b^2)^{1/2} t].$$

(*c*) Let $\mu$ be a constant. Show that $u = t^\lambda$ is a solution of

$$(1.17) \qquad u'' + \frac{\mu}{t^2} u = 0$$

if and only if $\lambda$ satisfies

$$(1.18) \qquad \lambda(\lambda - 1) + \mu = 0, \qquad \text{i.e.,} \quad \lambda = \tfrac{1}{2} \pm (\tfrac{1}{4} - \mu)^{1/2}.$$

Thus if $\mu \neq \frac{1}{4}$, the general solution of (1.17) is

$$(1.19) \quad u = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}, \qquad \mu \neq \tfrac{1}{4}, \qquad \text{and} \quad \lambda_1, \lambda_2 = \tfrac{1}{2} \pm (\tfrac{1}{4} - \mu)^{1/2}.$$

If $\mu$ is real and $\mu > \frac{1}{4}$, the nonreal exponents can be avoided by writing

$$(1.20) \qquad u = t^{1/2}[c_1 \cos (\mu - \tfrac{1}{4})^{1/2} \log t + c_2 \sin (\mu - \tfrac{1}{4})^{1/2} \log t].$$

Actually, the change of variables $u = t^{1/2}z$ and $t = e^s$ transforms (1.17) into

$$(1.21) \qquad \frac{d^2 z}{ds^2} + (\mu - \tfrac{1}{4})z = 0.$$

Thus by (*a*) the general solution of (1.17) is

$$u = t^{1/2}(c_1 + c_2 \log t) \qquad \text{or} \qquad u = c_1 t^{\lambda_1} + c_2 t^{\lambda_2} \tag{1.22}$$

according as $\mu = \frac{1}{4}$ or $\mu \neq \frac{1}{4}$.

*Exercise* 1.2. Consider the differential equation

$$u'' + q(t)u = 0. \tag{1.23}$$

The change of variables

$$t = e^s \qquad \text{and} \qquad u = t^{1/2}z \tag{1.24}$$

transforms (1.23) into

$$\frac{d^2z}{ds^2} + t^2\left[q(t) - \frac{1}{4t^2}\right]z = 0, \qquad \text{where} \quad t = e^s. \tag{1.25}$$

For a given constant $\mu$, consider the sequence of functions

$$q_0 = \mu - \tfrac{1}{4}, \qquad q_1(t) = \mu t^{-2}, \qquad q_2(t) = t^{-2}(\tfrac{1}{4} + \mu \log^{-2} t), \ldots,$$

defined by $t^2[q_n(t) - 1/4t^2] = q_{n-1}(s)$ if $t = e^s$, so that $q_n(t) = t^{-2}[\frac{1}{4} + q_{n-1}(\log t)]$ or

$$q_n(t) = t^{-2}\left[\tfrac{1}{4}\sum_{k=0}^{n-2}\left(\prod_{j=1}^{k}\log_j t\right)^{-2} + \mu\left(\prod_{j=1}^{n-1}\log_j t\right)^{-2}\right] \qquad \text{for} \quad n \geqq 1,$$

$\log_1 t = \log t$, $\log_j t = \log(\log_{j-1} t)$, and the empty product is 1. If $q(t) = q_n(t)$, $n > 0$, in (1.23), then the change of variables (1.24) reduces (1.23) to the case where $t, q_n(t)$ are replaced by $s, q_{n-1}(s)$. In particular, if $\mu$ is real and $q = q_n(t)$, $n \geqq 0$, then real-valued solutions of (1.23) have infinitely many zeros for large $t > 0$ if and only if $\mu > \frac{1}{4}$.

## 2. Basic Facts

Before considering more complicated matters, it is well to point out the consequences of Chapter IV (in particular, § IV 8) for the homogeneous and inhomogeneous equation

$$(p(t)u')' + q(t)u = 0, \tag{2.1}$$

$$(p(t)w')' + q(t)w = h(t). \tag{2.2}$$

To this end, the scalar equations (2.1) or (2.2) can be written as the binary vector equations

$$x' = A(t)x, \tag{2.3}$$

$$y' = A(t)y + \begin{pmatrix} 0 \\ h(t) \end{pmatrix}, \tag{2.4}$$

where $x = (x^1, x^2), y = (y^1, y^2)$ are the vectors $x = (u, p(t)u'), y = (w, p(t)w')$ and $A(t)$ is the $2 \times 2$ matrix

$$(2.5) \qquad A(t) = \begin{pmatrix} 0 & \dfrac{1}{p(t)} \\ -q(t) & 0 \end{pmatrix}.$$

Unless the contrary is stated, it is assumed that $p(t) \neq 0$, $q(t), h(t)$, and other coefficient functions are continuous, complex-valued functions on a $t$-interval $J$ (which may or may not be closed and/or bounded).

(i) If $t_0 \in J$ and $u_0$, $u_0'$ are arbitrary complex numbers, then the initial value problem (2.2) and

$$(2.6) \qquad w(t_0) = u_0, \qquad w'(t_0) = u_0'$$

has a unique solution which exists on all of $J$; Lemma IV 1.1.

(ii) In the particular case (2.1) of (2.2) and $u_0 = u_0' = 0$, the corresponding unique solution is $u(t) \equiv 0$. Hence, if $u(t) \not\equiv 0$ is a solution of (2.1), then the zeros of $u(t)$ cannot have a cluster point in $J$.

(iii) *Superposition Principles.* If $u(t), v(t)$ are solutions of (2.1) and $c_1$, $c_2$ are constants, then $c_1u(t) + c_2v(t)$ is a solution of (2.1). If $w_0(t)$ is a solution of (2.2), then $w_1(t)$ is also a solution of (2.2) if and only if $u = w_1(t) - w_0(t)$ is a solution of (2.1).

(iv) If $u(t)$, $v(t)$ are solutions of (2.1), then the corresponding vector solutions $x = (u(t), p(t)u'(t)), (v(t), p(t)v'(t))$ of (2.3) are linearly independent (at every value of $t$) if and only if $u(t)$, $v(t)$ are linearly independent in the sense that if $c_1$, $c_2$ are constants such that $c_1u(t) + c_2v(t) \equiv 0$, then $c_1 = c_2 = 0$; cf. § IV 8(iii).

(v) If $u(t), v(t)$ are solutions of (2.1), then there is a constant $c$, depending on $u(t)$ and $v(t)$, such that their Wronskian $W(t) = W(t; u, v)$ satisfies

$$(2.7) \qquad u(t)v'(t) - u'(t)v(t) = \frac{c}{p(t)}.$$

This follows from Theorem IV 1.2 since a solution matrix for (2.3) is

$$X(t) = \begin{pmatrix} u(t) & v(t) \\ p(t)u'(t) & p(t)v'(t) \end{pmatrix},$$

$\det X(t) = p(t)W(t)$ and $\operatorname{tr} A(t) = 0$; cf. § IV 8(iv). A simple direct proof is contained in the following paragraph.

(vi) *Lagrange Identity.* Consider the pair of relations

$$(2.8) \qquad (pu')' + qu = f, \qquad (pv')' + qv = g,$$

where $f = f(t)$, $g = g(t)$ are continuous functions on $J$. If the second is multiplied by $u$, the first by $v$, and the results subtracted, it follows that

$$(2.9) \qquad [p(uv' - u'v)]' = gu - fv$$

since $[p(uv' - u'v)]' = u(pv')' - v(pu')'$. The relation (2.9) is called the *Lagrange identity*. Its integrated form

$$(2.10) \qquad [p(uv' - u'v)]_a^t = \int_a^t (gu - fv)\, ds,$$

where $[a, t] \subset J$, is called *Green's formula*.

**(vii)** In particular, (v) shows that $u(t)$ and $v(t)$ are linearly independent solutions of (2.1) if and only if $c \neq 0$ in (2.7). In this case every solution of (2.1) is a linear combination $c_1u(t) + c_2v(t)$ of $u(t)$, $v(t)$ with constant coefficients.

**(viii)** If $p(t) \equiv$ const. [e.g., $p(t) \equiv 1$], the Wronskian of any pair of solutions $u(t)$, $v(t)$ of (2.1) is a constant.

**(ix)** According to the general theory of § IV 3, if one solution of $u(t) \not\equiv 0$ of (2.1) is known, the determination (at least, locally) of other solutions $v(t)$ of (2.1) are obtained by considering a certain scalar differential equation of first order. If $u(t) \neq 0$ on a subinterval $J'$ of $J$, the differential equation in question is (2.7), where $u$ is considered known and $v$ unknown. If (2.7) is divided by $u^2(t)$, the equation becomes

$$(2.11) \qquad \left(\frac{v}{u}\right)' = \frac{c}{p(t)u^2(t)}$$

and a quadrature gives

$$(2.12) \qquad v(t) = c_1u(t) + cu(t)\int_a^t \frac{ds}{p(s)u^2(s)},$$

if $a, t \in J'$; cf. § IV 8(iv). It is readily verified that if $c_1$, $c$ are arbitrary constants and $a, t \in J'$, then (2.12) is a solution of (2.1) satisfying (2.7) on any interval $J'$ where $u(t) \neq 0$.

**(x)** Let $u(t)$, $v(t)$ be solutions of (2.1) satisfying (2.7) with $c \neq 0$. For a fixed $s \in J$, the solution of (2.1) satisfying the initial conditions $u(s) = 0$, $p(s)u'(s) = 1$ is $c^{-1}[u(s)v(t) - u(t)v(s)]$. Hence the solution of (2.2) satisfying $w(t_0) = w'(t_0) = 0$ is

$$(2.13) \qquad w(t) = c^{-1}\int_{t_0}^t [u(s)v(t) - u(t)v(s)]h(s)\, ds;$$

cf. § IV 8(v) (or, more simply, verify this directly). The general solution of (2.2) is obtained by adding a general solution $c_1u(t) + c_2v(t)$ of (2.1) to (2.13) to give

$$(2.14) \quad w(t) = u(t)\left[c_1 - c^{-1}\int_{t_0}^t v(s)h(s)\, ds\right] + v(t)\left[c_2 + c^{-1}\int_{t_0}^t u(s)h(s)\, ds\right].$$

If the closed bounded interval $[a, b]$ is contained in $J$, then the choice

$$t_0 = a, \qquad c_1 = c^{-1}\int_a^b v(s)h(s)\,ds \qquad \text{and} \qquad c_2 = 0$$

reduces (2.14) to the particular solution

$$w(t) = c^{-1}\left[v(t)\int_a^t u(s)h(s)\,ds + u(t)\int_t^b v(s)h(s)\,ds\right]. \tag{2.15}$$

This can be written in the form

$$w(t) = \int_a^b G(t, s)h(s)\,ds, \tag{2.16}$$

where

$$\begin{aligned} G(t, s) &= c^{-1}v(t)u(s) \qquad \text{if} \quad a \leqq s \leqq t, \\ G(t, s) &= c^{-1}u(t)v(s) \qquad \text{if} \quad t \leqq s \leqq b. \end{aligned} \tag{2.17}$$

*Remark.* If $h(t)$ is (not necessarily continuous but) integrable over $[a, b]$, then $w(t)$ is a "solution" of (2.2) in the sense that $w(t)$ has a continuous derivative $w'$ such that $p(t)w'(t)$ is absolutely continuous and (2.2) holds except on a $t$-set of measure 0.

*Exercise* 2.1 Verify that if $\alpha$, $\beta$, $\gamma$, $\delta$ are constants such that

$$\alpha u(a) + \beta p(a)u'(a) = 0, \qquad \gamma v(b) + \delta p(b)v'(b) = 0,$$

then the particular solution (2.15) of (2.2) satisfies

$$\alpha w(a) + \beta p(a)w'(a) = 0, \qquad \gamma w(b) + \delta p(b)w'(b) = 0.$$

An extremely simple but important case occurs if $p \equiv 1$, $q \equiv 0$ so that (2.1) becomes $u'' = 0$. Then $u(t) = t - a$ and $v(t) = b - t$ are the solutions of (2.1) satisfying $u(a) = 0$, $v(b) = 0$, and (2.7) with $c = a - b$. Hence

$$w(t) = \frac{1}{a - b}\left[(b - t)\int_a^t (s - a)h(s)\,ds + (t - a)\int_t^b (b - s)h(s)\,ds\right] \tag{2.18}$$

is the solution of $w'' = h(t)$ satisfying $w(a) = w(b) = 0$.

*Exercise* 2.2. Let $[a, b] \subset J$. Show that most general function $G(t, s)$ defined for $a \leqq s, t \leqq b$ for which (2.16) is a solution of (2.2) for $a \leqq t \leqq b$ for every continuous function $h(t)$ is given by

$$G(t, s) = c^{-1}\sum_{k=1}^{2}\sum_{j=1}^{2} a_{jk}u_j(t)u_k(s) \qquad \text{if} \quad a \leqq s \leqq t,$$

$$G(t, s) = c^{-1}\sum_{k=1}^{2}\sum_{j=1}^{2} b_{jk}u_j(t)u_k(s) \qquad \text{if} \quad t \leqq s \leqq b,$$

where $A = (a_{jk})$, $B = (b_{jk})$ are constant matrices such that

$$B - A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $u_1 = u(t)$, $u_2 = v(t)$ are solutions of (2.1) satisfying (2.7) with $c \neq 0$. In this case, $G(t, s)$ is continuous for $a \leqq s, t \leqq b$.

*Exercise* 2.3. Let $a$ (and/or $b$) be a possibly infinite end point of $J$ which does not belong to $J$, so that $p(t)$, $q(t)$, $h(t)$ and $u(t)$, $v(t)$ need not have limits as $t \to a + 0$ (and/or $t \to b - 0$). Suppose, however, that $h, u, v$ have the property that the integrals in (2.15) are convergent (possibly, just conditionally). Then (2.15) is a solution of (2.2) on $J$. [This follows from the derivation of (2.15) or can be verified directly by substituting (2.15) into (2.2).]

**(xi)** *Variation of Constants.* In addition to (2.1), consider another equation

$$(2.19) \qquad (p_0(t)w')' + q_0(t)w = 0,$$

where $p_0(t) \neq 0$, $q_0(t)$ are also continuous in $J$. Correspondingly, (2.19) is equivalent to a first order system

$$(2.20) \qquad y' = A_0(t)y,$$

where

$$(2.21) \qquad y = (u, p_0(t)u') \quad \text{and} \quad A_0(t) = \begin{pmatrix} 0 & 1/p_0(t) \\ -q_0(t) & 0 \end{pmatrix}.$$

Let $u_0(t)$, $v_0(t)$ be linearly independent solutions of (2.19) such that

$$(2.22) \qquad Y(t) = \begin{pmatrix} u_0 & v_0 \\ p_0u_0' & p_0v_0' \end{pmatrix}$$

is a fundamental matrix for (2.20) with det $Y(t) \equiv 1$; i.e.,

$$p_0(u_0v_0' - u_0'v_0) = 1.$$

Hence

$$(2.23) \qquad Y^{-1}(t) = \begin{pmatrix} p_0v_0' & -v_0 \\ -p_0u_0' & u_0 \end{pmatrix}.$$

Consider the linear change of variables

$$(2.24) \qquad x = Y(t)y = \begin{pmatrix} u_0y^1 + v_0y^2 \\ p_0u_0'y^1 + p_0v_0'y^2 \end{pmatrix}$$

for the system (2.3). The resulting differential equation for the vector $y$ is

$$(2.25) \quad y' = C(t)y, \qquad \text{where} \quad C(t) = Y^{-1}(t)[A(t) - A_0(t)]Y(t);$$

cf. Theorem IV 2.1. A direct calculation using (2.5), (2.21), (2.22), and (2.23) shows that

$$(2.26)\quad C(t) = \left(\frac{1}{p} - \frac{1}{p_0}\right)p_0^2\begin{pmatrix} u_0'v_0' & v_0'^2 \\ -u_0'^2 & -u_0'v_0' \end{pmatrix} + (q - q_0)\begin{pmatrix} u_0v_0 & v_0^2 \\ -u_0^2 & -u_0v_0 \end{pmatrix}.$$

In the particular case, $p_0(t) = p(t)$, so that (2.19) reduces to

$$(2.27)\qquad (pw')' + q_0w = 0,$$

the *matrix* $C(t)$ *depends on* $u_0(t)$, $v_0(t)$ *but not on their derivatives.* Here, (2.1) or equivalently (2.3) is reduced to the binary system

$$(2.28)\qquad y' = (q - q_0)\begin{pmatrix} u_0v_0 & v_0^2 \\ -u_0^2 & -u_0v_0 \end{pmatrix}y.$$

*Exercise* 2.4. In order to interpret the significance of $y$, i.e., of the components $y^1$, $y^2$ of $y$ in (2.28) for a corresponding solution $u(t)$ of (2.1), write (2.1) as $(pw')' + q_0w = h(t)$, where $w = u(t)$, $h = [q_0(t) - q(t)]u(t)$. Then it is seen that the solution $u(t)$ of (2.1) is of the form (2.14) if $c = 1$ and $u(t)$, $v(t)$ are replaced by $u_0(t)$, $v_0(t)$. Using (2.24), where $p = p_0$ and $x$ is the binary vector $(u(t), p(t)u'(t))$, show that the coefficients of $u_0(t)$, $v_0(t)$ in this analogue of formula (2.14) are the component $y^1, y^2$ of the corresponding solution $y(t)$ of (2.28).

**(xii)** If we know a particular solution $u_0(t)$ of (2.27) which does not vanish on $J$, then we can determine linearly independent solutions by a quadrature [cf. (ix)] and hence obtain the matrix in (2.28). Actually this desired result can be obtained much more directly. Let (2.27) have a solution $w(t) \neq 0$ on the interval $J$. Change the dependent variable from $u$ to $z$ in (2.1), where

$$(2.29)\qquad u = w(t)z.$$

The differential equation satisfied by $z$ is

$$w(pz')' + 2pz'w' + [(pw')' + qw]z = 0.$$

If this is multiplied by $w$, it follows that

$$(2.30)\qquad (pw^2z')' + w[(pw')' + qw]z = 0$$

or, by (2.27),

$$(2.31)\qquad (pw^2z')' + w^2(q - q_0)z = 0;$$

i.e., (2.29) reduces (2.1) to (2.30) or (2.31). Instead of starting with a differential equation (2.27) and a solution $w(t)$, we can start with a function $w(t) \neq 0$ such that $w(t)$ has a continuous derivative $w'(t)$ and $p(t)w'(t)$ has a continuous derivative, in which case $q_0(t)$ is defined by (2.27), so $q_0 = -(pw')'/w$. The substitution (2.29) will also be called a *variation of constants.*

**(xiii)** *Liouville Substitution.* As a particular case, consider (2.1) with $p(t) \equiv 1$,

$$u'' + q(t)u = 0. \tag{2.32}$$

Suppose that $q(t)$ has a continuous second derivative, is real-valued, and does not vanish, say

$$\pm q(t) > 0, \qquad \text{where} \quad \pm = \operatorname{sgn} q(t) \tag{2.33}$$

is independent of $t$. Consider the variation of constants

$$u = w(t)z, \qquad \text{where} \quad w = |q(t)|^{-1/4} > 0. \tag{2.34}$$

Then (2.32) is reduced to (2.30), where $p \equiv 1$, i.e., to

$$(|q|^{-1/2} z')' \pm \left(|q|^{1/2} - \frac{q''}{4\,|q|^{3/2}} \pm \frac{5q'^2}{16\,|q|^{5/2}}\right) z = 0. \tag{2.35}$$

A change of independent variables $t \to s$ defined by

$$ds = \frac{dt}{|q|^{1/2}} \tag{2.36}$$

transforms (2.35) into

$$\frac{d^2 z}{ds^2} \pm f(s)z = 0, \tag{2.37}$$

where

$$f(s) = 1 - \frac{q''}{4\,|q|^2} \pm \frac{5q'^2}{16\,|q|^3} \tag{2.38}$$

and the argument of $q$ and its derivatives in (2.38) is $t = t(s)$, the inverse of the function $s = s(t)$ defined by (2.36) and a quadrature; cf. (1.7). In these formulae, a prime denotes differentiation with respect to $t$, so that $q' = dq/dt$.

The change of variables (2.34), (2.36) is the Liouville substitution. This substitution, or repeated applications of it, often leads to a differential equation of the type (2.37) in which $f(s)$ is "nearly" constant; cf. Exercise 8.3. For a simple extreme case of this remark, see Exercise 1.1($c$).

**(xiv)** *Riccati Equations.* Paragraphs (xi), (xii), and (xiii) concern the transformation of (2.1) into a different second order linear equation or into a suitable binary, first order linear system. (Other such transformations will be utilized later; cf. §§ 8–9.) Frequently, it is useful to transform (2.1) into a suitable *nonlinear* equation or system. In this direction, one of the most widely used devices is the following: Let

$$r = \frac{p(t)u'}{u}, \tag{2.39}$$

so that $r' = (pu')'/u - p^{-1}(pu'/u)^2$. Thus, if (2.1) is divided by $u$, the result can be written as

$$r' + \frac{r^2}{p(t)} + q(t)r = 0. \tag{2.40}$$

This is called the Riccati equation of (2.1). (In general, a differential equation of the form $r' = a(t)r^2 + b(t)r + c(t)$, where the right side is a quadratic polynomial in $r$, is called a *Riccati differential equation.*)

It will be left to the reader to verify that if $u(t)$ is a solution of (2.1) which does not vanish on a $t$-interval $J'$ ($\subset J$), then (2.39) is a solution of (2.40) on $J'$; conversely if $r = r(t)$ is a solution of (2.40) on a $t$-interval $J'$ ($\subset J$), then a quadrature of (2.39) gives

$$u = c \exp \int^t \frac{r(s)\, ds}{p(s)}, \tag{2.41}$$

a nonvanishing solution of (2.1) on $J'$.

*Exercise* 2.5. Verify that the substitution $r = u'/u$ transforms

$$u'' + g(t)u' + f(t)u = 0$$

into the Riccati equation

$$r' + r^2 + g(t)r + f(t) = 0.$$

**(xv)** *Prüfer Transformation.* In the case of an equation (2.1) with real-valued coefficients, the following transformation of (2.1) is often useful (cf. §§ 3, 5): Let $u = u(t) \not\equiv 0$ be a real-valued solution of (2.1) and let

$$\rho = (u^2 + p^2u'^2)^{1/2} > 0, \qquad \varphi = \arc\tan \frac{u}{pu'}. \tag{2.42}$$

Since $u$ and $u'$ cannot vanish simultaneously a suitable choice of $\varphi$ at some fixed point $t_0 \in J$ and the last part of (2.42) determine a continuously differentiable function $\varphi(t)$. The relations (2.42) transform (2.1) into

$$\varphi' = \frac{1}{p(t)} \cos^2 \varphi + q(t) \sin^2 \varphi, \tag{2.43}$$

$$\rho' = -\left[q(t) - \frac{1}{p(t)}\right]\rho \sin \varphi \cos \varphi. \tag{2.44}$$

The equation (2.43) involves only the one unknown function $\varphi$. If a solution $\varphi = \varphi(t)$ of (2.43) is known, a corresponding solution of (2.44) is obtained by a quadrature.

An advantage of (2.43) over (2.40) is that any solution of (2.43) exists on the whole interval $J$ where $p, q$ are continuous. This is clear from the relation between solutions of (2.1) and (2.43).

