A1. [**6b**] Vypočítejte limitu

$$\lim_{x\to+\infty}\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1)}\right)^{\frac{1}{1-\cos\frac{1}{x}}}$$

A2. [**5b**] Vypočítejte limitu

$$\lim_{x\to+\infty}\frac{\ln(x^2+1)+x+\sin x}{1-\ln(x^x)}$$

B1. [**6b**] Vypočítejte limitu

$$\lim_{x\to 0+}\left(1+\sin\sqrt{x}\right)^{\frac{1}{1-\sqrt[3]{1+x}}}$$

B2. [**5b**] Vypočítejte limitu

$$\lim_{x\to+\infty}\frac{\ln(e^{(x^2)}+1)-x}{\ln(e^x-1)-x^3+\cos x}$$

(A1) $\lim\limits_{x\to+\infty}\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)^{\frac{1}{1-\cos\frac{1}{x}}}$.

$$f(t)=\exp h(t);\quad h(t)=\frac{1}{1-\cos\frac{1}{x}}\cdot\ln\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)$$

úprava odmocniny: $\sqrt[3]{a}-\sqrt[3]{b}=\dfrac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}$

$$\Rightarrow\ \sqrt[3]{x^2}-\sqrt[3]{x^2+1}=\frac{-1}{\sqrt[3]{x^4}+\sqrt[3]{x^2(x^2+1)}+\sqrt[3]{(x^2+1)^2}}\to 0$$

neboť jmenovatel $\to+\infty$:

např. 1. člen: $\left.\begin{array}{l} x^4\to+\infty;\ x\to+\infty \\ \sqrt[3]{y}\to+\infty,\ y\to+\infty\end{array}\right\}$ VoLSF (b).

$$h(t)=\frac{\frac{1}{x^2}}{1-\cos\frac{1}{x}}\cdot\frac{\ln\left(1+\sqrt[3]{x^2}-\sqrt[3]{x^2+1}\right)}{\sqrt[3]{x^2}-\sqrt[3]{x^2+1}}\cdot\frac{-x^2}{\sqrt[3]{x^4}+\sqrt[3]{x^2(x^2+1)}+\sqrt[3]{(x^2+1)^2}}$$

$$=P_1(t)\cdot P_2(t)\cdot P_3(t)$$

$P_1(t)\to\underline{2}$; neboť $\dfrac{1-\cos y}{y^2}\to\dfrac{1}{2}$, $y\to 0$

$\left.\begin{array}{l}\frac{1}{x}\to 0;\ x\to+\infty \\ \neq 0 \text{ na } P(+\infty)\end{array}\right\}$. VoLSF (a)

$P_2(x) \longrightarrow 1$, neboť $\frac{\ln(1+y)}{y} \to 1$, $y \to 0$

„VoLSF(b)" $\sqrt[3]{x^2} - \sqrt[3]{x^2+1} \to 0$; $x \to +\infty$

$\neq 0$ na $P(+\infty)$

$$P_3(x) = \underbrace{\frac{-x^2}{\sqrt[3]{x^4}}}_{\substack{-x^{2/3} \\ \downarrow \\ -\infty}} \cdot \underbrace{\frac{1}{1+\sqrt[3]{1+\frac{1}{x^2}}+\sqrt[3]{\left(1+\frac{1}{x^2}\right)^2}}}_{\to \frac{1}{3}} \to -\infty \cdot \frac{1}{3} = -\infty.$$

VoAL;

$1+\frac{1}{x^2} \to 1$; $x \to +\infty$

$\sqrt[3]{y}$ spojitá v $y_0 = 1$

VoLSF(a).

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celkem $P_1(x) = P_1(x) \cdot P_2(x) \cdot P_3(x) \to 2 \cdot 1 \cdot (-\infty) = -\infty$

tedy $\boxed{f(x) \to 0, \quad x \to +\infty}$

A2 $\lim\limits_{x\to+\infty} \dfrac{\ln(x^2+1)+x+\sin x}{1-\ln(x^x)}$

$$\ln(x^2+1) = \ln x^2\left(1+\frac{1}{x^2}\right) = 2\ln x + \ln\left(1+\frac{1}{x^2}\right)$$

$$\ln(x^x) = x\cdot\ln x$$

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vytkneme „vedoucí člen": nahoře x

dole $x\cdot\ln x$

$$f(x) = \frac{1+2\frac{\ln x}{x}+\frac{1}{x}\ln\left(1+\frac{1}{x^2}\right)+\frac{\sin x}{x}}{-1+\frac{1}{x\cdot\ln x}}\cdot\frac{\cancel{x}}{\cancel{x}\cdot\ln x}$$

$$\to \frac{1+0+0\cdot 0+0}{-1+\frac{1}{+\infty}}\cdot\frac{1}{+\infty} = 0.$$

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podrobně: $\frac{\ln x}{x}\to 0$ (základní limita)