*Exercise* 2.6. Verify that if $\tau(t) > 0$ is continuous on $J$ and is locally of bounded variation (i.e., is of bounded variation on all closed, bounded subintervals of $J$) and if $u = u(t) \not\equiv 0$ is a real-valued solution of (2.1), then

$$\rho = (\tau^2u^2 + p^2u'^2)^{1/2} > 0, \qquad \varphi = \arc\tan \frac{\tau u}{pu'} \tag{2.45}$$

and a choice of $\varphi(t_0)$ for some $t_0 \in J$ determine continuous functions $\rho(t)$, $\varphi(t)$ which are locally of bounded variation and

$$d\varphi = \left(\frac{\tau}{p}\cos^2\varphi + \frac{q}{\tau}\sin^2\varphi\right) dt + (\sin\varphi\cos\varphi)\, d(\log\tau) \tag{2.46}$$

$$d(\log\rho) = -\left[\left(\frac{q}{\tau} - \frac{\tau}{p}\right)\sin\varphi\cos\varphi\right] dt + (\sin^2\varphi)\, d(\log\tau), \tag{2.47}$$

The relations (2.46), (2.47) are understood to mean that Riemann, Stieltjes integrals of both sides of these relations are equal. Conversely (continuous) solutions of (2.46)–(2.47) determine solutions of (2.1), via (2.45). Note that if $q(t) > 0$, $p(t) > 0$, and $q(t)p(t)$ is locally of bounded variation, then the choice $\tau(t) = p^{1/2}(t)q^{1/2}(t) > 0$ gives $q/\tau = \tau/p = p^{1/2}/q^{1/2}$ and reduces (2.45) and (2.46), (2.47) to

$$\rho = (pqu^2 + p^2u'^2)^{1/2} > 0, \qquad \varphi = \arc\tan\frac{q^{1/2}u}{p^{1/2}u'}, \tag{2.48}$$

and

$$d\varphi = \frac{q^{1/2}}{p^{1/2}}dt + (\tfrac{1}{2}\sin\varphi\cos\varphi)\, d(\log pq), \tag{2.49}$$

$$d(\log\rho) = (\tfrac{1}{2}\sin^2\varphi)\, d(\log pq). \tag{2.50}$$

## 3. Theorems of Sturm

In this section, we will consider only differential equations of the type (2.1) having real-valued, continuous coefficient functions $p(t) > 0$, $q(t)$. "Solution" will mean "real-valued, nontrivial ($\not\equiv 0$) solution." The object of interest will be the set of zeros of a solution $u(t)$. For the study of zeros of $u(t)$, the Prüfer transformation (2.42) is particularly useful since $u(t_0) = 0$ if and only if $\varphi(t_0) = 0 \mod \pi$.

**Lemma 3.1.** *Let $u(t) \not\equiv 0$ be a real-valued solution of* (2.1) *on $t_0 \leqq t \leqq t^0$, where $p(t) > 0$ and $q(t)$ are real-valued and continuous. Let $u(t)$ have exactly $n\ (\geqq 1)$ zeros $t_1 < t_2 < \cdots < t_n$ on $t_0 < t \leqq t^0$. Let $\varphi(t)$ be a continuous function defined by* (2.42) *and $0 \leqq \varphi(t_0) < \pi$. Then $\varphi(t_k) = k\pi$ and $\varphi(t) > k\pi$ for $t_k < t \leqq t^0$ for $k = 1, \ldots, n$.*

**Proof.** Note that at a $t$-value where $u = 0$, i.e., where $\varphi = 0 \bmod \pi$, (2.43) implies that $\varphi' = p(t) > 0$. Consequently $\varphi(t)$ is increasing in the neighborhoods of points where $\varphi(t) = j\pi$ for some integer $j$. It follows that if $t_0 \leqq a \leqq t^0$ and $j\pi \leqq \varphi(a)$, then $\varphi(t) > j\pi$ for $a < t \leqq t^0$; also if $j\pi \geqq \varphi(a)$, then $\varphi(t) < j\pi$ for $t_0 \leqq t < a$. This implies the assertion.

In the theorems of this section, two equations will be considered

$$(3.1_j) \qquad (p_j(t)u')' + q_j(t)u = 0, \qquad j = 1, 2,$$

where $p_j(t)$, $q_j(t)$ are real-valued continuous functions on an interval $J$, and

$$(3.2) \qquad p_1(t) \geqq p_2(t) > 0 \qquad \text{and} \qquad q_1(t) \leqq q_2(t).$$

In this case, $(3.1_2)$ is called a *Sturm majorant* for $(3.1_1)$ on $J$ and $(3.1_1)$ is a *Sturm minorant* for $(3.1_2)$. If, in addition,

$$(3.3_1) \qquad q_1(t) < q_2(t)$$

or

$$(3.3_2) \qquad p_1(t) > p_2(t) > 0 \qquad \text{and} \qquad q_2(t) \not\equiv 0$$

holds at some point $t$ of $J$, then $(3.1_2)$ is called a *strict Sturm majorant* for $(3.1_1)$ on $J$.

**Theorem 3.1 (Sturm's First Comparison Theorem).** *Let the coefficient functions in* $(3.1_j)$ *be continuous on an interval* $J: t_0 \leqq t \leqq t^0$ *and let* $(3.1_2)$ *be a Sturm majorant for* $(3.1_1)$. *Let* $u = u_1(t) \not\equiv 0$ *be a solution of* $(3.1_1)$ and *let* $u_1(t)$ *have exactly* $n$ $(\geqq 1)$ *zeros* $t = t_1 < t_2 < \cdots < t_n$ *on* $t_0 < t \leqq t^0$. *Let* $u = u_2(t) \not\equiv 0$ *be a solution of* $(3.1_2)$ *satisfying*

$$(3.4) \qquad \frac{p_1(t)u_1'(t)}{u_1(t)} \geqq \frac{p_2(t)u_2'(t)}{u_2(t)}$$

*at* $t = t_0$. (*The expression on the right* [*or left*] *of* (3.4) *at* $t = t_0$ *is considered to be* $+\infty$ *if* $u_2(t_0) = 0$ [*or* $u_1(t_0) = 0$]; *in particular,* (3.4) *holds at* $t = t_0$ *if* $u_1(t_0) = 0$.) *Then* $u_2(t)$ *has at least* $n$ *zeros on* $t_0 < t \leqq t_n$. *Furthermore* $u_2(t)$ *has at least* $n$ *zeros on* $t_0 < t < t_n$ *if either the inequality in* (3.4) *holds at* $t = t_0$ *or* $(3.1_2)$ *is a strict Sturm majorant for* $(3.1_1)$ *on* $t_0 \leqq t \leqq t_n$.

**Proof.** In view of (3.4), it is possible to define a pair of continuous functions $\varphi_1(t)$, $\varphi_2(t)$ on $t_0 \leqq t \leqq t^0$ by

$$(3.5) \quad \varphi_j(t) = \operatorname{arc\,tan} \frac{u_j(t)}{p_j(t)u_j'(t)} \qquad \text{and} \qquad 0 \leqq \varphi_1(t_0) \leqq \varphi_2(t_0) < \pi.$$

Then the analogue of (2.43) is

$$(3.6_j) \qquad \varphi_j' = \frac{1}{p_j(t)} \cos^2 \varphi_j + q_j(t) \sin^2 \varphi_j \equiv f_j(t, \varphi_j).$$

Since the continuous functions $f_j(t, \varphi_j)$ are smooth as functions of the variable $\varphi_j$, the solutions of (3.6) are uniquely determined by their initial conditions. It follows from (3.2) that $f_1(t, \varphi) \leqq f_2(t, \varphi)$ for $t_0 \leqq t \leqq t^0$ and all $\varphi$. Hence the last part of (3.5) and Corollary III 4.2 show that

$$(3.7) \qquad \varphi_1(t) \leqq \varphi_2(t) \quad \text{for} \quad t_0 \leqq t \leqq t^0.$$

In particular, $\varphi_1(t_n) = n\pi$ implies that $n\pi \leqq \varphi_2(t_n)$ and the first part of the theorem follows from Lemma 3.1.

In order to prove the last part of the theorem, suppose first that the sign of inequality holds in (3.4) at $t = t_0$. Then $\varphi_1(t_0) < \varphi_2(t_0)$. Let $\varphi_{20}(t)$ be the solution of $(3.6_2)$ satisfying the initial condition $\varphi_{20}(t_0) = \varphi_1(t_0)$, so that $\varphi_{20}(t_0) < \varphi_2(t_0)$. Since solutions of $(3.6_2)$ are uniquely determined by initial conditions, $\varphi_{20}(t) < \varphi_2(t)$ for $t_0 \leqq t \leqq t^0$. Thus the analogue of (3.7) gives $\varphi_1(t) \leqq \varphi_{20}(t) < \varphi_2(t)$, and so $\varphi_2(t_n) > n\pi$. Hence $u_2(t)$ has $n$ zeros on $t_0 < t < t_n$.

Consider the case that equality holds in (3.4) but either $(3.3_1)$ or $(3.3_2)$ holds at some point of $[t_0, t_n]$. Write $(3.6_2)$ as

$$\varphi_2' = \frac{1}{p_1}\cos^2\varphi_2 + q_1 \sin^2\varphi_2 + \epsilon(t),$$

where

$$\epsilon(t) = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)\cos^2\varphi_2 + (q_2 - q_1)\sin^2\varphi_2 \geqq 0.$$

If the assertion is false, it follows from the case just considered that $\varphi_1(t) = \varphi_2(t)$ for $t_0 \leqq t \leqq t_n$. Hence, $\varphi_1'(t) = \varphi_2'(t)$ and so $\epsilon(t) = 0$ for $t_0 \leqq t \leqq t_n$. Since $\sin\varphi_2(t) = 0$ only at the zeros of $u_2(t)$, it follows that $q_2(t) = q_1(t)$ for $t_0 \leqq t \leqq t_n$ and that $(p_2^{-1} - p_1^{-1})\cos^2\varphi_2 = 0$. Hence, $p_2^{-1}(t) - p_1^{-1}(t) > 0$ at some $t$ implies $\cos\varphi_2(t) = 0$; i.e., $u_2' = 0$. If $(3.3_1)$ does not hold at any $t$ on $[t_0, t_n]$, it follows that $(3.3_2)$ holds at some $t$ and hence on some subinterval of $[t_0, t_n]$. But then $u_2' = 0$ on this interval, thus $(p_2u_2')' = 0$ on this interval. But this contradicts $q_2(t) \not\equiv 0$ on this interval. This completes the proof.

**Corollary 3.1 (Sturm's Separation Theorem).** *Let* $(3.1_2)$ *be a Sturm majorant for* $(3.1_1)$ *on an interval J and let* $u = u_j(t) \not\equiv 0$ *be a real-valued solution of* $(3.1_j)$. *Let* $u_1(t)$ *vanish at a pair of points* $t = t_1, t_2\ (>t_1)$ *of J. Then* $u_2(t)$ *has at least one zero on* $[t_1, t_2]$. *In particular, if* $p_1 \equiv p_2, q_1 \equiv q_2$ *and* $u_1, u_2$ *are real-valued, linearly independent solutions of* $(3.1_1) \equiv (3.1_2)$, *then the zeros of* $u_1$ *separate and are separated by those of* $u_2$.

Note that the last statement of this theorem is meaningful since the zeros of $u_1$, $u_2$ do not have a cluster point on $J$; see § 2(ii). In addition, $u_1(t), u_2(t)$ cannot have a common zero $t = t_1$; otherwise, the uniqueness of the solutions of $(3.1_1)$ implies that $u_1(t) = cu_2(t)$ with $c = u_1'(t_1)/u_2'(t_1)$ [so that $u_1(t)$, $u_2(t)$ are not linearly independent].

*Exercise* 3.1. (*a*) [Another proof for Sturm's separation theorem when $p_1(t) \equiv p_2(t) > 0, q_2(t) \geqq q_1(t)$.] Suppose that $u_1(t) > 0$ for $t_1 < t < t_2$ and that the assertion is false, say $u_2(t) > 0$ for $t_1 \leqq t \leqq t_2$. Multiplying $(3.1_1)$ where $u = u_1$ by $u_2$ and $(3.1_2)$ where $u = u_2$ by $u_1$, subtracting, and integrating over $[t_1, t]$ gives

$$p(t)(u_1'u_2 - u_1u_2') \geqq 0 \qquad \text{for} \quad t_1 \leqq t \leqq t_2,$$

where $p = p_1 = p_2$; cf. the derivation of (2.9). This implies that $(u_1/u_2)' \geqq 0$; hence $u_1/u_2 > 0$ for $t_1 < t \leqq t_2$ (*b*) Reduce the case $p_1(t) \geqq p_2(t)$ to the case $p_1(t) \equiv p_2(t)$ by the device used below in the proof of Corollary 6.5.

*Exercise* 3.2. (*a*) In the differential equation

$$u'' + q(t)u = 0, \tag{3.8}$$

let $q(t)$ be real-valued, continuous, and satisfy $0 < m \leqq q(t) \leqq M$. If $u = u(t) \not\equiv 0$ is a solution with a pair of zeros $t = t_1$, $t_2(> t_1)$, then $\pi/m^{1/2} \geqq t_2 - t_1 \geqq \pi/M^{1/2}$. (*b*) Let $q(t)$ be continuous for $t \geqq 0$ and $q(t) \to 1$ as $t \to \infty$. Show that if $u = u(t) \not\equiv 0$ is a real-valued solution of (3.8), then the zeros of $u(t)$ form a sequence $(0 \leqq)\ t_1 < t_2 < \ldots$ such that $t_n - t_{n-1} \to \pi$ as $n \to \infty$. (*c*) Observe that real-valued solutions $u(t) \not\equiv 0$ of (1.17) have at most one zero for $t > 0$ if $\mu \leqq \frac{1}{4}$ and have infinitely many zeros for $t > 0$ if $\mu > \frac{1}{4}$. In the latter case, the zeros cluster at $t = 0$ and $t = \infty$. (*d*) Consider the Bessel equation

$$v'' + \frac{v'}{t} + \left(1 - \frac{\mu^2}{t^2}\right)v = 0, \tag{3.9}$$

where $\mu$ is a real parameter. The variations of constants $u = t^{1/2}v$ transforms (3.9) into

$$u'' + \left(1 - \frac{\alpha}{t^2}\right)u = 0, \qquad \text{where} \quad \alpha = \mu^2 - \tfrac{1}{4}. \tag{3.10}$$

Show that the zeros of a real-valued solution $v(t)$ of (3.9) on $t > 0$ form a sequence $t_1 < t_2 < \ldots$ such that $t_n - t_{n-1} \to \pi$ as $n \to \infty$.

**Theorem 3.2 (Sturm's Second Comparison Theorem).** *Assume the conditions of the first part of Theorem* 3.1 *and that $u_2(t)$ also has exactly n zeros on $t_0 < t \leqq t^0$. Then* (3.4) *holds at $t = t^0$ (where the expression on the right [or left] of* (3.4) *at $t = t^0$ is taken to be $+\infty$ if $u_2(t^0) = 0$ [or $u_1(t^0) = 0$]). Furthermore the sign of inequality holds at $t = t^0$ in* (3.4) *if the conditions of the last part of Theorem* 3.1 *hold.*

**Proof.** The proof of this assertion is essentially contained in the proof of Theorem 3.1 if it is noted that the assumption on the number of zeros of

$u_2(t)$ implies the last inequality in $n\pi \leqq \varphi_1(t^0) \leqq \varphi_2(t^0) < (n+1)\pi$. Also, the proof of Theorem 3.1 gives $\varphi_1(t^0) < \varphi_2(t^0)$ under the conditions of the last part of the theorem.

## 4. Sturm-Liouville Boundary Value Problems

This topic is one of the most important in the theory of second order linear equations. Since a full discussion of it would be very lengthy and since very complete treatments can be found in many books, only a few high points will be discussed here.

In the equation

$$(p(t)u')' + [q(t) + \lambda]u = 0, \tag{4.1$\lambda$}$$

let $p(t) > 0$, $q(t)$ be real-valued and continuous for $a \leqq t \leqq b$ and $\lambda$ a complex number. Let $\alpha, \beta$ be given real numbers and consider the problem of finding, if possible, a nontrivial ($\not\equiv 0$) solution of (4.1$\lambda$) satisfying the boundary conditions

$$u(a)\cos\alpha - p(a)u'(a)\sin\alpha = 0, \quad u(b)\cos\beta - p(b)u'(b)\sin\beta = 0. \tag{4.2}$$

*Exercise* 4.1. Show that if $\lambda$ is not real, then (4.1$\lambda$) and (4.2) do not have a nontrivial solution.

*Exercise* 4.2. Consider the following special cases of (4.1$\lambda$), (4.2):

$$u'' + \lambda u = 0, \qquad u(0) = u(\pi) = 0. \tag{4.3}$$

Show that this has a solution only if $\lambda = (n+1)^2$ for $n = 0, 1, \ldots$ and that the corresponding solution, up to a multiplicative constant, is $u = \sin(n+1)t$.

It will be shown that the results of Exercise 4.2 for the special case (4.3) are typical for the general situation (4.1$\lambda$), (4.2).

**Theorem 4.1.** *Let* $p(t) > 0$, $q(t)$ *be real-valued and continuous for* $a \leqq t \leqq b$. *Then there exists an unbounded sequence of real numbers* $\lambda_0 < \lambda_1 < \ldots$ *such that* (i) (4.1$\lambda$), (4.2) *has a nontrivial* ($\not\equiv 0$) *solution if and only if* $\lambda = \lambda_n$ *for some* $n$; (ii) *if* $\lambda = \lambda_n$ *and* $u = u_n(t) \not\equiv 0$ *is a solution of* (4.1$\lambda_n$), (4.2), *then* $u_n(t)$ *is unique up to a multiplicative constant, and* $u_n(t)$ *has exactly* $n$ *zeros on* $a < t < b$ *for* $n = 0, 1, \ldots$ ; (iii) *if* $n \neq m$, *then*

$$\int_a^b u_n(t)u_m(t)\,dt = 0; \tag{4.4}$$

(iv) *if* $\lambda$ *is a complex number* $\lambda \neq \lambda_n$ *for* $n = 0, 1, \ldots$, *then there exists a continuous function* $G(t, s; \lambda) = \bar{G}(s, t; \bar{\lambda})$ *for* $a \leqq s,\ t \leqq b$ *with the property that if* $h(t)$ *is any function integrable on* $a \leqq t \leqq b$, *then*

$$(p(t)w')' + [q(t) + \lambda]w = h(t) \tag{4.5$\lambda$}$$

*has a unique solution* $w = u(t)$ *satisfying*

(4.2′) $w(a)\cos\alpha - p(a)w'(a)\sin\alpha = 0,\ w(b)\cos\beta - p(b)w'(b)\sin\beta = 0$

*and* $w(t)$ *is given by*

$$w(t) = \int_a^b G(t, s; \lambda)h(s)\,ds, \tag{4.6}$$

*also* $G(t, s; \lambda)$ *is real-valued when* $\lambda$ *is real*; (v) *if* $\lambda = \lambda_n$ *and* $h(t)$ *is a function integrable on* $a \leqq t \leqq b$, *then* (4.5$\lambda_n$), (4.2′) *has a solution if and only if*

$$\int_a^b u_n(t)h(t)\,dt = 0; \tag{4.7}$$

*in this case, if* $w(t)$ *is a solution of* (4.5$\lambda_n$), (4.2′), *then* $w(t) + cu_n(t)$ *is also a solution and all solutions are of this form*; (vi) *if the functions* $u_n(t)$ *are chosen real-valued* (*uniquely up to a factor* $\pm 1$) *so as to satisfy*

$$\int_a^b u_n^2(t)\,dt = 1, \tag{4.8}$$

*then* $u_0(t), u_1(t), \ldots$ *form a complete orthonormal sequence for* $L^2(a, b)$; *i.e., if* $h(t) \in L^2(a, b)$, *then* $h(t)$ *has the Fourier series*

$$h(t) \sim \sum_{n=0}^{\infty} c_n u_n(t), \qquad \text{where} \quad c_n = \int_a^b h(t)u_n(t)\,dt \tag{4.9}$$

*and*

$$\int_a^b \left| h(t) - \sum_{h=0}^{n} c_k u_k(t) \right|^2 dt \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.10}$$

If $h(t)$ is not continuous in (iv) or (v), then a solution of (4.5$\lambda$) is to be interpreted as in the Remark in § 2(x).

Note the parallel of the assertions concerning the solvability of (4.5$\lambda$), (4.2′) with the corresponding situation for linear algebraic equations $(\lambda I - L)w = h$, where $L$ is a $d \times d$ Hermitian symmetric matrix, $I$ is the unit matrix and $w$, $h$ vectors: $(\lambda I - L)u = 0$ has a solution $u \neq 0$ if and only if $\lambda$ is an eigenvalue $\lambda_1, \ldots, \lambda_d$ of $L$; $\lambda_1, \ldots, \lambda_d$ are real; if $\lambda \neq \lambda_n$, then $(\lambda I - L)w = h$ has a unique solution $w$ for every $h$; finally, if $\lambda = \lambda_n$, then $(\lambda I - L)w = h$ has a solution $w$ if and only if $h$ is orthogonal (i.e., $u \cdot h = 0$) to all solutions $u$ of $(\lambda I - L)u = 0$.

**Proof.** This proof will only be sketched; details will be left to the reader.

***On (i) and (ii).*** In view of Exercise 4.1, it suffices to consider only real $\lambda$. Let $u(t, \lambda)$ be the solution of (4.1$\lambda$) satisfying the initial condition

$$u(a) = \sin\alpha, \qquad p(a)u'(a) = \cos\alpha, \tag{4.11}$$

so $u(t, \lambda)$ satisfies the first of the two conditions (4.2). It is clear that

(4.1$\lambda$), (4.2) has a solution ($\not\equiv 0$) if and only if $u(t, \lambda)$ satisfies the second condition in (4.2).

For fixed $\lambda$, define a continuous function $\varphi(t, \lambda)$ of $t$ on $[a, b]$ by

$$\varphi(t, \lambda) = \arctan \frac{u(t, \lambda)}{p(t)u'(t, \lambda)}, \qquad \varphi(a, \lambda) = \alpha. \tag{4.12}$$

Then $\varphi(t, \lambda)$ has a continuous derivative satisfying

$$\varphi' = \frac{1}{p(t)} \cos^2 \varphi + [q(t) + \lambda] \sin^2 \varphi, \qquad \varphi(a) = \alpha; \tag{4.13}$$

cf. § 2(xv). If follows from Theorem V 2.1 that the solution $\varphi = \varphi(t, \lambda)$ of (4.13) is a continuous function of $(t, \lambda)$ for $a \leqq t \leqq b$, $-\infty < \lambda < \infty$. The proof of the Sturm Comparison Theorem 3.1 shows that $\varphi(b, \lambda)$ is an increasing function of $\lambda$. Without loss of generality, it can be supposed that $\alpha$ satisfies $0 \leqq \alpha < \pi$. Note that

$$\varphi(b, \lambda) \to \infty \quad \text{as} \quad \lambda \to \infty. \tag{4.14}$$

In order to see this, introduce the new independent variable defined by $ds = dt/p(t)$ and $s(a) = 0$, so that (4.1$\lambda$) becomes

$$\ddot{u} + p(t)[q(t) + \lambda]u = 0, \qquad t = t(s), \qquad \dot{u} = \frac{du}{ds}. \tag{4.15}$$

If $M > 0$ is any number, $\lambda > 0$ can be chosen so large that $p(t)[q(t) + \lambda] \geqq M^2$ for $a \leqq t \leqq b$. Sturm's Comparison Theorem 3.1 applied to (4.15) and

$$\ddot{u} + M^2 u = 0$$

shows that if $n$ is arbitrary and $M$ is sufficiently large, then a nontrivial real-valued solution of (4.15) has at least $n$ zeros on the $s$-interval, $0 \leqq s \leqq \int_a^b dt/p(t)$; i.e., $\varphi(b, \lambda) \geqq n$ if $\lambda > 0$ is sufficiently large by Lemma 3.1.

It will be verified that

$$\varphi(b, \lambda) \to 0 \quad \text{as} \quad \lambda \to -\infty. \tag{4.16}$$

By Lemma 3.1, $\varphi(b, \lambda) \geqq 0$. Let $-\lambda > 0$ be so large that $p(t)[q(t) + \lambda] \leqq -M^2 < 0$. The solution of

$$\ddot{u} - M^2 u = 0$$

satisfying the analogue of (4.11), where $a = 0$ and $p \equiv 1$, is

$$u(s) = \sin \alpha \cosh Ms + \frac{1}{M} \cos \alpha \sinh Ms.$$

The analogue of $\varphi(t, \lambda)$ is

$$\psi(s, M) = \arctan \frac{u(s)}{\dot{u}(s)}, \qquad \psi(0, M) = \alpha.$$

For any fixed $s > 0$,

$$\frac{u(s)}{\dot{u}(s)} \to 0 \qquad \text{as} \quad M \to \infty;$$

hence $\psi(b_0, M) \to 0$ as $M \to \infty$, where $b_0 = \int_a^b dt/p(t)$. By Sturm's Comparison Theorem 3.1, $\varphi(b, \lambda) \leqq \psi(b_0, M)$. This proves (4.16)

The limit relations (4.14), (4.16) and the strict monotony of $\varphi(b, \lambda)$ as a function of $\lambda$ show that there exist $\lambda_0, \lambda_1, \ldots$ such that

$$\varphi(b, \lambda_n) = \beta + n\pi \qquad \text{for} \quad n = 0, 1, \ldots,$$

where it is supposed that $0 < \beta \leqq \pi$. Furthermore $\varphi(b, \lambda) \not\equiv \beta \bmod \pi$ unless $\lambda = \lambda_n$. This implies (i) and (ii).

***On (iii).*** In order to verify (iii), multiply $(4.1\lambda_n)$ by $u_m$, $(4.1\lambda_m)$ by $u_n$, subtract and integrate over $a \leqq t \leqq b$; i.e., apply the Green identity (2.10) to $f = -\lambda_n u_n(t)$, $g = -\lambda_m u_m(t)$.

***On (iv).*** See § 2(x) and Exercise 2.1. Choose $u = u(t, \lambda)$, and $v(t)$ as a solution of $(4.1\lambda)$ satisfying the second condition in (4.2).

***On (v).*** Suppose first that $(4.5\lambda_n)$, (4.2′) has a solution $w = w(t)$. Apply the Green identity (2.10) in the case where $q$ is replaced by $q + \lambda_n$, $f = h$, $w = u$, $v = u_n$, $g = 0$ in (2.8) in order to obtain (4.7).

Conversely, assume that (4.7) holds. Let $u(t) = u_n(t)$ and let $v(t)$ be a solution of $(4.1\lambda_n)$ linearly independent of $u_n(t)$, say $p(t)[uv' - u'v] = c \neq 0$. Then (2.15) is a solution of $(4.5\lambda_n)$. Furthermore $w(t)$ satisfies the first of the boundary conditions in (4.2′) since $u = u_n$ does; cf. Exercise 2.1. On the other hand, (4.7) and (2.15) show that $w(b) = w'(b) = 0$. Hence $w(t)$ is a solution $(4.5\lambda)$ satisfying the boundary conditions (4.2′).

***On (vi).*** Although the assertion (vi) is the main part of Theorem 4.1, it is a consequence of elementary theorems on completely continuous, self-adjoint operators on Hilbert space. For the sake of completeness, the proof of the necessary theorems will be sketched and (vi) will be deduced from them. A knowledge of Fourier series (involving, e.g., Bessel's inequality, Parseval's relation, and the theorem of Fischer-Riesz) will be assumed. In order to minimize the required discussion of topics on Hilbert space, some of the definitions or results, as stated, will involve redundant hypotheses.

Introduce the following notation and terminology:

$$(4.17) \qquad (f, g) = \int_a^b f(t)\bar{g}(t)\,dt, \qquad \|f\| = (f, f)^{1/2} \geqq 0,$$

where $f, g \in L^2(a, b)$. Thus $|(f, g)| \leqq \|f\| \cdot \|g\|$ and $\|f + g\| \leqq \|f\| + \|g\|$ by Schwarz's inequality. A sequence of functions $f_1(t), f_2(t), \ldots$ in $L^2(a, b)$ will be said to tend to $f(t)$ in $L^2(a, b)$ if $\|f_n - f\| \to 0$ as $n \to \infty$. They will be said to tend to $f(t)$ *weakly* in $L^2(a, b)$ if the sequence $\|f_1\|, \|f_2\|, \ldots$ is bounded and, for every $\varphi(t) \in L^2(a, b)$, $(f_n, \varphi) \to (f, \varphi)$ as $n \to \infty$. (In this last definition, the condition on $\|f_1\|, \|f_2\|, \ldots$ is redundant but this fact will not be needed below.) A subset $H$ of $L^2(a, b)$ is called a linear manifold if $f, g \in H$ implies that $c_1 f + c_2 g \in H$ for all constants $c_1, c_2$ and it is called *closed* if $f_n \in H$ for $n = 1, 2, \ldots, f \in L^2(a, b)$ and $\|f_n - f\| \to 0$ as $n \to \infty$ imply that $f \in H$. A linear manifold $H$ of $L^2(a, b)$ will be called *weakly closed* if $f_n \in H$ for $n = 1, 2, \ldots, f \in L^2(a, b)$ and $f_n \to f$ weakly as $n \to \infty$ imply that $f \in H$. (The fact that the notions of "closed" and "weakly closed" are equivalent for linear manifolds will not be needed here.)

**Lemma 4.1.** *Let $f_1, f_2, \ldots$ be a sequence of elements of $L^2(a, b)$ satisfying $\|f_n\| \leqq 1$. Then there exists an $f(t) \in L^2(a, b)$ and a subsequence $f_{n(1)}(t)$, $f_{n(2)}(t), \ldots$ of the given sequence such that $\|f\| \leqq 1$ and $f_{n(j)} \to f(t)$ weakly as $j \to \infty$.*

**Proof.** Without loss of generality, it can be supposed that $[a, b] = [0, \pi]$. Thus each $f_n(t)$ has a sine Fourier series

$$f_n(t) \sim \sum_{k=1}^{\infty} c_{nk} \sin kt,$$

where, by Parseval's relation, $\sum_k |c_{nk}|^2 = \|f_n\|^2 \leqq 1$. It follows from Cantor's diagonal process (Theorem I 2.1) that there exists a sequence of integers $1 \leqq n(1) < n(2) < \ldots$ such that

$$(4.18) \qquad c_k = \lim c_{n(j)k} \qquad \text{exists as} \quad j \to \infty \qquad \text{for} \quad k = 1, 2, \ldots.$$

Note that

$$\sum_{k=1}^{m} |c_k|^2 = \lim_{j \to \infty} \sum_{k=1}^{m} |c_{n(j)k}|^2 \leqq 1.$$

Hence $\Sigma\, |c_k|^2 \leqq 1$ and so, by the theorem of Fischer-Riesz, there exists an $f(t) \in L^2(a, b)$ such that

$$f(t) \sim \sum_{k=1}^{\infty} c_k \sin kt, \qquad \|f\|^2 = \sum |c_k|^2 \leqq 1.$$

It follows from (4.18) that $(f_{n(j)}, \varphi) \to (f, \varphi)$ as $j \to \infty$ holds if $\varphi = \sin kt$ for $k = 1, 2, \ldots$. Hence it holds for any sine polynomial $p(t) = a_1 \sin t + \cdots + a_m \sin mt$. For any $\varphi(t) \in L^2(0, \pi)$, there exists a sine polynomial $p(t)$ such that $\|\varphi - p\|$ is arbitrarily small and $|(f_{n(j)} - f, \varphi)| \leqq |(f_{n(j)} - f, p)| + |(f_{n(j)} - f, p - \varphi)|$, while $|(f_{n(j)} - f, p - \varphi)| \leqq \|f_{n(j)} - f\| \cdot \|p - \varphi\| \leqq 2\, \|p - \varphi\|$. Hence the lemma follows.

**Lemma 4.2.** *Let $G$ be a self-adjoint, linear operator defined on a weakly closed linear manifold $H$ of $L^2(a, b)$ satisfying $(Gh, h) = 0$ for all $h \in H$. Then $Gh = 0$ for all $h \in H$.*

To say that $G$ is a linear operator on $H$ means that to every $h \in H$ there is associated a unique element $w = Gh \in H$ and that if $w_j = Gh_j$ for $j = 1, 2$, then $c_1w_1 + c_2w_2 = G(c_1h_1 + c_2h_2)$ for all complex constants $c_1, c_2$. The assumption that $G$ is *self-adjoint* means that $(Gh, f) = (h, Gf)$ for all $f, h \in H$.

**Proof.** If $f, h \in L^2(a, b)$ and $c$ is a complex number, then $0 = (G(h + cf), h + cf) = 2 \operatorname{Re} \bar{c}(Gh, f)$ since $(Gf, f) = (Gh, h) = 0$. By the choices $c = 1$ and $c = i$, it follows that $(Gh, f) = 0$. On choosing $f = Gh$, it is seen that $Gh = 0$.

**Lemma 4.3.** *Let $G$ be a completely continuous, self-adjoint linear operator on a weakly closed linear manifold $H$ of $L^2(a, b)$ and let $Gh \neq 0$ for some $h \in H$. Then $G$ has at least one (real) eigenvalue $\mu \neq 0$; i.e., there exists a (real) number $\mu \neq 0$ and an $h_0 \in H$, $h_0 \neq 0$, such that $Gh_0 = \mu h_0$.*

A linear operator $G$ on $H$ is called *completely continuous* if $h_n, h \in H$ and $h_n \to h$ weakly as $n \to \infty$ imply that $\|Gh_n - Gh\| \to 0$ as $n \to \infty$.

**Proof.** It follows from Lemma 4.1, the complete continuity of $G$, and the fact that $H$ is weakly closed that $G$ is bounded, i.e., that there exists a constant $C$ such that $\|Gh\| \leqq C$ for all $h \in H$ satisfying $\|h\| \leqq 1$.

By Schwarz's inequality, $|(Gh, h)| \leqq \|Gh\| \cdot \|h\| \leqq C$ if $\|h\| \leqq 1$. Hence $\sup (Gh, h)$ and $\inf (Gh, h)$ for all $\|h\| \leqq 1$ exist and are finite. Since $Gh \neq 0$ for some $h \in H$, it follows from Lemma 4.2 that at least one of these two numbers is not zero. For the sake of definiteness, let $\mu = \sup (Gh, h) \neq 0$. The choice $h = 0$ shows that $\mu \geqq 0$, hence $\mu > 0$.

It will be shown that there exists an $h_0 \in H$ such that $(Gh_0, h_0) = \mu$ and $\|h_0\| \leqq 1$. For there exist elements $h_1, h_2, \ldots$ in $H$ such that $\|h_n\| \leqq 1$ and $(Gh_n, h_n) \to \mu$ as $n \to \infty$. In view of Lemma 4.1, we can suppose that there exists an $h_0 \in L^2(a, b)$ such that $h_n \to h_0$ weakly as $n \to \infty$ and $\|h_0\| \leqq 1$. Since $H$ is weakly closed, $h_0 \in H$. The complete continuity of $G$ shows that $\|Gh_n - Gh_0\| \to 0$ as $n \to \infty$. Also $(Gh_0, h_0) = (Gh_n, h_n) + 2 \operatorname{Re} (G(h_0 - h_n), h_n) + (G(h_0 - h_n), h_0 - h_n)$. From the boundedness of $G$ and Schwarz's inequality, we conclude, by letting $n \to \infty$, that $(Gh_0, h_0) = \mu$.

Note that $\mu \neq 0$ implies $h_0 \neq 0$. Also, since $\mu > 0$, it follows that $\|h_0\| = 1$, otherwise $(Gh, h) = \mu/\|h_0\|^2 > \mu$ for $h = h_0/\|h_0\|$ and $\|h\| = 1$.

In order to verify that $Gh_0 = \mu h_0$, let $h$ be any element of $H$ satisfying $\|h\| = 1$ and $(h_0, h) = 0$. Let $h_\epsilon = (h_0 + \epsilon h)/(1 + \epsilon^2)^{1/2}$ for a real $\epsilon$, so that $\|h_\epsilon\|^2 = 1$. Then the function

$$(Gh_\epsilon, h_\epsilon) = (1 + \epsilon^2)^{-1}\{(Gh_0, h_0) + 2\epsilon \operatorname{Re} (Gh_0, h) + \epsilon^2(Gh, h)\}$$

of $\epsilon$ has a maximum at $\epsilon = 0$ and hence $\operatorname{Re} (Gh_0, h) = 0$. Since $h$ can be replaced by $ih$, it follows that $(Gh_0, h) = 0$ for all $h \in H$ satisfying $(h_0, h) = 0$. In particular, $(Gh_0, h) = 0$ if $h = Gh_0 - \mu h_0$. This implies that

$\mu^2 = \|Gh_0\|^2$ and hence $\|Gh_0 - \mu h_0\|^2 = \|Gh_0\|^2 - 2(Gh_0, h_0) + \mu^2 = 0$. This proves $Gh_0 = \mu h_0$ and completes the proof of the lemma.

***Completion of Proof of (vi).*** A standard theorem on Fourier series implies that (vi) is false if and only if there exist functions $h(t) \in L^2(a, b)$, $\|h\| \neq 0$, having zero Fourier coefficients $(h, u_n) = 0$ for $n = 0, 1, \ldots$. Suppose, if possible, that (vi) is false and let $H$ denote the set of all elements $h(t) \in L^2(a, b)$ satisfying $(h, u_n) = 0$ for $n = 0, 1, \ldots$. Then $H$ is a weakly closed, linear manifold in $L^2(a, b)$ and contains elements $h \neq 0$.

Choose a real number $\lambda \neq \lambda_n$ for $n = 0, 1, \ldots$. Then (4.6) defines a linear operator $G$, $w = Gh$, on $L^2(a, b)$. This operator is self-adjoint since

$$(Gh, f) = \int_a^b \int_a^b G(t, s; \lambda) h(s) \bar{f}(t)\, ds\, dt = (h, Gf)$$

follows from the fact that $G(t, s; \lambda)$ is real-valued and $G(t, s; \lambda) = G(s, t; \lambda)$.

Also $G$ is completely continuous. In order to verify this, let $h_n \to h$ weakly as $n \to \infty$ and $w_n = Gh_n$, $w = Gh$, then

$$w_n(t) - w(t) = \int_a^b G(t, s; \lambda)[h_n(s) - h(s)]\, ds = (h_n - h, G(t, \cdot\,; \lambda))$$

tends to 0 as $n \to \infty$ for every fixed $t$. Furthermore, by Schwarz's inequality

$$|w_n(t) - w(t)|^2 \leqq 2C \int_a^b |G(t, s; \lambda)|^2\, ds \leqq \text{const.}$$

if $\|h_n\|^2 \leqq C$, $\|h(s)\|^2 \leqq C$. Thus $\|w_n - w\|^2 = \int |w_n(t) - w(t)|^2\, dt \to 0$ as $n \to \infty$ by Lebesgue's theorem on dominated convergence. [Actually, by Theorem I 2.2, $w_n(t) \to w(t)$ as $n \to \infty$ uniformly for $a \leqq t \leqq b$ since it is easily seen that the sequence $w_1, w_2, \ldots$ is uniformly bounded and equicontinuous.]

Finally, note that if $h \in H$, then $w = Gh$ is in $H$. In fact, $(h, u_n) = 0$ implies that $(w, u_n) = 0$ as can be seen by applying the Green identity (2.10) to $u = u_n$, $f = -\lambda_n u_n$, $v = w$, $g = -\lambda w + h$. Thus the restriction of $G$ to the weakly closed linear manifold $H$ gives a completely continuous, self-adjoint operator on $H$.

From (4.5$\lambda$) and (4.6), it is seen that $h \not\equiv 0$ implies that $w = Gh \not\equiv 0$. Since $H$ contains elements $h \neq 0$, Lemma 4.3 is applicable. Let $Gh_0 = \mu h_0$, where $h_0 \in H$, $\|h_0\| = 1$, $\mu \neq 0$. Thus, if $w_0 = Gh_0$, it follows from (4.5$\lambda$), (4.6) that $u = w_0(t) \not\equiv 0$ is a solution of (4.1 $\lambda - 1/\mu$) satisfying the boundary conditions (4.2). Hence, by part (i), there is a non-negative integer $k$ such that $\lambda - 1/\mu = \lambda_k$ and $w_0 = cu_k$ for some constant $c \neq 0$. But this contradicts $(w_0, u_n) = 0$ for $n = 0, 1, \ldots$ and proves the theorem.

*Exercise* 4.3. Let $p_0(t) > 0$, $r_0(t) > 0$ and $q_0(t)$ be real-valued continuous functions on an open bounded interval $a < t < b$. Let $\lambda_0 < \lambda_1 < \ldots$. Suppose that

$$(4.19) \qquad (p_0(t)u')' + [q_0(t) + \lambda_n r_0(t)]u = 0, \qquad n = 0, 1, \ldots,$$

has a (real) solution $u_n(t)$ on $a < t < b$ having at most $n$ zeros and such that the limits $\lim u_n(t)/u_0(t)$, $t \to a$ and $t \to b$, exist and are not zero. (*a*) Show that if $p_1(t) = 1/r_0(t)u_0^2(t) > 0$, $r_1(t) = 1/p_0(t)u_0^2(t) > 0$ and $q_1(t) = -\lambda_0/p_0(t)u_0^2(t)$, then $v_n(t) = p_0(u_0u'_{n+1} - u_0'u_{n+1})$ is a solution of

$$(4.20) \qquad (p_1(t)v')' + [q_1(t) + \lambda_{n+1}r_1(t)]v = 0, \qquad n = 0, 1, \ldots,$$

having at most $n$ zeros on $a < t < b$ and such that the limits $\lim v_n(t)/v_0(t)$, $t \to a$ and $t \to b$, exist and are not zero. (*b*) Show that there exist positive continuous functions $a_0(t), a_1(t), \ldots, a_{k-1}(t)$ on $a < t < b$, such that $u_0(t), \ldots, u_{k-1}(t)$ are solutions of the $k$th order linear differential equation

$$(4.21) \qquad (a_{k-1} \cdots \{a_2[a_1(a_0u)']'\}' \ldots)' = 0.$$

*Exercise* 4.4 (*Continuation*). (*a*) Let $p_0$, $r_0$, $q_0$, $\lambda_n$, $u_n$ be as in Exercise 4.3. Let $a < t_1 < \cdots < t_{k+1} < b$ and $\alpha_1, \ldots, \alpha_{k+1}$ be arbitrary numbers. Then there exists a unique set of constants $c_0, \ldots, c_k$ such that

$$(4.22) \quad c_0u_0(t_j) + \cdots + c_ku_k(t_j) = \alpha_j \qquad \text{for} \quad j = 1, \ldots, k + 1.$$

Use induction on $k$ (for *all* systems $u_0, u_1, \ldots$) or use Exercise IV 8.3. [This result is, of course, applicable to (real-valued) $u_n(t)$ in Theorem 4.1. If the functions $p_0, r_0, q_0$ have derivatives of sufficiently high order, then the interpolation property (4.22) can be generalized, as in Exercise IV 8.3(*d*).] (*b*) Let $a < t_0 < \cdots < t_n < b$. Then $D(t_0, \ldots, t_n) \equiv \det(u_j(t_k))$, where $j, k = 0, \ldots, n$, is different from 0. (*c*) Let $c_0, \ldots, c_n$ be real numbers and $U_n(t) = c_0u_0(t) + \cdots + c_nu_n(t)$. Then $U_n(t) \equiv 0$ if $U_n(t)$ vanishes at $n + 1$ distinct points of $a < t < b$, and if $U_n(t) \not\equiv 0$ vanishes at $n$ distinct points, then it changes sign at each. (*d*) Every real-valued continuous function $v(t)$ orthogonal to $u_0, \ldots, u_n$ on $[a, b]$ (i.e., $\int_a^b vu_j\,dt = 0$ for $j = 0, \ldots, n$) changes sign at least $n + 1$ times. (*e*) For any choice of constants $c_m, \ldots, c_n$, the function $c_mu_m(t) + \cdots + c_nu_n(t)$ changes sign at least $m$ times and at most $n$ times, where $m \leq n$.

## 5. Number of Zeros

This section will be concerned with zeros of real-valued solutions of an equation of the form

$$(5.1) \qquad u'' + q(t)u = 0.$$

**Theorem 5.1.** *Let $q(t)$ be real-valued and continuous for $a \leqq t \leqq b$. Let $m(t) \geqq 0$ be a continuous function for $a \leqq t \leqq b$ and*

$$\gamma_m = \inf \frac{m(t)}{(t-a)(b-t)} \quad \text{for} \quad a < t < b. \tag{5.2}$$

*If a real-valued solution $u(t) \not\equiv 0$ of (5.1) has two zeros, then*

$$\int_a^b m(t) q^+(t)\, dt > \gamma_m(b-a), \tag{5.3}$$

*where $q^+(t) = \max\,(q(t), 0)$; in particular,*

$$\int_a^b (t-a)(b-t) q^+(t)\, dt > b - a. \tag{5.4}$$

*Exercise* 5.1. Show that the inequality (5.3) is "sharp" in the sense that (5.3) need not hold if $\gamma_m$ is replaced by $\gamma_m + \epsilon$ for $\epsilon > 0$.