$\frac{1}{x}\cdot\ln\left(1+\frac{1}{x^2}\right)\to 0\cdot 0$; VoAL,

spojitost $\ln y$ v $y_0=1$

$\sin x\cdot\frac{1}{x}\to 0$: $|\sin x|\le 1$

$\frac{1}{x}\to 0$

$$\frac{1}{x\cdot\ln x}\to\frac{1}{\infty\cdot\infty}=0.$$

(B1) $\lim\limits_{x\to 0+} \left(1+\sin\sqrt{x}\right)^{\frac{1}{1-\sqrt[3]{1+x}}}$,

$$f(x) = \exp h(x); \quad h(x) = \frac{1}{1-\sqrt[3]{1+x}} \cdot \ln\left(1+\sin\sqrt{x}\right)$$

úprava odmocnin: $\dfrac{1}{\sqrt[3]{a}-\sqrt[3]{b}} = \dfrac{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}{a-b}$

$$\frac{1}{1-\sqrt[3]{1+x}} = \frac{1+\sqrt[3]{1+x}+\sqrt[3]{(1+x)^2}}{-x}$$

$$h(x) = \frac{\ln(1+\sin\sqrt{x})}{\sin\sqrt{x}} \cdot \frac{\sin\sqrt{x}}{\sqrt{x}} \cdot \frac{1+\sqrt[3]{1+x}+\sqrt[3]{(1+x)^2}}{-\sqrt{x}},$$

$$= R_1(x)\cdot R_2(x)\cdot R_3(x)$$

$\left.\begin{array}{l} \dfrac{\sin y}{y} \to 1, \; y\to 0 \\ \sqrt{x}\to 0; \; x\to 0+ \\ \quad > 0 \;\; \forall x\in P_+(0,\delta) \end{array}\right\}$ VoLSF(b) $\Rightarrow$ $R_2(x)\to 1$

$\left.\begin{array}{l} \dfrac{\ln(1+y)}{y} \to 1; \; y\to 0 \\ \sin\sqrt{x}\to 0; \; x\to 0+ \\ \quad \neq 0 \;\; \forall x\in P_+(0,\delta) \end{array}\right\}$ VoLSF(b) $\Rightarrow$ $R_1(x)\to 1$.

$R_3(x) = \left(1 + \sqrt[3]{1+x} + \sqrt[3]{(1+x)^2}\right) \cdot \frac{-1}{\sqrt{x}}$

1. čast $\to$ 3 ; VoAL ; $1+x \to 1$; $x \to 0+$ $\left.\begin{matrix} \\ \sqrt[3]{y} \text{ spoj. v } y_0 = 1 \end{matrix}\right\}$ VoLSF(a)

2. čast : $\sqrt{x} \to 0$; $x \to 0+$ $\left.\begin{matrix} \\ > 0 \text{ na } P_+(0,\delta) \end{matrix}\right\}$ $\Longrightarrow$ $\frac{1}{0+}$ typ $-\frac{1}{\sqrt{x}} \to -\infty$

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celkem: $h(x) = R_1(x) \cdot R_2(x) \cdot R_3(x) \longrightarrow 1 \cdot 1 \cdot (-\infty) = -\infty$

tedy $\boxed{f(x) \to 0 \ ; \ x \to 0+}$

B2) $\lim\limits_{x\to+\infty} \dfrac{\ln(e^{x^2}+1)-x}{\ln(e^{x}-1)-x^3+\cos x}$

$\ln(e^{x^2}+1) = \ln e^{x^2}(1+e^{-x^2}) = x^2+\ln(1+e^{-x^2})$;

$\ln(e^{x}-1) = x+\ln(1-e^{-x})$;

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vytkneme vedoucí člen: čitatel x^2

jmenovatel x^3

$$f(x) = \frac{x^2}{x^3}\cdot\frac{1+\frac{1}{x^2}\ln(1+e^{-x^2})-\frac{1}{x}}{-1+\frac{1}{x^2}+\frac{1}{x^3}\ln(1-e^{-x})+\frac{\cos x}{x^3}} \to 0\cdot\frac{1}{-1}=0$$

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podrobně: $\frac{x^2}{x^3}=\frac{1}{x}\to\frac{1}{+\infty}=0$ VoAL

$\frac{1}{x^2}\cdot\ln(1+e^{-x^2})\to 0\cdot\ln 1=0$

VoAL; $e^{-x^2}\to 0$, neboť $-x^2\to-\infty$

spojitost $\ln y$ v $y_0=1$

podobně $\frac{1}{x^3}\cdot\ln(1-e^{-x})$

konečně: $\left.\begin{array}{l}\cos x \text{ omezené} \\ \frac{1}{x^3}\to 0\end{array}\right\} \Rightarrow \frac{\cos x}{x^3}\to 0$