**Proof of Theorem 5.1.** Assume that (5.1) has a solution ($\not\equiv 0$) with two zeros on $[a, b]$. Since $q^+(t) \geqq q(t)$, the equation

$$u'' + q^-(t)u = 0 \tag{5.5}$$

is a Sturm majorant for (5.1) and hence has a solution $u(t) \not\equiv 0$ with two zeros $t = \alpha$, $\beta$ on $[a, b]$; cf. Theorem 3.1. Since $u'' = -q^- u$, it follows that

$$(\beta - \alpha)u(t) = (\beta - t)\int_\alpha^t (s-\alpha)q^+(s)u(s)\, ds + (t-\alpha)\int_t^\beta (\beta - s)q^+(s)u(s)\, ds;$$

cf. Exercise 2.1, in particular (2.18). Suppose that $\alpha$, $\beta$ are successive zeros of $u$ and that $u(t) > 0$ for $\alpha < t < \beta$. Choose $t = t_0$ so that $u(t)_0 = \max u(t)$ on $(\alpha, \beta)$. The right side is increased if $u(s)$ is replaced by $u(t_0)$. Thus dividing by $u(t_0) > 0$ gives

$$\beta - \alpha < (\beta - t)\int_\alpha^t (s-\alpha) q^+(s)\, ds + (t-\alpha)\int_t^\beta (\beta - s) q^+(s)\, ds,$$

where $t = t_0$. Since $\beta - t \leqq \beta - s$ for $t \geqq s$ and $t - \alpha \leqq s - \alpha$ for $s \geqq t$,

$$\beta - \alpha < \int_\alpha^\beta (\beta - s)(s - \alpha) q^+(s)\, ds. \tag{5.6}$$

Finally, note that $(t-a)(b-t)/(b-a) \geqq (t-\alpha)(\beta-t)/(\beta-\alpha)$ for $a \leqq \alpha \leqq t \leqq \beta \leqq b$; in fact, differentiation with respect to $\beta$ and $\alpha$ shows that $(t-\alpha)(\beta-t)/(\beta-\alpha)$ increases with $\beta$ if $t \geqq \alpha$ and decreases with $\alpha$ if $t \leqq \beta$. Hence (5.4) follows from the last display. The relation (5.3) is a consequence of (5.2) and (5.4). This proves Theorem 5.1.

Since $(t - a)(b - t) \leqq (b - a)^2/4$, the choice $m(t) = 1$ in Theorem 5.1 gives the following:

**Corollary 5.1 (Lyapunov).** *Let $q(t)$ be real-valued and continuous on $a \leqq t \leqq b$. A necessary condition for* (5.1) *to have a solution $u(t) \not\equiv 0$ possessing two zeros is that*

$$\int_a^b q^+(t)\,dt > \frac{4}{b-a}\,. \tag{5.7}$$

*Exercise* 5.2. Let $q(t) \geqq 0$ be continuous on $a \leqq t \leqq b$ and let (5.1) have a solution $u(t)$ vanishing at $t = a, b$ and $u(t) > 0$ in $(a, b)$. (*a*) Use (5.7) to show that $\int_a^b q(t)\,dt > 2M/A$, where $M = \max u(t)$ and $A = \int_a^b u(t)\,dt$. (*b*) Show that the factor 2 of $M/A$ cannot be replaced by a larger constant.

*Exercise* 5.3. (*a*) Consider a differential equation $u'' + g(t)u' + f(t)u = 0$ with real-valued continuous coefficients on $0 \leqq t \leqq b$ having a solution $u(t) \not\equiv 0$ vanishing at $t = 0, b$. Show that

$$b < \int_0^b t(b-t)f^+(t)\,dt + \max\left\{\int_0^b t\,|g|\,dt, \qquad \int_0^b (b-t)\,|g|\,dt\right\}.$$

(*b*) In particular, if $|g| \leqq M_1$ and $|f| \leqq M_2$, then $1 < M_1b/2 + M_2b^2/6$. But this inequality can easily be improved by the use of Wirtinger's inequality $\int_0^b u^2\,dt \leqq (b/\pi)^2 \int_0^b u'^2\,dt$ (which can be proved by assuming $b = \pi$, expanding $u$ into a Fourier sine series, and applying Parseval's relation for $u, u'$). Show that $1 \leqq M_1b/\pi + M_2b^2/\pi^2$. (*c*) The result of part (*b*) can further be improved to $1 \leqq 2M_1b/\pi^2 + M_2b^2/\pi^2$. See Opial [3]. (*d*) An analogous result for a $d$th order equation, $d \geqq 2$, is as follows: Let the differential equation $u^{(d)} + p_1(t)u^{(d-1)} + \cdots + p_d(t)u = 0$ have continuous coefficients for $0 \leqq t \leqq b$ and a solution $u(t) \not\equiv 0$ with $d$ zeros on $[0, b]$. Let $|p_j(t)| \leqq M_j$. Then $1 < M_1b + M_2b^2/2! + \cdots + M_{d-1}b^{d-1}/(d-1)! + (M_db^d/d!)[(d-1)^{d-1}/d^d]$.

When $q(t) = q^+(t)$ is a positive constant on $[0, T]$, the number $N$ of zeros of a solution ($\not\equiv 0$) of (5.1) on $(0, T]$ obviously satisfies

$$\pi N \leqq (q^+)^{1/2}T \equiv \int_0^T (q^+)^{1/2}\,dt \leqq \left(T\int_0^T q^+(t)\,dt\right)^{1/2}, \tag{5.8}$$

where the last inequality follows from Schwartz's inequality. It turns out that a similar inequality holds for nonconstant, continuous $q(t)$:

**Corollary 5.2.** *Let $q(t)$ be real-valued and continuous for $0 \leqq t \leqq T$.*

*Let* $u(t) \not\equiv 0$ *be a solution of* (5.1) *and* $N$ *the number of its zeros on* $0 < t \leqq T$. *Then*

$$N < \frac{1}{2}\left(T\int_0^T q^+(t)\,dt\right)^{1/2} + 1. \tag{5.9}$$

**Proof.** In order to prove this, let $N \geqq 2$ and let the $N$ zeros of $u$ on $(0, T]$ be $(0 <)\ t_1 < t_2 < \cdots < t_N\ (\leqq T)$. By Corollary 5.1,

$$\int_u^v q^+(t)\,dt > \frac{4}{v-u} \qquad \text{if} \quad u = t_k, \qquad v = t_{k+1}, \tag{5.10}$$

for $k = 1, \ldots, N-1$. Since the harmonic mean of $N-1$ positive numbers is majorized by their arithmetic mean,

$$\left[\frac{1}{N-1}\sum_{k=1}^{N-1}\frac{1}{t_{k+1}-t_k}\right]^{-1} \leqq \frac{1}{N-1}\sum_{k=1}^{N-1}(t_{k+1}-t_k) = \frac{t_N - t_1}{N-1}.$$

Thus adding (5.10) for $k = 1, \ldots, N-1$ gives

$$\int_{t_1}^{t_N} q^+(t)\,dt > \frac{4(N-1)^2}{t_N - t_1} \geqq \frac{4(N-1)^2}{T},$$

hence (5.9).

*Exercise* 5.4. Show that $N$ also satisfies

$$N < \int_0^T tq^+(t)\,dt + 1. \tag{5.11}$$

To this end, use (5.3) with $m(t) = t - a$ in place of (5.7).

Note that if $q(t)$ is a positive constant, then

$$\left|\pi N - \int_0^T q^{1/2}\,dt\right| \leqq \pi.$$

An analogous inequality holds under mild assumptions on nonconstant $q$:

**Theorem 5.2.** *Let* $q(t) > 0$ *be continuous and of bounded variation on* $0 \leqq t \leqq T$. *Let* $u(t) \not\equiv 0$ *be a real-valued solution of* (5.1) *and* $N$ *the number of its zeros on* $0 < t \leqq T$. *Then*

$$\left|\pi N - \int_0^T q^{1/2}(t)\,dt\right| \leqq \pi + \frac{1}{4}\int_0^T \frac{|dq(t)|}{q(t)}. \tag{5.12}$$

**Proof.** In terms of $u(t)$ define a continuous function $\varphi(t)$ by

$$\varphi(t) = \arctan\frac{q^{1/2}(t)u}{u'}, \qquad 0 \leqq \varphi(0) < \pi.$$

Then [cf. Exercise 2.6; in particular (2.49) where $p(t) \equiv 1$]

$$\varphi(T) = \varphi(0) + \int_0^T q^{1/2}(t)\,dt + \frac{1}{4}\int_0^T \sin 2\varphi(t)\,d(\log q).$$

By Lemma 3.1, $N$ is the greatest integer not exceeding $\varphi(T)/\pi$, so that $\pi N \leqq \varphi(T) \leqq \pi(N+1)$. This implies (5.12).

*Exercise* 5.5. (*a*) Let $q(t)$ be continuous on $0 \leqq t \leqq T$. Let $u(t) \not\equiv 0$ be a real-valued solution and $N$ the number of its zeros on $0 < t \leqq T$. Show that

$$|\pi N - T| \leqq \pi + \int_0^T |1 - q(t)|\, dt.$$

(*b*) If, in addition, $q(t) > 0$ has a continuous second derivative, then

$$\left|\pi N - \int_0^T q^{1/2}(t)\, dt\right| \leqq \pi + \int_0^T \left|\frac{5q'^2}{16q^{5/2}} - \frac{q''}{4q^{3/2}}\right| dt.$$

**Corollary 5.3.** *Let $q(t) > 0$ be continuous and of bounded variation on $[0, T]$ for every $T > 0$. Suppose also that*

$$\text{(5.13)} \qquad \int_0^T q^{-1}\, |dq| = o\left(\int_0^T q^{1/2}\, dt\right) \quad \text{as} \quad T \to \infty;$$

*e.g., suppose that $q(t)$ has a continuous derivative $q'(t)$ satisfying*

$$\text{(5.14)} \qquad q'(t) = o(q^{3/2}(t)) \quad \text{as} \quad t \to \infty.$$

*Let $u(t) \not\equiv 0$ be a real-valued solution of* (5.1) *and $N(T)$ the number of its zeros on $0 < t \leqq T$. Then*

$$\text{(5.15)} \qquad \pi N(T) \sim \int_0^T q^{1/2}(t)\, dt \quad \text{as} \quad T \to \infty.$$

This is clear from (5.13) and the formula (5.12) in Theorem 5.2. It should be mentioned that if, e.g., $q$ is monotone and $q(t) \to \infty$ as $t \to \infty$, then (5.14) imposes no restriction on the rapidity of growth of $q(t)$ but is a condition on the regularity of growth. This can be seen from the fact that the integral

$$\int^T \frac{q'\, dt}{q^{3/2}} = \frac{2}{q^{1/2}(T)} + \text{const.}$$

tends to a limit as $T \to \infty$; thus, in general, $q'/q^{3/2}$ is "small" for large $t$.

The conditions of Corollary 5.3 for the validity of (5.15) can be lightened somewhat, as is shown by the following exercises.

*Exercise* 5.6. (*a*) Let $q(t) > 0$ be continuous for $t \geqq 0$ and satisfy

$$\text{(5.16)} \qquad \sup_{s \leqq t < \infty} \frac{|\log q(t)/q(s)|}{1 + \int_s^t q^{1/2}(r)\, dr} \to 0 \quad \text{as} \quad s \to \infty.$$

Let $u(t) \not\equiv 0$ be a solution of (5.1) and $N(T)$ the number of its zeros on $0 < t \leqq T$. Then (5.15) holds. (*b*) Necessary and sufficient for (5.16) is the following pair of conditions: $\int^{\infty} q^{1/2}\,dt = \infty$ and $q(t + cq^{-1/2}(t))/q(t) \to 1$ as $t \to \infty$ holds uniformly on every fixed bounded $c$-interval on $-\infty < c < \infty$.

*Exercise* 5.7. Part (*b*) of the last exercise can be generalized as follows: Let $q(t) > 0$ be continuous for $t \geqq 0$. Let $m(t) > 0$ be continuous for $t > 0$ and satisfy $[m(t)/m(s)]^{\pm 1} \leqq C(t/s)^{\gamma}$ for $0 < s < t < \infty$ and some pair of non-negative constants $C$, $\gamma$. Necessary and sufficient for

$$\sup_{s \leqq t < \infty} \frac{|\log q(t)/q(s)|}{1 + \int_s^t m(q(r))\,dr} \to 0 \qquad \text{as} \quad s \to \infty$$

is that $\int^{\infty} m(q(t))\,dt = \infty$ and that $q(t + c/m[q(t)])/q(t) \to 1$ as $\min\,[t, t + c/m(q(t))] \to \infty$ holds uniformly on every bounded $c$-interval on $-\infty < c < \infty$.

An estimate for $N$ of a type very different from those just given is the following:

**Theorem 5.3.** *Let $p(t) > 0$, $q(t)$ be real-valued and continuous for $0 \leqq t \leqq T$. Let $u(t)$, $v(t)$ be real-valued solutions of*

$$(pu')' + qu = 0 \tag{5.17}$$

*satisfying*

$$p(t)[u'(t)v(t) - u(t)v'(t)] = c > 0. \tag{5.18}$$

*Let $N$ be the number of zeros of $u(t)$ on $0 < t \leqq T$. Then*

$$\left| \pi N - c\int_0^T \frac{dt}{p(t)[u^2(t) + v^2(t)]} \right| \leqq \pi. \tag{5.19}$$

**Proof.** Let $\alpha$ be an arbitrary real number. Consider the solutions

$$u^*(t) = u(t)\cos\alpha + v(t)\sin\alpha, \qquad v^*(t) = -u(t)\sin\alpha + v(t)\cos\alpha$$

of (5.17). They satisfy

$$u^2 + v^2 = u^{*2} + v^{*2}, \qquad p[u^{*\prime}v^* - u^*v^{*\prime}] = c > 0. \tag{5.20}$$

Choose $\alpha$ so that $u^*(0) = 0$ and let $N^*$ be the number of zeros of $u^*(t)$ on $0 < t \leqq T$.

Since (5.20) implies that $u^*$, $v^*$ are linearly independent, they have no common zeros. Hence it is possible to define a continuous function by

$$\varphi(t) = \arc\tan \frac{u^*(t)}{v^*(t)} \qquad \text{and} \quad \varphi(0) = 0. \tag{5.21}$$

This function is continuously differentiable and, by (5.20),

$$\varphi'(t) = \frac{c}{p(t)[u^2(t) + v^2(t)]} > 0. \tag{5.22}$$

Hence $\varphi(t)$ is increasing; also $\varphi(t) = 0 \bmod \pi$ if and only if $u(t) = 0$. Thus $N^*$ is the greatest integer not exceeding $\varphi(T)/\pi$ and a quadrature of (5.22) gives

$$\pi N^* \leqq c\int_0^T \frac{dt}{p(t)[u^2(t) + v^2(t)]} < \pi(N^* + 1).$$

Sturm's separation theorem implies $N^* \leqq N \leqq N^* + 1$, thus (5.19) follows.

*Exercise* 5.8. Let $p(t)$, $q(t)$, $u(t)$, $v(t)$, and $N$ be as in Theorem 5.3 and, in addition, let $q(t) \geqq 0$. Show that

$$\left|\pi N - c\int_0^T \frac{q(t)\,dt}{p^2(t)[u'^2(t) + v'^2(t)]}\right| \leqq 2\pi \tag{5.23}$$

(If $q > 0$, the relations (5.19) and (5.23) are particular cases of "duality" in which $(u, u', q, dt)$ are replaced by $(pu', -u, 1/q, q\,dt)$; cf. Lemma XIV 3.1.)

*Exercise* 5.9. (*a*) Let $q(t)$ be continuous for $t \geqq 0$. Using (5.9) and (5.19), show that if all solutions of $u'' + q(t)u = 0$ are bounded, then, for large $t$,

$$\frac{1}{t}\int_0^t q^+(s)\,ds \geqq \text{const.} > 0. \tag{5.24}$$

Replacing $u$, $v$ in (5.19) by $u/\epsilon$, $\epsilon v$, show that if, in addition, a nontrivial solution $u(t) \to 0$ as $t \to \infty$, then

$$\frac{1}{t}\int_0^t q^+(s)\,ds \to \infty \qquad \text{as} \quad t \to \infty. \tag{5.25}$$

(*b*) Let $q(t) \geqq 0$ for $t \geqq 0$. Using (5.9) and (5.23), show that if the first derivatives of all solutions of $u'' + q(t)u = 0$ are bounded, then, for large $t$,

$$\frac{1}{t}\int_0^t q^+(s)\,ds \leqq \text{const.} \tag{5.26}$$

If, in addition, $u'(t) \to 0$ as $t \to \infty$ for some solution $u(t) \not\equiv 0$, then

$$\frac{1}{t}\int_0^t q^+(s)\,ds \to 0 \qquad \text{as} \quad t \to \infty. \tag{5.27}$$

(*c*) Generalize (*a*) [or (*b*)] for the case when $u'' + qu = 0$ is replaced by $(pu')' + qu = 0$ and the assumption that solutions [or derivatives of solutions] are bounded is replaced by the assumption that all solutions satisfy $u(t) = O(1/\Phi(t))$ [or $u'(t) = O(1/\Phi(t))$], where $\Phi(t) > 0$ is continuous.

## 6. Nonoscillatory Equations and Principal Solutions

A homogeneous, linear second order equation with real-valued coefficient functions defined on an interval $J$ is said to be *oscillatory on J* if one (and/or every) real-valued solution ($\not\equiv 0$) has infinitely many zeros on $J$. Conversely, when every solution ($\not\equiv 0$) has at most a finite number of zeros on $J$, it is said to be *nonoscillatory on $J$*. In the latter case, the equation is said to be *disconjugate on $J$* if every solution ($\not\equiv 0$) has at most one zero on $J$. If $t = \omega$ is a (possibly infinite) endpoint of $J$ which does not belong to $J$, then the equation is said to be *oscillatory at $t = \omega$* if one (and/or every) real-valued solution ($\not\equiv 0$) has an infinite sequence of zeros clustering at $t = \omega$. Otherwise it is called *nonoscillatory at $t = \omega$*.

Extensions of many of the results of this section to higher order equations or more general systems will be indicated in §§ 10, 11 of the Appendix.

**Theorem 6.1.** *Let $p(t) > 0$, $q(t)$ be real-valued, continuous functions on a $t$-interval $J$. Then*

$$(p(t)u')' + q(t)u = 0 \tag{6.1}$$

*is disconjugate on $J$ if and only if for every pair of distinct points $t_1, t_2 \in J$ and arbitrary numbers $u_1, u_2$; there exists a unique solution $u = u^*(t)$ of* (6.1) *satisfying*

$$u^*(t_1) = u_1 \quad \text{and} \quad u^*(t_2) = u^2; \tag{6.2}$$

*or, equivalently, if and only if every pair of linearly independent solutions $u(t), v(t)$ of* (6.1) *satisfy*

$$u(t_1)v(t_2) - u(t_2)v(t_1) \neq 0 \tag{6.3}$$

*for distinct points $t_1, t_2 \in J$.*

**Proof.** Let $u(t)$, $v(t)$ be a pair of linearly independent solutions of (6.1). Then any solution $u^*(t)$ is of the form $u^* = c_1u(t) + c_2v(t)$. This solution satisfies (6.2) if and only if

$$c_1u(t_1) + c_2v(t_1) = u_1, \qquad c_1u(t_2) + c_2v(t_2) = u_2.$$

These linear equations for $c_1$, $c_2$ have a solution for all $u_1$, $u_2$ if and only if (6.3) holds. In addition, they have a solution for all $u_1$, $u_2$ if and only if the only solution of

$$c_1u(t_1) + c_2v(t_1) = 0, \qquad c_1u(t_2) + c_2v(t_2) = 0$$

is $c_1 = c_2 = 0$; i.e., if and only if the only solution $u^*(t)$ of (6.1) with two zeros $t = t_1, t_2$ is $u^*(t) \equiv 0$.

**Corollary 6.1.** *Let $p(t) > 0, q(t)$ be as in Theorem* 6.1. *If $J$ is open or is closed and bounded, then* (6.1) *is disconjugate on $J$ if and only if* (6.1) *has a solution satisfying $u(t) > 0$ on $J$. If $J$ is a half-closed interval or a closed half-line, then* (6.1) *is disconjugate on $J$ if and only if there exists a solution $u(t) > 0$ on the interior of $J$.*

The example $u'' + u = 0$ on $J$: $0 \leqq t < \pi$ shows that, in the last part of the theorem, there need not exist a solution $u(t) > 0$ on $J$.

*Exercise* 6.1. Deduce Corollary 6.1 from Theorem 6.1 (another proof follows from Exercise 6.6).

*Exercise* 6.2. Let $p(t) > 0$, $q(t) \geqq 0$ be continuous on an interval $J$: $a \leqq t < \omega$ $(\leqq \infty)$ such that $\int^{\omega} dt/p(t) = \infty$, then (6.1) is disconjugate on $J$ if and only if it has a solution $u(t)$ such that $u(t) > 0$, $u'(t) \geqq 0$ for $a < t < \omega$.

A very useful criterion for (6.1) to be disconjugate is a "variational principle" to be stated as the next theorem. A real-valued function $\eta(t)$ on the subinterval $[a, b]$ of $J$ will be said to be *admissible of class* $A_1(a, b)$ [or $A_2(a, b)$] if (i) $\eta(a) = \eta(b) = 0$, and (ii$_1$) $\eta(t)$ is absolutely continuous and its derivative $\eta'(t)$ is of class $L^2$ on $a \leqq t \leqq b$ [or (ii$_2$) $\eta(t)$ is continuously differentiable and $p(t)\eta'(t)$ is continuously differentiable on $a \leqq t \leqq b$]. Put

$$(6.4) \qquad I(\eta; a, b) = \int_a^b (p\eta'^2 - q\eta^2)\, dt \qquad \text{for} \quad \eta \in A_1(a, b).$$

If $\eta$ is admissible $A_2(a, b)$, the first term can be integrated by parts and it is seen that

$$(6.5) \qquad I(\eta; a, b) = -\int_a^b \eta[(p\eta')' + q\eta]\, dt \qquad \text{for} \quad \eta \in A_2(a, b).$$

**Theorem 6.2.** *Let* $p(t) > 0$, $q(t)$ *be real-valued continuous functions on a* $t$*-interval* $J$. *Then* (6.1) *is disconjugate on* $J$ *if and only if, for every closed bounded subinterval* $a \leqq t \leqq b$ *of* $J$, *the functional* (6.4) *is positive-definite on* $A_1(a, b)$ [*or* $A_2(a, b)$]; *i.e.,* $I(\eta; a, b) \geqq 0$ *for* $\eta \in A_1(a, b)$ [*or* $\eta \in A_2(a, b)$] *and* $I(\eta; a, b) = 0$ *if and only if* $\eta \equiv 0$.

The "only if" half of the theorem is stronger for $A_1(a, b)$ and the "if" half is stronger for $A_2(a, b)$.

**Proof ("Only if").** Suppose that (6.1) is disconjugate on $a \leqq t \leqq b$. Then, by Corollary 6.1, there is a solution $u(t) > 0$ on $a \leqq t \leqq b$. If $\eta(t) \in A_1(a, b)$, put $\zeta(t) = \eta(t)/u(t)$. Then

$$(6.6) \qquad I(\eta; a, b) = \int_a^b [\zeta^2(pu'^2 - qu^2) + p(u^2\zeta'^2 + 2\zeta\zeta' uu')]\, dt.$$

An integration by parts [integrating $u'$ and differentiating $(pu')\zeta^2$] shows that the first term is

$$\int_a^b \zeta^2 pu'^2\, dt = [\zeta^2 puu']_a^b - \int_a^b [\zeta^2 u(pu')' + 2p\zeta\zeta' uu']\, dt.$$

The integrated terms vanish since $\eta(a) = \eta(b) = 0$ imply that $\zeta(a) = \zeta(b) = 0$. The last two formula lines and $\zeta^2 u[(pu')' + qu] = 0$ give

$$(6.7) \qquad I(\eta; a, b) = \int_a^b pu^2\zeta'^2\, dt \qquad \text{for} \quad \eta = u\zeta \in A_1(a, b).$$

It is clear that $I(\eta; a, b) \geqq 0$ and $I(\eta; a, b) = 0$ if and only if $\zeta(t) \equiv 0$. This proves the "only if" part of the theorem.

**Proof** ("**If**"). Suppose that $I(\eta; a, b)$ is positive definite on $A_2(a, b)$ for every $[a, b] \subset J$. Let $\eta(t)$ be a solution of (6.1) having two zeros $t = a, b \in J$. It will be shown that $\eta(t) \equiv 0$. In fact $\eta(t) \in A_2(a, b)$; thus (6.5) holds. Hence, $I(\eta; a, b) = 0$ because $\eta$ is a solution of (6.1). Since (6.4) is positive definite on $A_2(a, b)$, it follows that $\eta(t) \equiv 0$. This implies that (6.1) is disconjugate on $J$ and completes the proof of the theorem.

*Exercise* 6.3. Suppose that $J$ is not a closed bounded interval. Show that, in Theorem 6.2, (6.1) is disconjugate on $J$ if $I(\eta; a, b) \geqq 0$ for all $[a, b] \subset J$ and all $\eta \in A_2(a, b)$.

*Exercise* 6.4. Deduce Sturm's separation theorem (Corollary 3.1) from Theorem 6.2.

If $P$ is a constant positive definite Hermitian matrix, then there exists a positive definite Hermitian matrix $P_1$ which is the "square root" of $P$ in the sense that $P = P_1^2 = P_1^* P_1$; cf. Exercise XIV 1.2. An analogue of this algebraic fact will be obtained for the differential operator

$$L[\eta] = -(p(t)\eta')' - q(t)\eta.$$

Note that (6.5) can be written as

$$I(\eta; a, b) = (L[\eta], \eta) \qquad \text{for} \quad \eta \in A_2(a, b);$$

cf. (4.17). Also, (6.7) can be written as

$$I(\eta; a, b) = \int_a^b \frac{p(\eta' u - \eta u')^2}{u^2}\, dt.$$

In addition to the quadratic functional (6.4), consider the bilinear form

$$I(\eta_1, \eta_2; a, b) = \int_a^b (p\eta_1'\eta_2' - q\eta_1\eta_2)\, dt$$

for $\eta_1, \eta_2 \in A_1(a, b)$. If $\eta_1 \in A_2(a, b)$, an integration by parts shows that

$$I(\eta_1, \eta_2; a, b) = -\int_a^b \eta_2[(p\eta_1')' + q(\eta_1)]\, dt = (L[\eta_1], \eta_2).$$

If $u(t)$ is a solution of (6.1) and $u(t) > 0$ on $[a, b]$, it is readily verified that, for $\eta_1, \eta_2 \in A_2(a, b)$ and $\zeta_1 = \eta_1/u$, $\zeta_2 = \eta/u$,

$$I(\eta_1, \eta_2; a, b) = \int_a^b pu^2\zeta_1'\zeta_2'\, dt.$$

or

$$I(\eta_1, \eta_2; a, b) = \int_a^b \frac{p(\eta_1' u - \eta_1 u')(\eta_2' u - \eta_2 u')}{u^2}\, dt$$

Thus if the first order differential operator $L_1$ is defined by

$$L_1[\eta] = \frac{p^{1/2}(t)[\eta' u(t) - \eta u'(t)]}{u(t)} = p^{1/2} u \left(\frac{\eta}{u}\right)',$$

i.e., by

$$L_1[\eta] = p^{1/2}(t)\eta' - \frac{p^{1/2}(t)u'(t)}{u(t)}\eta = p^{1/2} u \left(\frac{\eta}{u}\right)', \tag{6.8}$$

then it follows that

$$(L[\eta_1], \eta_2) = (L_1[\eta_1], L_1[\eta_2]) \quad \text{for} \quad \eta_1, \eta_2 \in A_2(a, b). \tag{6.9}$$

Consequently, if $L$ [i.e., (6.4)] is positive definite on $A_2(a, b)$, so that there exists a positive solution $u(t) > 0$ of (6.1) on $[a, b]$, then formally

$$L = L_1{}^* L_1.$$

In fact this relation is not only formally correct but is correct in the following sense:

**Corollary 6.2.** *Let $p(t) > 0$, $q(t)$ be continuous on $J$ and let* (6.1) *have a solution $u(t) > 0$ on $J$. Let $L_1$ be defined by* (6.8) *and $L_1{}^*$ its formal adjoint*

$$L_1{}^*[\eta] = -(p^{1/2}(t)\eta)' - \frac{p^{1/2}(t)u'(t)}{u(t)}\eta = -\frac{(p^{1/2}u\eta)'}{u};$$

*cf.* § IV 8 (viii). *Then*

$$L[\eta] = L_1{}^*\{L_1[\eta]\}$$

*for all continuously differentiable functions $\eta$ for which $p(t)\eta'$ is absolutely continuous* (*i.e., for all $\eta$ for which $L[\eta]$ is usually defined*).

This can be deduced from the identity (6.9) or, more easily, by a straightforward verification. See Appendix for generalizations of this result.

Theorem 5.3 and its proof have the following consequence.

**Theorem 6.3.** *Let $p(t) > 0$, $q(t)$ be real-valued and continuous on a $t$-interval $J$. Then $J$ is nonoscillatory on $J$ if and only if every pair of linearly independent solutions $u(t)$, $v(t)$ of* (6.1) *satisfy*

$$\int_J \frac{dt}{p(t)(|u|^2 + |v|^2)} < \infty.$$

*Furthermore,* (6.1) *is disconjugate on $J$ if and only if*

$$|c| \int_a^b \frac{dt}{p(t)(u^2 + v^2)} < \pi$$

*for every pair of real-valued solutions $u(t)$, $v(t)$ satisfying $p(u'v - uv') = c \neq 0$ and every interval $[a, b] \subset J$.*

If $J$ is a half-open interval, say $J: a \leqq t < \omega$ ($\leqq \infty$) and (6.1) is non-oscillatory at $t = \omega$, then (6.1) has real-valued solutions $u(t)$ for which $\int^{\omega} dt/pu^2$ is convergent and solutions for which it is divergent. The latter type of solution will be called a *principal solution* of (6.1) at $t = \omega$.

**Theorem 6.4.** *Let* $p(t) > 0$, $q(t)$ *be real-valued and continuous on* $J$: $a \leqq t < \omega$ ($\leqq \infty$) *and such that* (6.1) *is nonoscillatory at* $t = \omega$. *Then there exists a real-valued solution* $u = u_0(t)$ *of* (6.1) *which is uniquely determined up to a constant factor by any one of the following conditions in which* $u_1(t)$ *denotes an arbitrary real-valued solution linearly independent of* $u_0(t)$: (i) $u_0$, $u_1$ *satisfy*

$$(6.10) \qquad \frac{u_0(t)}{u_1(t)} \to 0 \quad \text{as} \quad t \to \omega;$$

(ii) $u_0$, $u_1$ *satisfy*

$$(6.11_0) \quad \int^{\omega} \frac{dt}{p(t)u_0^2(t)} = \infty \quad \text{and} \quad (6.11_1) \quad \int^{\omega} \frac{dt}{p(t)u_1^2(t)} < \infty;$$

(iii) *if* $T \in J$ *exceeds the largest zero, if any, of* $u_0(t)$ *and if* $u_1(T) \neq 0$, *then* $u_1(t)$ *has one or no zero on* $T < t < \omega$ *according as*

$$(6.12_0) \quad \frac{u_1'}{u_1} < \frac{u_0'}{u_0} \quad \text{or} \quad (6.12_1) \quad \frac{u_1'}{u_1} > \frac{u_0'}{u_0}$$

*holds at* $t = T$; *in particular,* $(6.12_1)$ *holds for all* $t$ ($\in J$) *near* $\omega$.

It is understood that in (6.10) and (6.11) only $t$-values exceeding the largest zeros, if any, of $u_0$, $u_1$ are considered. A solution $u_0(t)$ satisfying one (and/or) all of the conditions (i), (ii), (iii) will be called a *principal solution* of (6.1) (at $t = \omega$). A solution $u(t)$ linearly independent of $u_0(t)$ will be termed a *nonprincipal solution* of (6.1) (at $t = \omega$). In view of (6.10), (6.11), the terms "principal" and "nonprincipal" might well be replaced by "small" and "large." The expressions "small," "large" will not be used in this context because of the relative nature of these terms. Consider, e.g., the equations $u'' - u = 0$, $u'' = 0$ and $u'' + u/4t^2 = 0$ for $t \geqq 1$. Examples of principal and nonprincipal solutions at $t = \infty$ for the first equation are $u = e^{-t}$ and $u = e^t$; for the second, $u = 1$ and $u = t$; for the third, $u = t^{1/2}$ and $u = t^{1/2} \log t$; cf. Exercise 1.1. The proof of (ii) will lead to the following:

**Corollary 6.3.** *Assume the conditions of Theorem* 6.4. *Let* $u = u(t) \not\equiv 0$ *be any real-valued solution of* (6.1) *and let* $t = T$ *exceed its last zero. Then*

$$(6.13) \qquad u_1(t) = u(t) \int_T^t \frac{ds}{p(s)u^2(s)}$$

*is a nonprincipal solution of* (6.1) *on* $T \leqq t < \omega$. *If, in addition,* $u(t)$ *is a nonprincipal solution of* (6.1), *then*

$$u_0(t) = u(t)\int_t^\omega \frac{ds}{p(s)u^2(s)} \tag{6.14}$$

*is a principal solution on* $T \leqq t < \omega$.

**Proof of Theorem 6.4 and Corollary 6.3**

***On*** (*i*). Let $u(t)$, $v(t)$ be a pair of real-valued linearly independent solutions of (6.1) such that

$$p(u'v - uv') = c \neq 0. \tag{6.15}$$

If $T$ exceeds the largest zero, if any, of $v(t)$, then (6.15) is equivalent to

$$\left(\frac{u}{v}\right)' = \frac{c}{pv^2} \neq 0, \tag{6.16}$$

for $T \leqq t < \omega$. Hence $u/v$ is monotone on this $t$-range and so

$$C = \lim_{t\to\omega} \frac{u(t)}{v(t)} \tag{6.17}$$

exists if $C = \pm\infty$ is allowed.

It will be shown that $u, v$ can be chosen so that $C = 0$ in (6.17). If this is granted and if $u(t)$ is called $u_0(t)$, then (i) holds. In fact, a solution $u_1(t)$ is linearly independent of $u_0(t)$ if and only if it is of the form $u_1(t) = c_0u_0(t) + c_1v(t)$ and $c_1 \neq 0$; in which case, $C = 0$ implies that $u_1 = [c_1 + o(1)]v(t)$; thus $u_0 = o(u_1)$ as $t \to \omega$.

If $C = \pm\infty$ in (6.17) and if $u, v$ are interchanged, then (6.17) holds with $C = 0$. If $|C| < \infty$ and if $u(t) - Cv(t)$ is renamed $u(t)$, then (6.15) still holds and (6.17) holds with $C = 0$. This proves (i).

***On*** (*ii*). Note that (6.16), (6.17) give

$$C = \frac{u(T)}{v(T)} + c\int_T^\omega \frac{ds}{p(s)v^2(s)}$$

whether or not $|C| = \infty$ or $|C| < \infty$. If $u, v$ is a pair $u_0, u_1$, so that $C = 0$, then ($6.11_1$) holds. If $u, v$ is a pair $u_1, u_0$, so that $C = \pm\infty$, then ($6.11_0$) holds.

***On Corollary 6.3.*** Note that if $u(t)$ is a solution of (6.1) and $u(t) \neq 0$ for $T \leqq t < \omega$, then (6.13) defines a solution $u_1(t)$ linearly independent of

$u(t)$ and that the same is true of (6.14) when the integral is convergent; see § 2 (ix). By (i), this implies Corollary 6.3.

***On (iii).*** Since $u_0$, $u_1$ can be replaced by $-u_0$, $-u_1$, respectively, without affecting the zeros of $u_1$ or the inequalities (6.12), it can be supposed that

(6.18) $$u_0(t) > 0 \quad \text{for} \quad T \leqq t < \omega \quad \text{and} \quad u_1(T) > 0.$$

Multiplying (6.12) by $u_0(T)u_1(T) > 0$ shows that the case (6.15), where $(u, v) = (u_1, u_0)$ holds with $c < 0$ or $c > 0$ according as $(6.12_0)$ or $(6.12_1)$ holds. Hence $u_1(t)/u_0(t) \to \mp\infty$ as $t \to \omega$ according as $(6.12_0)$ or $(6.12_1)$ holds. Since $u_1(T)/u_0(T) > 0$ and, by the Sturm separation theorem, $u_1$ has at most one zero on $T < t < \omega$, the statement concerning the zeros of $u_1$ on $T < t < \omega$ follows.

It remains to show that property (iii) is characteristic of a principal solution; i.e., if $u_0(t)$ has the property (iii) for every solution $u_1(t)$ linearly independent of $u_0(t)$, then $u_0(t)$ is a principal solution. In particular $(6.12_1)$ holds for $t$ ($\in J$) near $\omega$. Consequently $|u_0(t)| \leqq \text{const.}\, |u_1(t)|$ for $t \to \omega$. This is a contradiction if $u_0(t)$ is not a principal solution and $u_1(t)$ is chosen to be a principal solution.

*Exercise* 6.5. Assume (i) that the conditions of Theorem 6.4 hold; (ii) that (6.1) has a nonvanishing real-valued solution for $(a \leqq)\ T \leqq t < \omega$; and (iii) that $u_{0r}(t)$ is the unique solution of (6.1) satisfying $u_{0r}(T) = 1$, $u_{0r}(r) = 0$, where $T < r < \omega$; cf. Theorem 6.1. (*a*) Show that $u_0(t) = \lim u_{0r}(t)$ exists as $r \to \omega$ uniformly on compact intervals of $J$ and is the principal solution of (6.1) at $t = \omega$ satisfying $u_0(T) = 1$. (*b*) Show that (*a*) is false if condition (ii) is relaxed to the condition that (6.1) is disconjugate on $T \leqq t < \omega$.

*Exercise* 6.6. Let $p(t) > 0$, $q(t)$ be real-valued and continuous functions such that (6.1) is disconjugate on a $t$-interval $J$ having $t = \omega$ $(\leqq \infty)$ as right endpoint. Let $u_0(t)$ be a principal solution of (6.1) at $t = \omega$. Then $u_0(t) \neq 0$ on the interior of $J$.

Sturm's comparison theorem implies that "$q(t) \leqq 0$ on $J$" is sufficient for (6.1) to be disconjugate on $J$. In this case, we can give some additional information about a principal solution.

**Corollary 6.4.** *Let $p(t) > 0$, $q(t) \leqq 0$ be continuous on $J: a \leqq t < \omega$. Then* (6.1) *has a principal solution satisfying*

(6.19) $$u_0(t) > 0, \quad u_0'(t) \leqq 0 \quad \text{for} \quad a \leqq t < \omega$$

*and a nonprincipal solution $u_1(t)$ such that*

(6.20) $$u_1(t) > 0, \quad u_1'(t) > 0 \quad \text{for} \quad a \leqq t < \omega.$$

*Exercise* 6.7. (*a*) In Corollary 6.4, the conditions (6.19) uniquely determine $u_0(t)$, up to a constant factor, if and only if

$$(6.21)\qquad \int^{\omega} \frac{dt}{p(t)} = \infty \qquad \text{or} \qquad -\int^{\omega} q(t)\,dt = \infty.$$

(*b*) Assume the first part of (6.21). Using Corollary 9.1, show that a principal solution in Corollary 6.4 satisfies $u_0(t) \to 0$ as $t \to \omega$ if and only if $-\int^{\infty} q(t) \left(\int^{t} dr/p(r)\right) dt = \infty$.

For generalizations, related results, and a different proof of Corollary 6.4, see XIV §§ 1, 2.

**Proof.** Assume first that $p(t) \equiv 1$, so that (6.1) is of the form

$$(6.22)\qquad u'' + q(t)u = 0,$$

where $q \leqq 0$. Hence the graph of a solution $u = u(t)$ of (6.22) in the $(t, u)$-plane is concave upwards when $u(t) > 0$. Let $u(t)$ be the solution of (6.22) determined by $u(a) = 1$, $u'(a) = 1$. Then $u = u(t)$ has a graph which is concave upward for $a \leqq t < \omega$. In particular, $u(t) > u(a) = 1$, $u'(t) \geqq u'(a) = 1$; so that $u(t) \geqq 1 + t$. Thus $\int^{\omega} dt/u^2(t)$ is convergent, and so $u(t)$ is a nonprincipal solution of (6.22). By Corollary 6.3,

$$u_0(t) = u(t)\int_t^{\omega} \frac{ds}{u^2(s)} > 0$$

is a principal solution of (6.22). Differentiating this formula gives

$$u_0'(t) = u'(t)\int_t^{\omega} \frac{ds}{u^2(s)} - \frac{1}{u(t)}$$

Since $u'(t)$ is nondecreasing,

$$u_0'(t) \leqq \int_t^{\omega} u'(s)\, \frac{ds}{u^2(s)} - \frac{1}{u(t)} = -\lim_{s\to\omega} \frac{1}{u(s)} \leqq 0.$$

This gives (6.19). The case $p(t) > 0$ can be reduced to the case $p(t) \equiv 1$ by the change of independent variables (1.7). This completes the proof.

*Exercise* 6.8. Give a proof of the part of Corollary 6.4 concerning $u_0(t)$ along the following lines: Let $a < T < \omega$ and let $u_T(t)$ be the solution of (6.1) satisfying $u_T(a) = 1$, $u_T(T) = 0$; cf. Theorem 6.1. Show that $u_0(t) = \lim u_T(t)$ exists as $T \to \omega$ uniformly on compact intervals of $[a, \omega)$, is a principal solution of (6.1) and satisfies (6.19); cf. Exercise 6.5.

**Corollary 6.5.** *In the two differential equations*

$$(6.23_j)\qquad (p_j(t)u') + q_j(t)u = 0,$$

*where* $j = 1, 2$, *let* $p_j(t) > 0$, $q_j(t)$ *be real-valued and continuous on* $J: a \leqq t < \omega$; *let* $(6.23_2)$ *be a Sturm majorant for* $(6.23_1)$, *i.e.,*

$$p_1 \geqq p_2 > 0 \quad \text{and} \quad q_1 \leqq q_2; \tag{6.24}$$

*let* $(6.23_2)$ *be disconjugate* [*so that* $(6.23_1)$ *is also*]. *Let* $u_2(t) \not\equiv 0$ *be a real-valued solution of* $(6.23_2)$. *Then* $(6.23_1)$ *has principal and nonprincipal solutions,* $u_{10}(t)$ *and* $u_{11}(t)$, *which satisfy*

$$\frac{p_1u_{10}'}{u_{10}} \leqq \frac{p_2u_2'}{u_2} \leqq \frac{p_1u_{11}'}{u_{11}} \tag{6.25}$$

*for all* $t$ *beyond the last zero, if any, of* $u_2(t)$.

The rough content of this corollary is that the principal [nonprincipal] solutions of $(6.23_1)$ are smaller [larger] than the principal [nonprincipal] solutions of $(6.23_2)$. If $p_1 \equiv p_2$ and $u_2$, $u_{10}$, $u_{11}$ are normalized by suitable constant factors, (6.25) implies that $u_{10} \leqq u_2 \leqq u_{11}$ for $t$ near $\omega$.

*Exercise* 6.9. In Corollary 6.5, the principal solutions $u_{10}$ of $(6.23_1)$ satisfy $\int^{\omega} u_{10}^2(q_2 - q_1)\, dt < \infty$. In particular, if $q \leqq 0$ in (6.1), then a principal solution $u_0$ of (6.1) satisfies $\int^{\omega} u_0^2\, |q|\, ds < \infty$.

**Proof.**

***Case 1*** **($p_1 \equiv p_2$).** Suppose that $u_2(t) > 0$ for $T \leqq t < \omega$. Make the variations of constants $u = u_2z$ in $(6.23_1)$. Then $(6.23_1)$ is transformed [cf. (2.31) of § 2 (xii)] into

$$(p_1u_2^2z')' + u_2^2(q_1 - q_2)z = 0, \tag{6.26}$$

where $q_1 - q_2 \leqq 0$ and

$$\frac{u'}{u} = \frac{u_2'}{u_2} + \frac{z'}{z}. \tag{6.27}$$

By Corollary 6.4, (6.26) has solutions $z_0(t)$, $z_1(t)$ satisfying $z_0 > 0$, $z_0' \leqq 0$, and $z_1 > 0$, $z_1' > 0$ for $T \leqq t < \omega$. The desired solutions of $(6.23_1)$ are $u_{10} = u_2z_0$, $u_{11} = u_2z_1$.

***Case 2*** **($p_1 \not\equiv p_2$).** The function $r = p_2u_2'/u_2$ satisfies the Riccati equation $r' + r^2/p_2 + q_2 = 0$ belonging to $(6.23_2)$; cf. § 2 (xiv). This equation can be written as

$$r' + \frac{r^2}{p_1} + q_0 = 0, \tag{6.28}$$

where $q_0 = q_2 + (1/p_2 - 1/p_1)(p_2u_2'/u_2)^2 \geqq q_2 \geqq q_1$. But (6.28) is the Riccati equation belonging to

$$(p_1u')' + q_0u = 0, \tag{6.29}$$

which is a Sturm majorant for $(6.23_1)$. In addition, (6.29) has the solution [cf. § 2 (xiv)]

$$u = \exp \int_T^t \left(\frac{r}{p_1}\right) ds = \exp \int_T^t \left(\frac{p_2 u_2'}{p_1 u_2}\right) ds$$

satisfying

$$\frac{p_1 u'}{u'} = \frac{p_2 u_2'}{u_2}.$$

Thus application of the Case 1 to $(6.23_1)$, (6.29) gives the desired result.

*Exercise* 6.10. In the differential equations

$$(6.30_j) \qquad u'' + g_j(t)u' - f_j(t)u = 0,$$

where $j = 1, 2$, let $f_j, g_j$ be continuous for $0 \leqq t < \omega$ $(\leqq \infty)$; let $0 \leqq f_1(t) \leqq f_2(t)$ and $g_1(t) \leqq g_2(t)$; let $u_1(t)$ be a solution of $(6.30_1)$ satisfying $u_1(0) = 1$ and $u_1(t) > 0$, $u_1'(t) \leqq 0$ for $0 \leqq t < \omega$; cf. Corollary 6.4. Then $(6.30_2)$ has a solution $u_2(t)$ satisfying $u_2(0) = 1$, $u_2'(t) \leqq 0$ and $0 < u_2(t) \leqq u_1(t)$ for $0 \leqq t < \omega$ [in fact, satisfying $u_2(0) = 1$ and $0 \leqq u_2/u_1 \leqq 1$, $(u_2/u_1)' \leqq 0$ for $0 \leqq t < \omega$].

The following is a "selection" or "continuity" theorem for principal solutions:

**Corollary 6.6.** *Let $p_1(t), p_2(t), \ldots, p_\infty(t)$ and $q_1(t), q_2(t), \ldots, q_\infty(t)$ be continuous functions for $a \leqq t < \omega$ satisfying*

$$(6.31_j) \qquad p_j(t) > 0, \quad q_j(t) \leqq 0 \quad \text{for} \quad a \leqq t < \omega \quad \text{and } j = 1, 2, \ldots, \infty$$

*and*

$$(6.32) \qquad p_j(t) \to p_\infty(t), \qquad q_j(t) \to q_\infty(t) \qquad \text{as} \quad j \to \infty$$

*uniformly on every closed interval of $a \leqq t < \omega$. For $1 \leqq j < \infty$, let $u_{j0}(t)$ be a principal solution of*

$$(6.33_j) \qquad (p_j u')' + q_j(t)u = 0$$

*satisfying* (6.19) *and*

$$(6.34) \qquad u(a) = 1.$$

*Then there exists a sequence of positive integers $j(1) < j(2) < \cdots$ such that*

$$(6.35) \qquad u_\infty(t) = \lim_{n\to\infty} u_{j0}(t), \qquad \text{where} \quad j = j(n),$$

*exists uniformly on every closed interval of $a \leqq t < \omega$ and is a solution of* $(6.33_\infty)$ *satisfying* (6.19) *and* (6.34).

Of course, a selection is unnecessary (i.e., $j(n) = n$ is permitted) if $(6.33_\infty)$ has a unique solution satisfying (6.19) and (6.34); cf. Exercise 6.7.

*Exercise* 6.11. This corollary is false if the condition $q_j(t) \leqq 0$ is replaced by the assumption that $(6.33_j)$ is nonoscillatory and (6.19) is deleted from both assumption and assertion.

**Proof.** Let $u_{j1}(t)$ be the solution of $(6.33_j)$ determined by

$$u_{j1}(a) = 1, \qquad p_j(a)u'_{j1}(a) = 1. \tag{6.36}$$

Then (6.20) holds and $u_{j1}(t)$ is a nonprincipal solution of $(6.33_j)$; cf. the proof of Corollary 6.4. Hence, by Corollary 6.3, the principal solution $u_{j0}(t)$ of $(6.33_j)$ satisfying (6.34) is given by

$$C_j u_{j0}(t) = u_{j1}(t)\int_t^\omega \frac{ds}{p_j(s)u_{j1}^2(s)} \qquad \text{for} \quad a \leqq t < \omega, \tag{6.37}$$

where

$$C_j = \int_a^\omega \frac{ds}{p_j(s)u_{j1}^2(s)}. \tag{6.38}$$

Differentiation of (6.37) gives

$$0 \geqq C_j u'_{j0}(t) = u'_{j1}(t)\int_t^\omega \frac{ds}{p_j u_{j1}^2} - \frac{1}{p_j(t)u_{j1}(t)},$$

so that, if $t = a$,

$$0 \geqq p_j(a)u'_{j0}(a) = 1 - \frac{1}{C_j}.$$

Thus the sequence $p_j(a)u'_{j0}(a), j = 1, 2, \ldots$, is bounded if

$$C_j \geqq \text{const.} > 0 \qquad \text{for} \quad j = 1, 2, \ldots \tag{6.39}$$

In order to verify (6.39), note that (6.36) and the assumption on (6.32) imply that $u_{j1}(t) \to u_{\infty 1}(t)$ as $j \to \infty$ uniformly on closed intervals of $a \leqq t < \omega$. Thus, by (6.38),

$$C_j > \int_a^T \frac{ds}{p_j u_{j1}^2} \to \int_a^T \frac{ds}{p_\infty u_{\infty 1}^2} \qquad \text{as} \quad j \to \infty$$

for any fixed $T$, $a < T < \omega$. This implies (6.39).

Since the sequence of numbers $u_{j0}(a) = 1$ and $u'_{j0}(a)$ for $j = 1, 2, \ldots$, are bounded, there exist subsequences which have limits. If $j(1) < j(2) < \cdots$ are the indices of such a subsequence and

$$1 = \lim u_{j0}(a), \qquad u'_{\infty 0} = \lim u'_{j0}(a) \qquad \text{for} \quad j = j(n) \to \infty,$$

then the assumption on (6.32) implies (6.35) uniformly on every interval $[a, T] \subset [a, \omega)$, where $u_\infty(t)$ is the solution of $(6.33_\infty)$ satisfying $u_\infty(a) = 1$, $u_\infty'(a) = u'_{\infty 0}$. The solution $u_\infty(t)$ clearly satisfies (6.19) and (6.34). This proves Corollary 6.6.

## 7. Nonoscillation Theorems

This section will be concerned with conditions, necessary and/or sufficient, for

$$u'' + q(t)u = 0 \tag{7.1}$$

to be nonoscillatory. In view of the Sturm comparison theorem, the simplest (and one of the most important) sufficient conditions for (7.1) to be nonoscillatory [or oscillatory] is for (7.1) to possess a nonoscillatory [or oscillatory] Sturm majorant [minorant]. For example, if $q(t) \leqq 0$ [so that $u'' = 0$ is a Sturm majorant for (7.1)], then (7.1) is nonoscillatory. If $q(t) = \mu t^{-2}$, then (7.1) is nonoscillatory or oscillatory at $t = \infty$ according as $\mu \leqq \frac{1}{4}$ or $\mu > \frac{1}{4}$; see Exercise 1.1(*c*). This gives the following criteria:

**Theorem 7.1.** *Let $q(t)$ be real-valued and continuous for large $t > 0$. If*

$$-\infty \leqq \limsup_{t\to\infty} t^2 q(t) < \frac{1}{4} \qquad \left[\text{or} \quad \infty \geqq \liminf_{t\to\infty} t^2 q(t) > \frac{1}{4}\right], \tag{7.2}$$

*then* (7.1) *is nonoscillatory* [*or oscillatory*] *at $t = \infty$.*

If, e.g., $t^2 q(t) \to \frac{1}{4}$ as $t \to \infty$, then Theorem 7.1 does not apply. In this case, Exercise 1.2 shows that (7.2) can be replaced by

$$-\infty \leqq \limsup_{t\to\infty} t^2 \log^2 t \left[q(t) - \frac{1}{4t^2}\right] < \frac{1}{4}$$

or

$$\infty \geqq \liminf_{t\to\infty} t^2 \log^2 t \left[q(t) - \frac{1}{4t^2}\right] > \frac{1}{4}$$

In fact, the sequence of functions in Exercise 1.2 gives a scale of tests for (7.1) to be nonoscillatory or oscillatory at $t = \infty$.

The criterion given by Sturm's comparison theorem can be cast in the following convenient form:

**Theorem 7.2** *Let $q(t)$ be real-valued and continuous for $J : a \leqq t < \omega\ (\leqq \infty)$. Then* (7.1) *is disconjugate on $J$ if and only if there exists a continuously differentiable function $r(t)$ for $a < t < \omega$ such that*

$$r' + r^2 + q(t) \leqq 0. \tag{7.3}$$

*Exercise* 7.1. Formulate analogues of Theorem 7.2 when $J$ is open or $J$ is closed and bounded.

*Remark.* It is clear from § 1 that analogues of Theorem 7.2 remain valid if (7.1) is replaced by an equation of the form $(pu')' + qu = 0$ or $u'' + gu' + fu = 0$ provided that (7.3) is replaced by the corresponding

Riccati differential inequality $r' + r^2/p + q \leqq 0$ or $r' + r^2 + gr + f \leqq 0$, respectively.

**Proof.** First, if (7.1) is disconjugate on $J$, then (7.1) has a solution $u = u_0(t) > 0$ for $a < t < \omega$; see Corollary 6.1. In this case, $r = u_0'/u_0$ satisfies the Riccati equation

$$r' + r^2 + q(t) = 0 \tag{7.4}$$

for $a < t < \omega$. This proves the "only if" part of the theorem.

If there exists a continuously differentiable function $r(t)$ satisfying (7.3), let $q_0(t) \leqq 0$ denote the left side of (7.3) for $a < t < \omega$, so that $r' + r^2 + q - q_0 = 0$. Then

$$u'' + [q(t) - q_0(t)]\, u = 0$$

is a Sturm majorant for (7.1) on $a < t < \omega$ and, by § 2 (xiv), possesses the positive solution $u = \exp \int_c^t r(s)\, ds$, where $a < c < \omega$. This shows that (7.1) is disconjugate on $a < t < \omega$. In order to complete the proof, we must show that if $u_1(t) \not\equiv 0$ is the solution of (7.1) satisfying $u_1(a) = 0$ and $u_1'(a) = 1$, then $u_1(t) \neq 0$ for $a < t < \omega$. Suppose that this is not the case, so that $u_1(t_0) = 0$ for some $t_0$, $a < t_0 < \omega$. Since $u_1$ changes sign at $t = t_0$ and solutions of (7.1) depend continuously on initial conditions, it follows that if $\epsilon > 0$ is sufficiently small, then the solution of (7.1) satisfying $u(a + \epsilon) = 0$, $u'(a + \epsilon) = 1$ has a zero near $t_0$. This contradicts the fact that (7.1) is disconjugate on $a < t < \omega$ and proves the theorem.

*Exercise* 7.2. (*a*) Using the Remark following Theorem 7.2, show that if, in the differential equations

$$u'' + g_j(t)u' + f_j(t)u = 0, \tag{7.5$_j$}$$

where $j = 1, 2$, the coefficient functions are real-valued and continuous on $J: a \leqq t < \omega$ $(\leqq \infty)$ such that

$$g_1(t) \leqq g_2(t), \qquad f_1(t) \leqq f_2(t) \tag{7.6}$$

and if (7.5$_2$) has a solution $u(t)$ satisfying $u > 0$, $u' \geqq 0$ for $a < t < \omega$, then (7.5$_1$) is disconjugate on $J$. [For an application in Exercise 7.9, note that the conditions on (7.5$_2$) hold if (7.5$_2$) is disconjugate on $J$, $f_2(t) \geqq 0$ and $\int^{\omega} \left[\exp - \int_a^t g_2(s)\, ds\right] dt = \infty$; cf. Exercise 6.2.] (*b*) Let $f(t)$ be continuous and $g(t)$ continuously differentiable real-valued functions on $a \leqq t \leqq b$. Then

$$u'' + g(t)u' + f(t)u = 0$$

is disconjugate on $[a, b]$ if there exists a real number $c$ such that

$$f(t) - cg'(t) + c(c-1)g^2(t) \leqq 0$$

for $a \leqq t \leqq b$.

**Corollary 7.1.** *Let $q(t)$ be real-valued and continuous on $J: a \leqq t < \omega$, $C$ a constant, and*

$$Q(t) = C - \int_a^t q(s)\, ds. \tag{7.7}$$

*If the differential equation*

$$u'' + 4Q^2(t)u = 0 \tag{7.8}$$

*is disconjugate on $J$, then* (7.1) *is disconjugate on $J$.*

*Exercise* 7.3. Show that this corollary is false if the 4 in (7.8) is replaced by a constant $\gamma < 4$.

**Proof of Corollary 7.1.** In the Riccati equation (7.4) belonging to (7.1), introduce the new variable

$$\rho = r - Q, \tag{7.9}$$

so that $\rho' = r' + q$, and (7.4) becomes

$$\rho' + \rho^2 + 2Q\rho + Q^2 = 0. \tag{7.10}$$

Since $2Q\rho \leqq \rho^2 + Q^2$, a solution of

$$\rho' + 2(\rho^2 + Q^2) = 0 \tag{7.11}$$

on some interval satisfies

$$\rho' + \rho^2 + 2Q\rho + Q^2 \leqq 0. \tag{7.12}$$

The differential equation (7.11) can be written as

$$\sigma' + \sigma^2 + 4Q^2 = 0 \qquad \text{if} \quad \sigma = 2\rho. \tag{7.13}$$

Finally, (7.13) is the Riccati equation for (7.8).

Thus if (7.8) has a solution $u(t) > 0$ on $J$, then $\sigma = u'/u$ satisfies (7.13). Hence $\rho = \frac{1}{2}\sigma$ satisfies (7.12) and $r = \rho + Q$ is a solution of the differential inequality (7.3) on $J$. In virtue of Theorem 7.2, this proves the corollary.

*Exercise* 7.4. A counterpart of Corollary 7.1 can be stated as follows: Let $q(t)$ be real-valued and continuous for $0 \leqq t \leqq b$. Let $a$ be fixed, $0 \leqq a < b$. Suppose that

$$Q(t) = \int_0^t q(s)\, ds$$

has the properties that $Q(t) \geqq 0$ for $a \leqq t \leqq b$ and that if $z(t)$ is a solution of $z'' + Q^2(t)z = 0$, $z'(a) = 0$, then $z(t)$ has a zero on $a < t \leqq b$. Then a solution $u(t)$ of (7.1) satisfying $u'(0) = 0$ has a zero on $0 < t \leqq b$.

One of the main results on equations (7.1) which are nonoscillatory at $t = \infty$ will be based on the following lemma.

**Lemma 7.1.** *Let $q(t)$ be real-valued and continuous on $0 \leqq t < \infty$ with the property that* (7.1) *is nonoscillatory at $t = \infty$. Then a necessary and sufficient condition that*

$$\int^{\infty} \left(\frac{u'}{u}\right)^2 dt < \infty \tag{7.14}$$

*holds for one (and/or every) real-valued solution $u(t) \not\equiv 0$ of* (7.1) *is that*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_0^t q(s)\, ds\right) dt = C \qquad \text{exists} \tag{7.15}$$

*(as a finite number).*

*Remark.* For the application of this lemma, it is important to note that the proof will show that condition (7.15) can be relaxed to

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_0^t q(s)\, ds\right) dt > -\infty. \tag{7.16}$$

In other words, when (7.1) is nonoscillatory at $t = \infty$, then (7.16) implies (7.15); in fact, it implies the stronger relation

$$\frac{1}{T}\int_0^T \left| C - \int_0^t q(s)\, ds \right|^2 dt \to 0 \qquad \text{as} \quad T \to \infty. \tag{7.17}$$

*Exercise* 7.5. Let $q(t)$ be as in Lemma 7.1. Show that

$$\frac{u'}{u} \to 0 \qquad \text{as} \quad t \to \infty \tag{7.18}$$

holds for one (and/or every) real-valued relation $u(t) \not\equiv 0$ of (7.1) if and only if

$$\sup_{0<\sigma<\infty} \frac{1}{1+\sigma}\left| \int_t^{t+\sigma} q(s)\, ds \right| \to 0 \qquad \text{as} \quad t \to \infty. \tag{7.19}$$

[Note that (7.19) holds if, e.g., $q(t) \to 0$ as $t \to \infty$ or $\int^{\infty} |q(s)|^{\gamma}\, ds < \infty$ for some $\gamma \geqq 1$.]

**Proof.** Suppose first that (7.14) holds for a real-valued solution $u(t) \not\equiv 0$ of (7.1). Let $t = a$ exceed the largest zero, if any, of $u(t)$. Put $r = u'/u$ for $t \geqq a$, so that $r$ satisfies the Riccati equation (7.4). A quadrature gives

$$r(t) + \int_a^t r^2(s)\, ds = r(a) - \int_a^t q(s)\, ds \tag{7.20}$$

for $t \geqq a$. Then (7.14) implies that (7.20) can be written as

$$(7.21) \qquad r(t) - \int_t^\infty r^2(s)\,ds = C - \int_0^t q(s)\,ds,$$

where $C = r(a) - \int_a^\infty r^2(s)\,ds + \int_0^a q(s)\,ds$. By (7.14),

$$\frac{1}{T}\int_a^T r^2(t)\,dt, \qquad \frac{1}{T}\int_a^T \left(\int_t^\infty r^2(s)\,ds\right)^2 dt \to 0 \qquad \text{as} \quad T \to \infty.$$

Hence (7.21) implies (7.17) [by virtue of the inequality $(\alpha + \beta)^2 \leqq 2(\alpha^2 + \beta^2)$ for real numbers $\alpha$, $\beta$]. Since Schwarz's inequality [cf. (7.22)] shows that (7.15) is a consequence of (7.17), it follows that (7.15) is necessary for (7.14).

In order to prove the converse, assume (7.16), that $u(t) \not\equiv 0$ is any real-valued solution of (7.1), and that $u(t) > 0$ for $t \geqq a$. Then (7.20) holds for $r = u'/u$ and a quadrature of (7.20) gives

$$\frac{1}{t}\int_a^t r(s)\,ds + \frac{1}{t}\int_a^t \left(\int_a^s r^2(\sigma)\,d\sigma\right) ds$$
$$= \frac{1}{t}\,r(a)(t - a) - \frac{1}{t}\int_a^t \left(\int_a^s q(\sigma)\,d\sigma\right) ds.$$

The assumption (7.16) implies that the right side is bounded from above. Suppose, if possible, that (7.14) does not hold, then the second term on the left tends to $\infty$ as $t \to \infty$, thus

$$-\frac{1}{t}\int_a^t r(s)\,ds \geqq \frac{1}{2t}\int_a^t \left(\int_a^s r^2(\sigma)\,d\sigma\right) ds \qquad \text{for large } t.$$

Schwarz's inequality implies

$$(7.22) \qquad \left|\frac{1}{t}\int_a^t r(s)\,ds\right| \leqq \left(\frac{1}{t}\int_a^t r^2(s)\,ds\right)^{1/2}$$

and, consequently,

$$4t\int_a^t r^2(s)\,ds \geqq \left(\int_a^t \left(\int_a^s r^2(\sigma)\,d\sigma\right) ds\right)^2 \qquad \text{for large } t.$$

This can be written as

$$4tS' \geqq S^2, \qquad \text{where} \quad S(t) = \int_a^t \left(\int_a^s r^2(\sigma)\,d\sigma\right) ds \to \infty$$

as $t \to \infty$. A quadrature gives

$$\text{const.} - \frac{4}{S(t)} \geqq \log t \qquad \text{for large } t.$$

This contradiction shows that the hypotheses that (7.14) fails to hold is untenable and proves the theorem.

**Theorem 7.3.** *Let $q(t)$ be real-valued and continuous for* $0 \leqq t < \infty$. *A necessary condition for* (7.1) *to be nonoscillatory at* $t = \infty$ *is that either*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_0^t q(s)\,ds\right) dt = -\infty \tag{7.23}$$

*or that* (7.15) *holds* [*and, in the latter case,* (7.17) *holds*].

It follows, e.g., that if $q(t) \geqq 0$, then, in order for (7.1) to be nonoscillatory at $t = \infty$, it is necessary that $\int^\infty q(t)\,dt < \infty$. In fact, as is seen from Exercise 7.8, it is necessary that $\int^\infty t^\gamma q(t)\,dt < \infty$ for every $\gamma < 1$.

**Proof.** Suppose that (7.1) is nonoscillatory at $t = \infty$ and that (7.23) fails to hold, so that (7.16) holds. The validity of (7.17) must be verified. But this is clear from the proof of Lemma 7.1 which shows that, on the one hand, (7.16) implies (7.14) for every real-valued solution $u(t) \not\equiv 0$ of (7.1) and, on the other hand, that (7.14) for some solution assures (7.17).

*Exercise* 7.6. Let $q(t)$ be as in Theorem 7.3 and, in addition, satisfy

$$q(t) \to 0 \quad \text{as} \quad t \to \infty \tag{7.24}$$

or, more generally, (7.19). Then a necessary condition for (7.1) to be nonoscillatory at $t = \infty$ is that either

$$\int_0^T q(t)\,dt \to -\infty \quad \text{as} \quad T \to \infty \tag{7.25}$$

or that

$$\int_0^\infty q(t)\,dt = \lim_{T\to\infty}\int_0^T q(t)\,dt \quad \text{converges} \tag{7.26}$$

(possibly conditionally).

*Exercise* 7.7. (*a*) Give examples to show that (7.15) in Theorem 7.3 is compatible with each of the possibilities

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_0^t q(s)\,ds\right) dt \quad \text{is} \quad -\infty, \quad \text{finite}, \quad \text{or} \quad +\infty. \tag{7.27}$$

(*b*) Show that if, in Theorem 7.3, $q(t)$ is half-bounded or, more generally, if there exists an $\epsilon > 0$ such that

$$\int_t^{s+t} q(\sigma)\,d\sigma \quad \text{is half-bounded for} \quad 0 \leqq t < \infty, \quad 0 \leqq s \leqq \epsilon,$$

then a necessary condition that (7.1) be nonoscillatory is that either (7.15) or (7.25) hold. See Hartman [11].

Changes of variables in (7.1) followed by applications of Theorem 7.3 (and its consequences) give new necessary conditions for (7.1) to be nonoscillatory. This is illustrated by the following exercise.

*Exercise* 7.8. (*a*) Introduce the new independent and dependent variables $s = t^\gamma$, $\gamma > 0$ and $z = t^{(\gamma-1)/2}u$, and state necessary conditions for the resulting equation and/or (7.1) to be nonoscillatory at $t = \infty$. (*b*) In particular, show that if $q(t) \geqq 0$ and (7.1) is nonoscillatory at $t = \infty$, then $\int^\infty t^{1-\gamma}q(t)\,dt < \infty$ for all $\gamma > 0$.

The next result gives a conclusion very different from (7.17) in Theorem 7.3 in the case (7.15).

**Theorem 7.4.** *Let $q(t)$ be as in Theorem* 7.3 *such that* (7.1) *is nonoscillatory at $t = \infty$ and* (7.23) *does not hold* [*so that* (7.15) *does*]. *Then*

$$\int^\infty \exp\left(-\gamma\int_0^t Q(s)\,ds\right)dt = \infty \qquad \text{for} \quad 0 < \gamma \leqq 4, \tag{7.28}$$

*where*

$$Q(t) = C - \int_0^t q(s)\,ds. \tag{7.29}$$

In applications, interesting cases of this theorem occur if (7.26) holds, so that

$$Q(t) = \int_t^\infty q(s)\,ds. \tag{7.30}$$

It is readily verified from $q(t) = \mu/t^2$, $t \geqq 1$, that the "4" in (7.28) cannot be replaced by a larger constant. It is rather curious that the proof of Corollary 7.1 and Theorem 7.4 depend on the inequality $2Q\rho \leqq \rho^2 + Q^2$. In the proof of Corollary 7.1, this inequality is used to deduce (7.12) from (7.11); in the proof of Theorem 7.4, it is used to deduce

$$\rho' + 4Q\rho \leqq 0 \tag{7.31}$$

from (7.10).

**Proof.** Let $u = u(t) \not\equiv 0$ be a real-valued solution of (7.1) and suppose $T$ is so large that $u(t) \neq 0$ for $t \geqq T$. Since it is assumed that (7.15) holds, the relation (7.14) holds. Thus if $r = u'/u$, a quadrature of the corresponding Riccati equation gives (7.21) as in the proof of Lemma 7.1. Rewrite (7.21) as $r(t) = \rho(t) + Q(t)$, where

$$\rho(t) = \int_t^\infty r^2(s)\,ds. \tag{7.32}$$

Since $\rho' = -r^2 = -(\rho + Q)^2$, the equation (7.10) holds. This gives (7.31). In particular, if $Q(t) \geqq 0$, then

$$\rho' + \gamma Q\rho \leqq 0 \qquad \text{for} \quad 0 < \gamma \leqq 4. \tag{7.33}$$

Note that if $Q < 0$, then (7.33) holds since $\rho' \leqq 0$, $\rho \geqq 0$. Hence (7.33) holds for $t \geqq T$. Since the result to be proved is trivial if $q(t) \equiv 0$ for large $t$, it can be supposed that this is not the case. Hence $r \not\equiv 0$ for large $t$ and so, $\rho(t) > 0$. Consequently, (7.33) gives

$$\rho(t) \leqq \rho(T) \exp\left(-\gamma \int_T^t Q(s)\, ds\right) \quad \text{for} \quad t \geqq T. \tag{7.34}$$

Suppose, if possible, that (7.28) fails to hold, then (7.32), (7.34) show that

$$\int^\infty \rho(t)\, dt < \infty, \qquad \text{hence} \qquad \int^\infty t r^2(t)\, dt < \infty$$

holds for $r = u'/u$, where $u(t) \not\equiv 0$ is an arbitrary real-valued solution of (7.1). It will be shown that this leads to a contradiction. To this end, note that

$$\log \frac{u(t)}{u(T)} = \int_T^t r(s)\, ds.$$

Thus Schwarz's inequality gives

$$\left[\log \frac{u(t)}{u(T)}\right]^2 \leqq \left(\int_T^t s r^2(s)\, ds\right)\left(\int_T^t \frac{ds}{s}\right).$$

Consequently there exist constants $c_0$, $c$ such that $|u(t)| \leqq c_0 \exp c(\log t)^{1/2}$ for large $t$. It follows that

$$\int^\infty \frac{dt}{u^2(t)} \geqq \frac{1}{c_0^{\,2}} \int^\infty \exp\left[-2c(\log t)^{1/2}\right] dt = \infty$$

for all real-valued solutions ($\not\equiv 0$) of (7.1). This contradicts the existence (Theorem 6.4) of nonprincipal solutions and completes the proof.

*Exercise* 7.9. In the differential equations

$$u'' + q_j(t)u = 0, \tag{7.35$_j$}$$

where $j = 1, 2$, let $q(t)$ be real-valued and continuous for large $t$ and such that

$$Q_j(t) = \int_t^\infty q_j(s)\, ds = \lim_{T\to\infty} \int_t^T$$

converges (possibly conditionally), $|Q_1(t)| \leqq Q_2(t)$, and (7.35$_2$) is nonoscillatory at $t = \infty$. Show that (7.35$_1$) is nonoscillatory at $t = \infty$.

## 8. Asymptotic Integrations. Elliptic Cases

In the next two sections, we will consider the problem of the asymptotic integration of equations

$$u'' + q(t)u = 0, \tag{8.1}$$

where $q(t)$ is continuous for large $t$. Except for the last part of this section, the main interest will center around the situations where the coefficient $q(t)$ is nearly a constant or (8.1) can be reduced to this case. The last part of this section (see Exercises 8.6, 8.8) deals with bounds for $|u'|$ when $q(t)$ is bounded from above.

When $q(t)$ is a constant, say $\lambda$, and $\lambda$ is real and positive, then the solutions are, roughly speaking, of the same order of magnitude. On the other hand, if $\lambda$ is not real and positive, then essentially there is one small solution, as $t \to \infty$, and the other solutions are large. These facts indicate that different techniques will be needed when $q(t)$ is nearly a constant $\lambda$, and $\lambda$ is or is not real and positive. In this section, the first case will be considered.

**Theorem 8.1.** *In the differential equations* (8.1) *and*

$$w'' + q_0(t)w = 0, \tag{8.2}$$

*let* $q(t)$, $q_0(t)$ *be continuous, complex-valued functions for* $0 \leqq t < \infty$ *satisfying*

$$\int^{\infty} |w(t)|^2 \, |q_0(t) - q(t)| \, dt < \infty \tag{8.3}$$

*for every solution* $w(t)$ *of* (8.2). *Let* $u_0(t)$, $v_0(t)$ *be linearly independent solutions of* (8.2). *Then to every solution* $u(t)$ *of* (8.1), *there corresponds at least one pair of constants* $\alpha$, $\beta$ *such that*

$$\begin{aligned} u(t) &= [\alpha + o(1)]u_0(t) + [\beta + o(1)]v_0(t), \\ u'(t) &= [\alpha + o(1)]u_0'(t) + [\beta + o(1)]v_0'(t), \end{aligned} \tag{8.4}$$

*as* $t \to \infty$; *conversely, to every pair of constants* $\alpha$, $\beta$, *there exists at least one solution* $u(t)$ *of* (8.1) *satisfying* (8.4).

Note that for a given $u(t)$, (8.4) might hold for more than one pair of constants $(\alpha, \beta)$. This is true, e.g., if $v_0(t) = o(u_0(t))$ as $t \to \infty$.

An interesting aspect of Theorem 8.1 is the fact that the main condition (8.3) does not involve the derivatives $w'(t)$ of solutions $w(t)$ of (8.2). This advantage is lost if (8.1) or (8.2) is replaced by a more complicated equation as in the Exercise 8.4 below.

**Proof.** It can be supposed that $\det Y(t) = 1$, where

$$Y(t) = \begin{pmatrix} u_0 & v_0 \\ u_0' & v_0' \end{pmatrix}.$$

Write (8.1) as a first order system $x' = A(t)x$ for the binary vectors $x = (u, u')$; cf. (2.5). Then the variations of constants $x = Y(t)y$ reduce the system $x' = A(t)x$, say to $y' = C(t)y$, in (2.28); cf. § 2(xi). Thus

Theorem 8.1 follows from the linear case of Theorem X 1.2, cf. Exercise X 1.4.

**Corollary 8.1.** *Let $q(t)$ be a continuous complex-valued function on $0 \leqq t < \infty$ satisfying*

$$\int^\infty |1 - q(t)|\, dt < \infty. \tag{8.5}$$

*Then if $\alpha$, $\beta$ are constants, there exists one and only one solution $u(t)$ of* (8.1) *satisfying the asymptotic relations*

$$\begin{aligned} u &= [\alpha + o(1)] \cos t + [\beta + o(1)] \sin t, \\ u' &= -[\alpha + o(1)] \sin t + [\beta + o(1)] \cos t. \end{aligned} \tag{8.6}$$

The relations (8.6) can also be written as $u = \delta \cos [t + \gamma + o(1)]$, $u' = -\delta \sin [t + \gamma + o(1)]$ as $t \to \infty$ for some constants $\gamma$ and $\delta$.

*Exercise* 8.1. Show that if $\alpha$, $\beta$ are constants, there exists a unique solution $v(t)$ of the Bessel equation $t^2 v'' + tv' + (t^2 - \mu^2)v = 0$ for $t > 0$ such that $u(t) = t^{1/2} v(t)$ satisfies (8.6) as $t \to \infty$.

*Exercise* 8.2. Show that the conclusion of Corollary 8.1 is correct if (8.5) is relaxed to the following conditions in which $f(t) = 1 - q(t)$: the integrals

$$g_0(t) = \int_t^\infty f(s)\, ds, \quad g_1(t) = \int_t^\infty f(s) \cos 2s\, ds, \quad g_2(t) = \int_t^\infty f(s) \sin 2s\, ds$$

exist as (possibly conditional) improper Riemann integrals $\left(\int^\infty = \lim \int^T \text{ as } T \to \infty\right)$ and $\int^\infty |g_k(t) f(t)|\, dt < \infty$ for $k = 0, 1, 2$.

*Exercise* 8.3. (*a*) Let $q(t)$ be a positive function on $0 \leqq t < \infty$ possessing a continuous second derivative and such that

$$\int^\infty q^{1/2}(t)\, dt = \infty \quad \text{and} \quad \int^\infty \left| \frac{5q'^2}{16q^3} - \frac{q''}{4q^2} \right| q^{1/2}\, dt < \infty. \tag{8.7}$$

Then the assertion of Corollary 8.1 remains valid if (8.6) is replaced by

$$q^{1/4} u = [\alpha + o(1)] \cos \int_0^t q^{1/2}(s)\, ds + [\beta + o(1)] \sin \int_0^t q^{1/2}(s)\, ds,$$

$$(q^{1/4} u)' q^{-1/2} = -[\alpha + o(1)] \sin \int_0^t q^{1/2}(s)\, ds + [\beta + o(1)] \cos \int_0^t q^{1/2}(s)\, ds.$$

(*b*) Show that (8.7) in (*a*) holds if $0 \neq \alpha > -\frac{1}{2}$, $q(t) \geqq \text{const.} > 0$ for $0 \leqq t < \infty$, and $f(t) = q^\alpha(t) \geqq 0$ has a continuous second derivative such that $\int^\infty |f''(t)|\, dt < \infty$. [In fact, for the validity of the conclusion of (*a*), it can be merely supposed that $f(t)$ has a continuous first derivative which is of bounded variation on $0 \leqq t < \infty$, i.e., $\int^{\infty} |df'(t)| < \infty$; e.g., $f'(t)$ is monotone and bounded. This last refinement follows from the first part by approximating $q(t)$ by suitable smooth functions.]

*Exercise* 8.4. (*a*) In the differential equations

$$(p_j u')' + r_j u' + q_j u = 0, \qquad j = 0, 1, \tag{8.8$_j$}$$

let $p_j(t) \neq 0$, $q_j(t)$, $r_j(t)$ be continuous complex-valued functions for $0 \leqq t < \infty$ such that

$$\int^{\infty} |w|^2 |q_1 - q_0| \cdot \left| \exp \int^t \frac{r_0\, ds}{p_0} \right| dt < \infty,$$

$$\int^{\infty} |p_0 w'|^2 \left| \frac{1}{p_1} - \frac{1}{p_0} \right| \cdot \left| \exp \int^t \frac{r_0\, ds}{p_0} \right| dt < \infty, \tag{8.9}$$

$$\int^{\infty} \left| \frac{r_0}{p_0} - \frac{r_1}{p_1} \right| \left( 1 + \left| p_0 w w' \exp \int^t \frac{r_0\, ds}{p_0} \right| \right) dt < \infty$$

hold for all solutions $u = w(t)$ of $(8.8_0)$. Let $u_0(t), v_0(t)$ be linearly independent solutions of $(8.8_0)$. Then to every solution $u(t)$ of $(8.8_1)$, there corresponds at least one pair of constants $\alpha$, $\beta$ such that (8.4) holds; conversely, if $\alpha$, $\beta$ are constants, then there is at least one solution $u(t)$ of $(8.8_1)$ satisfying (8.4). (*b*) In the differential equation (8.1), let $q(t) \neq 0$ be a continuous, complex-valued function for $0 \leqq t < \infty$ such that $q(t)$ is of bounded variation over $0 \leqq t < \infty$ $\left(\text{i.e., } \int_0^{\infty} |dq| < \infty\right)$; $c_0 = \lim q(t)$, $t \to \infty$, is a positive constant; and the solutions $u(t)$ of (8.1) are bounded (e.g., if $q(t)$ is real-valued or, more generally, $\int^{\infty} |\mathrm{Im}\, q(t)|\, dt < \infty$, then solutions $u(t)$ and their derivatives $u'(t)$ are bounded). Let $\alpha$, $\beta$ be constants. Then (8.1) has a unique solution $u(t)$ satisfying, as $t \to \infty$,

$$u = [\alpha + o(1)] \cos \int_0^t q^{1/2}(s)\, ds + [\beta + o(1)] \sin \int_0^t q^{1/2}(s)\, ds,$$

(8.10)

$$u' = [\alpha + o(1)] c_0^{1/2} \sin \int_0^t q^{1/2}(s)\, ds + [\beta + o(1)] c_0^{1/2} \cos \int_0^t q^{1/2}(s)\, ds,$$

where $q^{1/2}(t)$ is any fixed continuous determination of the square root of $q$.

*Exercise* 8.5. Let $f(t)$ be a nonvanishing (possibly complex-valued) function for $t \geqq 0$ having a continuous derivative satisfying

$$\int^{\infty} \left| d\left(\frac{f'}{f^{3/2}}\right) \right| < \infty, \qquad \gamma = \lim_{t \to \infty} \frac{f'}{4f^{3/2}}, \qquad \text{and} \quad \gamma^2 \neq 1.$$

Suppose further that

$$\text{(8.11)} \qquad \exp \pm i\int^t f^{1/2}\left(1 - \frac{f'^2}{16f^3}\right)^{1/2} dr \qquad \text{are bounded}$$

as $t \to \infty$. Then the differential equation

$$\text{(8.12)} \qquad u'' + f(t)u = 0$$

has a pair of solutions satisfying, as $t \to \infty$,

$$u \sim f^{-1/4}(t) \exp \pm i\int^t f^{1/2}\left(1 - \frac{f'^2}{16f^3}\right)^{1/2} dr,$$

$$u' \sim [-\gamma \pm i(1 - \gamma^2)^{1/2}]f^{1/2}u.$$

Here all powers of $f(t)$ that occur can be assumed to be integral (positive or negative) powers of a fixed continuous fourth root $f^{1/4}(t)$ of $f(t)$. Condition (8.11) is trivially satisfied if $f(t)$ is real-valued and positive and $0 < \gamma^2 < 1$.

The object of the next exercise is to obtain bounds for derivatives of solutions of (8.1) or, more generally, the inhomogeneous equation

$$\text{(8.13)} \qquad u'' + q(t)u = f(t).$$

*Exercise* 8.6. Let $q(t)$, $f(t)$ be continuous real-valued functions on $0 \leqq t \leqq t_0$. Let the positive constants $\epsilon$, $1/\theta > 1$, $C$ be such that

$$\text{(8.14)} \qquad 0 < \epsilon \leqq \frac{\theta}{C}$$

and

$$\text{(8.15)} \qquad \int_a^b q(\tau)\, d\tau \leqq C \qquad \text{if} \quad b - a \leqq \frac{\theta}{C}$$

and $0 \leqq a < b \leqq t_0$. [The inequality (8.15) holds, e.g., is $q(t) \leqq C^2/\theta$ for $0 \leqq t \leqq t_0$.] Let $u = u(t)$ be a real-valued solution of (8.13). Consider the case

$$uu' \geqq 0 \text{ at } t = T,\ 0 \leqq T \leqq t_0 - \theta/C,\ S = T + \theta/C,\ U = T + \epsilon$$

or the case

$$uu' \leqq 0 \text{ at } t = T,\ \theta/C \leqq T \leqq t_0,\ S = T - \theta/C,\ U = T - \epsilon.$$

(*a*) Show that, in either case,

$$\text{(8.16)} \quad |u'(T)| \leqq \frac{1}{1-\theta}\left|\int_T^U |f|\, d\tau\right| + \max\left[\frac{C\,|u(T)|}{1-\theta},\ \left(C + \frac{1}{\epsilon}\right)|u(U)|\right].$$

(*b*) Show that (8.16) holds if $|u(U)|$ is replaced by $2\epsilon^{-1}\left|\int_T^U |u|\,d\tau\right|$. (*c*) Part (*a*) implies that if $\theta/C \leqq t \leqq t_0 - \theta/C$, then

$$(8.17)\qquad |u'(t)| \leqq \frac{1}{1-\theta}\int_{t-\epsilon}^{t+\epsilon} |f|\,d\tau + C_0(\epsilon)\sum_{j=-1}^{1} |u(t+j\epsilon)|,$$

where $C_0(\epsilon) = \max\,(C/(1-\theta),\ C + 1/\epsilon)$. (*d*) Put

$$(8.18)\qquad r(t) = |u'(t)| + C|u(t)|.$$

Show that there exists a nondecreasing function $K(\Delta)$ for $0 < \Delta < \min\,(\theta, 1-\theta)/C$ such that

$$(8.19)\qquad r(t) \leqq K(\Delta)\left\{r(a) + r(b) + \int_a^b |f|\,d\tau\right\} \qquad \text{for} \quad a \leqq t \leqq b$$

if $b - a = \Delta$ and $0 \leqq a < b \leqq t_0$.

The results of the last exercise can be extended to an equation of the form

$$(8.20)\qquad u'' + p(t)u' + q(t)u = f(t);$$

see Exercise 8.8. In fact, the results for (8.20) can be derived from those on (8.13) by the use of the lemma given in the next exercise which has nothing to do with differential equations.

*Exercise* 8.7. Let $h(t) \geqq 0$ be of bounded variation and $g(t)$ continuous on an interval $a \leqq t \leqq b$. Then

$$(8.21)\qquad \int_a^b h(t)\,dg(t) \leqq (\inf h + \operatorname{var} h) \sup_{a \leqq \alpha < \beta \leqq b} \int_\alpha^\beta dg(t),$$

where the integrals are Riemann-Stieltjes integrals and var $h$ denotes the total variation of $h(t)$ on $a \leqq t \leqq b$.

*Exercise* 8.8. Let $p(t), q(t), f(t)$ be continuous real-valued functions on $0 \leqq t \leqq t_0$ and $u(t)$ a real-valued solution of (8.20). Let $1/\theta > 1$, $C$ be positive constants such that (8.15) holds. Consider the two cases in Exercise 8.6, with (8.14) replaced by

$$(8.22)\qquad 0 < \epsilon \leqq \theta/CE^2, \qquad \text{where} \quad E = \exp\left|\int_T^S |p(\tau)|\,d\tau\right|.$$

(*a*) Then parts (*a*), (*b*) of Exercise 8.6 hold if $|u'(T)|$ in (8.16) is replaced by $|u'(T)|/E$. (*b*) Part (*c*) of Exercise 8.6 holds if $|u'(t)|$ in (8.17) is replaced by $|u'(t)|/E$, where

$$(8.23)\qquad 0 < \epsilon \leqq \theta/CE^2 \qquad \text{and}$$

$$E = \max \exp \int_t^{t+\theta/C} |p(\tau)|\,d\tau \qquad \text{for} \quad 0 \leqq t \leqq t_0 - \theta/C.$$

(*c*) Part (*d*) of Exercise 8.6 holds if $\Delta$ is restricted to $0 < \Delta < \min(\theta, 1 - \theta)/CE^2$, where $E$ is defined in (8.23).

## 9. Asymptotic Integrations. Nonelliptic Cases

Asymptotic integrations of $u'' + q(t)u = 0$, where $q(t)$ is "nearly" a real, but not positive constant, can be based on Chapter X as were the results of the last section. Instead a different technique will be used in this section; this technique takes greater advantage of the special structure of the second order equation

$$(p(t)u')' + q(t)u = 0. \tag{9.1}$$

This equation is equivalent to a binary system of the form

$$v' = \beta(t)z, \qquad z' = \gamma(t)v \tag{9.2}$$

in which the diagonal elements vanish. [This system cannot be reduced to an equation of the form (9.1) unless either $\beta(t)$ or $\gamma(t)$ does not vanish.] The main results on (9.1) will be based on lemmas dealing with (9.2).

A system of the form (9.2) on $0 \leqq t < \omega\ (\leqq \infty)$ will be called of *type Z* at $t = \omega$ if

$$z(\omega) = \lim z(t) \qquad \text{exists as} \quad t \to \omega$$

for every solution $(v(t), z(t))$, and $z(\omega) \neq 0$ for some solution. It is easy to see that (9.2) is of type $Z$ if and only if there exist linearly independent solutions $(v_j(t), z_j(t))$, $j = 0, 1$, such that $\lim z_0(t) = 0$ and $\lim z_1(t) = 1$.

**Lemma 9.1.** *Let $\beta(t)$, $\gamma(t)$ be continuous complex-valued functions for $0 \leqq t < \omega\ (\leqq \infty)$. Suppose that*

$$(9.3_1)\quad \int^{\omega} |\gamma(t)|\, dt < \infty; \qquad (9.3_2)\quad \int^{\omega} |\gamma(t)| \left(\int^{t} |\beta(s)|\, ds\right) dt < \infty$$

*or, more generally, that*

$$(9.4_1)\quad \int^{\omega} \gamma(t)\, dt = \lim_{T \to \omega} \int^{T} \gamma(t)\, dt \qquad \text{exists}$$

(*possibly conditionally*) *and that*

$$(9.4_2)\quad \int^{\omega} |\beta(s)|\, \Gamma(s)\, ds < \infty, \qquad \text{where} \quad \Gamma(s) = \sup_{s \leqq t < \omega} \left| \int_{t}^{\omega} \gamma(r)\, dr \right|.$$

*Then* (9.2) *is of type Z.*

Unless $\beta(t) \equiv 0$, the condition $(9.3_2)$ implies $(9.3_1)$. If the order of integration is reversed, it is seen that $(9.3_2)$ is equivalent to

$$\int^{\omega} |\beta(s)| \left(\int_{s}^{\omega} |\gamma(t)|\, dt\right) ds < \infty.$$

This shows that (9.3) implies (9.4). Lemma 9.1 has a partial converse.

**Lemma 9.2.** *If $\beta(t)$ and $\gamma(t)$, where $0 \leqq t < \omega$ ($\leqq \infty$), are continuous real-valued functions which do not change sign (i.e., $\beta \geqq 0$ or $\beta \leqq 0$ and $\gamma \geqq 0$ or $\gamma \leqq 0$) and if* (9.2) *is of type Z, then* $(9.3_1)$–$(9.3_2)$ *hold.*

*Exercise* 9.1. Generalize Lemma 9.1 to the case where (9.2) is replaced by a $d$-dimensional system of the form $v^{j\prime} = \gamma_j(t)v^{j+1}$, where $j = 1, \ldots, d$ and $v^{d+1} = v^1$.

**Proof of Lemma 9.1.** Two quadratures of (9.2) give

$$(9.5) \quad v(t) = \int_T^t \beta(s)z(s)\,ds + c_1, \qquad c_1 = v(T),$$

$$(9.6) \quad z(t) = \int_T^t \gamma(s)\int_T^s \beta(r)z(r)\,dr\,ds + c_1\int_T^t \gamma(s)\,ds + c_2, \quad c_2 = z(T).$$

On interchanging the order of integration, the last formula becomes

$$(9.7) \qquad z(t) = \int_T^t \beta(r)z(r)\int_r^t \gamma(s)\,ds\,dr + c_1\int_T^t \gamma(s)\,ds + c_2.$$

If $t \geqq T$, then $T \leqq r \leqq t$ and the definition of $\Gamma$ in $(9.4_2)$ imply that

$$(9.8) \qquad \left|\int_r^t \gamma(s)\,ds\right| \leqq \left|\int_r^\omega \gamma(s)\,ds\right| + \left|\int_t^\omega \gamma(s)\,ds\right| \leqq 2\Gamma(r).$$

Consequently

$$|z(t)| \leqq 2\int_T^t |\beta(s)|\,\Gamma(s)\,|z(s)|\,ds + C,$$

where

$$(9.9) \qquad C = 2\,|c_1|\,\Gamma(T) + |c_2|.$$

By Gronwall's inequality (Theorem III 1.1),

$$(9.10) \quad |z(t)| \leqq C \exp 2\int_T^t |\beta(s)|\,\Gamma(s)\,ds \leqq C \exp 2\int_T^\omega |\beta(s)|\,\Gamma(s)\,ds$$

for $T \leqq t < \omega$. Hence $(9.4_2)$ implies that $z(t)$ is bounded. The relations (9.7) and $(9.4_2)$ then show that $z(\omega) = \lim z(t)$ as $t \to \omega$ exists.

The limit $z(\omega)$ is obtained by writing $t = \omega$ in (9.7). In order to show that $z(\omega) \neq 0$ for some solution of (9.2), choose the initial conditions $c_1 = v(T) = 0$ and $c_2 = z(T) = 1$ in (9.5), (9.6). Thus $C = 1$ in (9.9) and (9.10) and so (9.7), (9.8), and (9.10) give

$$|z(\omega) - 1| \leqq 2\left(\int_T^\omega |\beta(r)|\,\Gamma(r)\,dr\right) \exp 2\int_T^\omega |\beta(s)|\,\Gamma(s)\,ds.$$

Since the right side tends to 0 as $T \to \omega$, it follows that if $T$ is sufficiently near to $\omega$, then $z(\omega) \neq 0$. This proves Lemma 9.1.

**Proof of Lemma 9.2.** Let $(v(t), z(t))$ be a solution of (9.2) such that $z(\omega) = 1$. It can also be supposed that $v(T) = 0$ for some $T$. Otherwise it is possible to add to $(v(t), z(t))$ a suitable multiple of a solution $(v_0(t), z_0(t)) \not\equiv 0$ for which $z(\omega) = 0$. In fact $v_0(t) \equiv 0$ cannot hold, for then (9.2) shows that $z_0(t) \equiv z_0(\omega) = 0$.

Thus $c_1 = 0$ in (9.6) and $z(\omega) = 1$ shows that $(9.3_2)$ holds (since $\beta$, $\gamma$ do not change sign). If $\beta(t) \not\equiv 0$ for $t$ near $\omega$, $(9.3_1)$ follows. If, however, $\beta(t) \equiv 0$ and (9.2) is of type $Z$, then $(9.3_1)$ holds when $\gamma$ does not change signs. This completes the proof.

Let $(v(t), z(t))$, $(v_1(t), z_1(t))$ be solutions of (9.2). Then

$$z_1(t)v(t) - v_1(t)z(t) = c_0 \tag{9.11}$$

is a constant. This follows from Theorem IV 1.2 (or can easily be verified by differentiation). If $z_1(t) \neq 0$ and (9.11) is multiplied by $\gamma(t)/z_1^2(t)$, it is seen from (9.2) that $(z/z_1)' = c_0\gamma/z_1^2$, and hence there is a constant $c_1$ such that

$$z(t) = c_1 z_1(t) + c_0 z_1(t) \int_T^t \frac{\gamma(s)\, ds}{z_1^2(s)} \tag{9.12}$$

if $z_1 \neq 0$ on the interval $[T, t]$. Similarly, if $v_1 \neq 0$ for $[T, t]$ then $(v/v_1)' = -c_0\beta/v_1^2$ and

$$v(t) = c_1 v_1(t) - c_0 v_1(t) \int_T^t \frac{\beta(s)\, ds}{v_1^2(s)}. \tag{9.13}$$

Conversely, if $z_1 \neq 0$ [or $v_1 \neq 0$] in the $t$-interval $[T, t]$, then (9.12) [or (9.13)] and (9.11) define a solution $(v(t), z(t))$ of (9.2).

*Exercise* 9.2. Suppose that (9.2) is of type $Z$ and that $(v_1(t), z_1(t))$ is a solution of (9.2) satisfying $z_1(\omega) = 1$. (*a*) Show that

$$\int^\omega \frac{\gamma(t)\, dt}{z_1^2(t)} = \lim_{T \to \omega} \int^T \frac{\gamma(t)\, dt}{z_1^2(t)} \qquad \text{exists}$$

and that (9.2) has a solution $(v_0(t), z_0(t))$ in which

$$z_0(t) = z_1(t) \int_t^\omega \frac{\gamma(s)\, ds}{z_1^2(s)} \tag{9.14}$$

for $t$ near $\omega$. (*b*) If $(v(t), z(t))$ is any solution of (9.2), then

$$v(t) = O\left(1 + \int^t |\beta(s)|\, ds\right) \qquad \text{as} \quad t \to \omega.$$

If, in addition, (9.3) holds, then (9.14) satisfies

$$z_0(t) = O\left(\int_t^\omega |\gamma(s)|\, ds\right), \tag{9.15}$$

$c = \lim v_0(t)$ exists as $t \to \omega$, and

$$(9.16) \qquad v_0(t) = c + O\left(\int_t^\omega |\beta(s)| \int_s^\omega |\gamma(r)|\, dr\, ds\right) \qquad \text{as} \quad t \to \omega.$$

Also, if $\gamma(t)$ is real-valued and does not change signs, then (9.15) can be improved to

$$(9.17) \qquad z_0(t) \sim \int_t^\omega \gamma(s)\, ds.$$

**Lemma 9.3.** *Let $\beta(t)$, $\gamma(t)$ be as in Lemma* 9.1. *In addition, suppose that $\beta(t) \geqq 0$ and that*

$$(9.18) \qquad \int^\omega \beta(t)\, dt = \infty.$$

*Then* (9.2) *has a pair of solutions $(v_j(t), z_j(t))$ for $j = 0, 1$, satisfying, as $t \to \omega$,*

$$(9.19_0) \qquad v_0 \sim 1, \qquad z_0 = o\left(\frac{1}{\int^t \beta(s)\, ds}\right);$$

$$(9.19_1) \qquad v_1 \sim \int^t \beta(s)\, ds, \qquad z_1 \sim 1.$$

This has a partial converse.

**Lemma 9.4.** *Let $\beta(t)$, $\gamma(t)$ be continuous real-valued functions such that $\beta(t) \geqq 0$ satisfies* (9.18) *and $\gamma(t)$ does not change signs. Let* (9.2) *have a solution satisfying either* $(9.19_0)$ *or* $(9.19_1)$. *Then* (9.3) *holds* [*so that* (9.2) *has solutions satisfying* $(9.19_0)$ *and* $(9.19_1)$].

*Exercise* 9.3. Prove Lemma 9.4.

**Proof of Lemma 9.3.** By Lemma 9.1, (9.2) has a solution $(v_1(t), z_1(t))$ such that $z_1(\omega) = 1$. Thus the first part of $(9.19_1)$ follows from the first equation in (9.2). Note that

$$(9.20) \qquad \int^t \frac{\beta(s)\, ds}{\left(\int^s \beta(r)\, dr\right)^2} = \text{const.} - \frac{1}{\int^t \beta(s)\, ds}$$

tends to const. as $t \to \omega$ by (9.18). Consequently, the integral $c_1 = \int_T^\omega \beta(s)\, ds / v_1^2(s)$ is absolutely convergent (for $T$ near $\omega$). It follows from (9.13) with the choice $c_0 = 1$, that (9.2) has a solution $(v, z) = (v_0, z_0)$ satisfying (9.11) with $c_0 = 1$ and

$$v_0(t) = v_1(t) \int_t^\omega \frac{\beta(s)\, ds}{v_1^2(s)}.$$

Then $v_0 \sim 1$ follows from the first part of $(9.19_1)$ and (9.20). Letting

$(v, z, c_0) = (v_0, z_0, 1)$ in (9.11) and solving for $z_0$ gives the last part of $(9.19_0)$. This completes the proof.

**Theorem 9.1.** *Let $p(t)$ be a positive and $q_0(t)$ a real-valued continuous function for $0 \leqq t < \omega$ such that*

$$(9.21) \qquad (p(t)x')' + q_0(t)x = 0$$

*is nonoscillatory at $t = \omega$ and let $x_0(t)$, $x_1(t)$ be principal, nonprincipal solutions of* (9.21); *cf.* § 6. *Suppose that $q(t)$ is a continuous complex-valued function satisfying*

$$(9.22) \qquad \int^{\omega} |x_0(t)x_1(t)| \cdot |q(t) - q_0(t)| \, dt < \infty$$

*or, more generally,*

$$(9.23_1) \qquad \int^{\omega} (q - q_0)x_0^2 \, dt = \lim_{T \to \omega} \int^{T} (q - q_0)x_0^2 \, dt \qquad \text{exists,}$$

$$(9.23_2) \qquad \int^{\omega} \frac{\Gamma(s)\, ds}{p(s)x_0^2(s)} < \infty, \qquad \text{where} \quad \Gamma(s) = \sup_{s \leqq t < \omega} \left| \int_t^{\omega} (q - q_0)x_0^2 \, dr \right|.$$

*Then* (9.1) *has a pair of solutions $u_0(t)$, $u_1(t)$ satisfying, as $t \to \omega$,*

$$(9.24) \qquad u_j \sim x_j,$$

$$(9.25) \qquad \frac{pu_j'}{u_j} = \frac{px_j'}{x_j} + o\left(\frac{1}{|x_0x_1|}\right),$$

*for $j = 0, 1$.*

*Exercise* 9.4. Verify that if $q(t)$ is real-valued, $q(t) - q_0(t)$ does not change signs, and (9.1) has a solution $u_j(t)$ satisfying (9.24)–(9.25) for either $j = 0$ or $j = 1$, then (9.22) holds.

Condition (9.22) in Theorem 9.1 should be compared with (8.3) in Theorem 8.1. The analogue of (8.3) is the stronger condition

$$\int^{\omega} |x_1|^2 \cdot |q - q_0| \, dt < \infty \qquad \text{since} \quad x_0 = o(x_1) \qquad \text{as} \quad t \to \omega.$$

*Remark.* It will be clear from the proof of Theorem 9.1 that if $q_0(t)$ is complex-valued but has a pair of solutions asymptotically proportional to real-valued positive functions $x_0(t)$, $x_1(t)$ satisfying (9.22) [or (9.23)] and $\int^{\omega} ds/px_1^2 < \infty, \int^{\omega} ds/px_0^2 = \infty,\ x_1 \sim x_0 \int^{t} ds/px_0^2$, then Theorem 9.1 remains valid.

*Exercise* 9.5. Let $p(t) \neq 0$, $q(t)$, $q_0(t)$ be continuous complex-valued functions for $0 \leqq t < \omega$ $(\leqq \infty)$ such that (9.21) has a solution $x_1(t)$ which does not vanish for large $t$ and satisfies

$$\int^{\omega} \frac{dt}{p(t)x_1^2(t)} = \lim \int^{T} \qquad \text{as} \quad T \to \omega \qquad \text{exists}$$

and

$$\int^{\omega} |q - q_0| \cdot |x_1|^2 \Gamma \, dt < \infty, \qquad \text{where} \quad \Gamma(t) = \sup_{t \leq s < \omega} \left| \int_s^{\omega} \frac{dr}{px_1^2} \right|.$$

Then (9.1) has a pair of nontrivial solutions $u_0(t)$, $u_1(t)$ such that

$$u_0 = o(|u_1|), \qquad u_1 \sim x_1$$

$$\frac{pu_1'}{u_1} = \frac{px_1'}{x_1} + O\left(\frac{1}{|x_1|^2} \int^t |q - q_0| \cdot |x_1|^2 \, ds\right) \qquad \text{as} \quad t \to \omega.$$

**Proof of Theorem 9.1.** The variations of constants $u = x_0(t)v$ reduces (9.1) to

(9.26) $$(px_0^2 v')' + x_0^2(q - q_0)v = 0$$

for $t$ near $\omega$; cf. (2.31). Write this as a system (9.2), where

(9.27) $$z = px_0^2 v', \qquad \beta = \frac{1}{px_0^2}, \qquad \gamma = -x_0^2(q - q_0).$$

It will be verified that Lemma 9.3 is applicable. Note that condition (9.18) holds since $x_0(t)$ is a principal solution of (9.21); Theorem 6.4. A nonprincipal solution $x_1(t)$ of (9.21) is given by

(9.28) $$x_1(t) = x_0(t) \int^t \frac{ds}{p(s)x_0^2(s)} = x_0(t) \int^t \beta(s) \, ds$$

and any other nonprincipal solution is a constant times $[1 + o(1)]x_1(t)$ as $t \to \omega$; Corollary 6.3. The condition (9.4) is equivalent to (9.23).

Thus Lemma 9.3 is applicable. Let $(v_0, z_0)$, $(v_1, z_1)$ be the corresponding solutions of (9.2) and $u_0 = x_0 v_0$, $u_1 = x_0 v_1$ the corresponding solutions of (9.1). Then the first part of ($9.19_j$) for $j = 0, 1$ gives (9.24) for $j = 0, 1$. Note that $u = x_0 v$ implies that $pu'/u = px_0'/x_0 + pv'/v$, so that, by (9.27), $pu'/u = px_0'/x_0 + z/x_0^2 v$. Since $z_0/v_0 = o\left(1 \Big/ \int^t \beta(s) \, ds\right)$, the case $j = 0$ of (9.25) follows. Also, $z_1/x_0^2 v_1 = [1 + o(1)]/x_0^2 \int^t \beta(s) \, ds = [1 + o(1)]/x_0 x_1$ and, from (9.28), $px_1'/x_1 = px_0'/x_0 + 1/x_0 x_1$. Consequently, the case $j = 1$ of (9.25) holds. This proves the theorem.

**Corollary 9.1.** *In the equation*

(9.29) $$u'' - q(t)u = 0,$$

*let $q(t)$ be a continuous complex-valued function for large $t$ satisfying*

(9.30) $$\int^{\infty} t \, |q(t)| \, dt < \infty$$

*or, more generally,*

$$(9.31)\quad Q(t) \equiv \int_t^\infty q(s)\,ds = \lim_{T\to\infty}\int_t^T \text{ exists and } \int^\infty \sup_{t\leq r<\infty} |Q(r)|\,dt < \infty$$

*Then* (9.29) *has a pair of solutions* $u_0(t)$, $u_1(t)$ *satisfying, as* $t \to \infty$,

$$(9.32)\qquad u_0(t) \sim 1, \qquad u_0'(t) = o\left(\frac{1}{t}\right),$$

$$(9.33)\qquad u_1(t) \sim t, \qquad u_1'(t) \sim 1.$$

*Conversely, if* $q(t)$ *is real-valued and does not change signs and if* (9.29) *has a solution satisfying* (9.32) *or* (9.33), *then* (9.30) *holds.*

The first part of the corollary follows from Theorem 9.1, where (9.29) and $x'' = 0$ are identified with (9.1) and (9.21), respectively. The latter has the solutions $x_0(t) = 1$, $x_1(t) = t$. [Under the condition (9.30), the existence of $u_0$, $u_1$ is also contained in Theorem X 17.1.] The last part of the corollary follows from Lemma 9.4 or Exercise 9.4.

**Corollary 9.2.** *In the equation*

$$(9.34)\qquad u'' - [\lambda^2 + q(t)]u = 0,$$

*let* $\lambda > 0$ *and* $q(t)$ *be a complex-valued continuous function for large* $t$ *satisfying*

$$(9.35)\qquad \int^\infty |q(t)|\,dt < \infty$$

*or, more generally,*

$$(9.36)\qquad \int^\infty q(s)e^{-2\lambda s}\,ds = \lim_{T\to\infty}\int^T \quad \text{exists and}$$

$$\int^\infty e^{2\lambda t} \sup_{t\leq s<\infty} \left| \int_s^\infty q(r)e^{-2\lambda r}\,dr \right| dt < \infty.$$

*Then* (9.34) *has solutions* $u_0(t)$, $u_1(t)$ *satisfying*

$$(9.37)\qquad u_0 \sim -u_0'/\lambda \sim e^{-\lambda t}, \qquad u_1 \sim u_1'/\lambda \sim e^{\lambda t}.$$

*Conversely, if* $q(t)$ *is real-valued and does not change signs and if* (9.34) *has a solution* $u_0(t)$ *or* $u_1(t)$ *satisfying the corresponding conditions in* (9.37), *then* (9.35) *holds.*

The first part follows from Theorem 9.1 if (9.34) and $x'' - \lambda^2 x = 0$ are identified with (9.1) and (9.21), respectively. The latter has solutions $x_0(t) = e^{-\lambda t}$, $x_1(t) = e^{\lambda t}$. [Under condition (9.35), the existence of $u_0$, $u_1$ is also implied by Theorem X 17.2.]

*Exercise* 9.6. Let $q(t) > 0$ be a positive function on $0 \leqq t < \infty$ possessing a continuous second derivative and satisfying

$$\int^{\infty} q^{\frac{1}{2}}(t)\,dt = \infty \quad \text{and} \quad \int^{\infty} \left| \frac{5q'^2}{16q^3} - \frac{q''}{4q^2} \right| q^{\frac{1}{2}}\,dt < \infty.$$

Then $u'' - q(t)u = 0$ has a pair of solutions satisfying

$$q^{\frac{1}{4}}u \sim \exp \pm \int^t q^{\frac{1}{2}}(s)\,ds, \qquad q^{-\frac{1}{2}}(q^{\frac{1}{4}}u)' \sim \pm q^{\frac{1}{4}}u$$

as $t \to \infty$. (Compare this with Exercise X 17.5.)

*Exercise* 9.7. Find asymptotic formulae for the principal and non-principal solutions of Weber's equation

$$u'' + tu' - 2\lambda u = 0$$

(where $\lambda$ is a real number) by first eliminating the middle term using the analogue of substitution (1.9) and then applying Exercise 9.6 to the resulting equation; cf. Exercise X 17.6.

**Corollary 9.3.** *In equation* (9.29), *let* $q(t)$ *be a continuous complex-valued function for large* $t$ *such that* $Q(t)$ *in* (9.31) *satisfies*

$$\int^{\infty} Q(t)\,dt = \lim \int^{T} \qquad \text{exists as} \quad T \to \infty.$$

*Then a sufficient condition for* (9.29) *to have solutions* $u_0(t)$, $u_1(t)$ *satisfying*

$$u_0(t) \sim 1, \qquad u_0'(t) = o(1), \tag{9.38}$$

$$u_1(t) \sim t, \qquad u_1'(t) \sim 1, \tag{9.39}$$

*as* $t \to \infty$, *is that*

$$\int^{\infty} t\,|Q(t)|^2\,dt < \infty. \tag{9.40}$$

*This condition is also necessary if* $q(t)$ *is real-valued.*

**Proof.** It is easily verified that

$$x_0(t) = \exp\left(-\int^t Q(s)\,ds\right) \tag{9.41}$$

is a solution

$$x'' - [q(t) + Q^2(t)]x = 0. \tag{9.42}$$

One of the conditions on $Q$ implies that $\lim x_0(t)$ exists as $t \to \infty$ and is not 0. Correspondingly, the solution of (9.42) given by

$$x_1(t) = x_0(t) \int^t \frac{ds}{x_0^2(s)} \tag{9.43}$$

is asymptotically proportional to $t$, as $t \to \infty$.

Thus (9.42) has solutions asymptotically proportional to the (positive) functions 1, $t$. Hence if (9.29) and (9.42) are identified with (9.1), (9.21), respectively, and if (9.40) holds, the Remark following Theorem 9.1 shows that the conclusions of that theorem are valid. Consequently (9.29) has solutions $u_0$, $u_1$ satisfying $u_0 \sim x_0$, $u_1 \sim x_1$ as $t \to \infty$. The analogues of (9.25) are

$$\frac{u_0'}{u_0} = -Q + o\left(\frac{1}{t}\right), \qquad \frac{u_1'}{u_1} = -Q + 1 + o\left(\frac{1}{x_0 x_1}\right).$$

Since $Q(t) \to 0$ as $t \to \infty$, it is clear that certain constant multiples of $u_0$, $u_1$ satisfy (9.38), (9.39). The last part of the theorem follows from the fact that $q + Q^2 \geqq q$ when $q$ is real-valued; cf. Exercise 9.4.

By the use of a simple change of variables, a theorem about (9.29) for "small" $q(t)$ can be transcribed into a theorem about (9.34) for "small" $q(t)$, and conversely:

**Lemma 9.5.** *Let $q(t)$ be a continuous complex-valued function for large $t$. Then the change of variables, where $\lambda > 0$,*

(9.44) $$u = ve^{-\lambda t}, \qquad s = (2\lambda)^{-1}e^{2\lambda t}$$

*transforms* (9.34) *into*

(9.45) $$\frac{d^2v}{ds^2} - e^{-4\lambda t}q(t)v = 0;$$

*while the change of variables*

(9.46) $$u = t^{1/2}v = e^s v, \qquad s = \tfrac{1}{2}\log t$$

*transforms* (9.29) *into*

(9.47) $$\frac{d^2v}{ds^2} - [1 + 4t^2q(t)]v = 0.$$

*Exercise* 9.8. Verify this lemma.

*Exercise* 9.9. (*a*) Let $\lambda > 0$ and $q(t)$ be a continuous complex-valued function for large $t$ such that

$$Q_\lambda(t) = \int_t^\infty q(s)e^{-2\lambda s}\,ds = \lim_{T\to\infty}\int_t^T \quad \text{exists,}$$

$$\int^\infty Q_\lambda(t)e^{2\lambda t}\,dt = \lim_{T\to\infty}\int^T \quad \text{exists and} \quad \int^\infty |Q(t)|^2\,e^{4\lambda t}\,dt < \infty.$$

Then $u'' - [\lambda^2 + q(t)]u = 0$ has a pair of solutions satisfying, as $t \to \infty$,

$$u \sim e^{\pm\lambda t}, \qquad \frac{u'}{u} = \pm\lambda + e^{2\lambda t}Q_\lambda(t) + o(1).$$

(*b*) Let $q(t)$ be a continuous complex-valued function for $t \geqq 0$ such that $\int^{\infty} t^{2p-1} |q(t)|^p \, dt < \infty$ for some $p$ on the range $1 \leqq p \leqq 2$. Then $u'' - q(t)u = 0$ has a solution satisfying

$$u \sim \exp - \int^t sq(s)\, ds \quad \text{and} \quad \frac{u'}{u} = o\left(\frac{1}{t}\right)$$

and a solution satisfying

$$u \sim t \exp \int^t sq(s)\, ds \quad \text{and} \quad \frac{u'}{u} \sim \frac{1}{t},$$

as $t \to \infty$.

# APPENDIX: DISCONJUGATE SYSTEMS

## 10. Disconjugate Systems

This appendix deals with systems of equations of the form

$$(10.1) \qquad [P(t)x' + R(t)x]' - [R^*(t)x' - Q(t)x] = 0$$

or, more generally, systems of the form

$$(10.2) \qquad x' = A(t)x + B(t)y, \qquad y' = C(t)x - A^*(t)y.$$

Here $x$, $y$ are $d$-dimensional vectors; $A(t)$, $B(t)$, $C(t)$, $P(t)$, $Q(t)$, $R(t)$ are $d \times d$ matrices (with real- or complex-valued entries) continuous on a $t$-interval $J$. The object is to obtain generalizations of some of the results of § 6. The difficulty arises from the fact that the theorems of Sturm in § 3 do not have complete analogues.

In dealing with (10.1), it will usually be assumed that

$$(10.3) \qquad P = P^* \quad \text{and} \quad Q = Q^*,$$

$$(10.4) \qquad \det P \neq 0.$$

If the vector $y$ is defined by

$$(10.5) \qquad y = P(t)x' + R(t)x,$$

then (10.1) is of the form (10.2), where

$$(10.6) \qquad A = -P^{-1}R, \qquad B = P^{-1}, \qquad C = -Q - R^*P^{-1}R;$$

so that

$$(10.7) \qquad B = B^* \quad \text{and} \quad C = C^*,$$

$$(10.8) \qquad \det B \neq 0.$$