Infinite-Dimensional Dynamical Systems

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Infinite-dimensional dynamical systems

1.1 Semigroups

Our abstract 'infinite-dimensional dynamical systems' are semigroups defined on Banach spaces; more usually Hilbert spaces.

Given a Banach space $\mathscr{B}$, a semigroup on $\mathscr{B}$ is a family $\{S(t): \ t \geq 0\}$ of mappings from $\mathscr{B}$ into itself with the properties:

$$S(0) = \mathrm{id}_{\mathscr{B}} \tag{1.1}$$

$$S(t+s) = S(t)S(s) = S(s)S(t) \qquad \text{for all} \qquad t, s \geq 0 \tag{1.2}$$

$$S(t)u_0 \text{ is continuous in both } t \text{ and } u_0. \tag{1.3}$$

Despite the notation this semigroup need not be linear (and in all interesting examples will be nonlinear).

Usually, our semigroups will be generated by the solutions of some partial differential equation. If $u(x,t)$ is the solution at time t corresponding to the initial condition $u(x,0) = u_0(x)$ for any $u_0(\cdot) \in \mathscr{B}$, provided that $u(\cdot,t) \in \mathscr{B}$ we can define $S(t): \mathscr{B} \to \mathscr{B}$ by

$$S(t)u_0(\cdot) = u(\cdot,t).$$

To make the notation more manageable, we consistently suppress the dependence on x, and so write $u(t) = S(t)u_0$. After all, the x dependence is implicit in the fact that $u_0 \in \mathscr{B}$ (if $\mathscr{B}$ is a function space).

1.2 Examples

1.2.1 ODEs

If $f:\mathbb{R}^n\to\mathbb{R}^n$ is locally Lipschitz,

$$|f(x)-f(y)|\le L(R)|x-y| \qquad \text{for all} \qquad |x|,|y|\le R$$

then solutions of

$$\dot{x}=f(x) \qquad \text{with} \qquad x(0)=x_0$$

exist locally on some time interval $[0,T(|x_0|))$. Solutions exist for all time provided that they do not blow up.

In this case we can take $\mathscr{B}=\mathbb{R}^n$ and let $S(t)x_0=x(t;x_0)$, where $x(t;x_0)$ is the solution of the equation at time t when the initial condition is x_0.

1.2.2 Reaction-diffusion equations

Consider the following scalar reaction-diffusion equation on a smooth bounded domain $\Omega\subset\mathbb{R}^m$,

$$u_t-\Delta u=f(u),$$

subject to Dirichlet boundary conditions ($u|_{\partial\Omega}=0$). Recall that for functions in $H_0^1(\Omega)$ (essentially '$u\in H^1(\Omega)$ with $u|_{\partial\Omega}=0$') we have the Poincaré inequality

$$\|u\|\le c\|\nabla u\| \qquad \text{for all} \qquad u\in H_0^1(\Omega),$$

which means in particular that $\|\nabla u\|$ gives a norm on $H_0^1(\Omega)$ which is equivalent to the standard H^1 norm.

We impose the following conditions on f:

$$-k-\alpha_1|s|^p\le f(s)s\le k-\alpha_2|s|^p \tag{1.4}$$

$$f'(s)\le l, \tag{1.5}$$

where $\alpha_1,\alpha_2,k>0$ and $p\ge 2$.

Under these conditions, the equations define a semigroup on $L^2(\Omega)$. A formal proof of existence and uniqueness of solutions would involve the method of Galerkin approximations (the lower bound in (1.4) is important in enabling the technical steps of the Galerkin procedure to be justified), but at its heart is the following ('formal') estimate, showing that if $u_0 \in L^2(\Omega)$ then $u(t) \in L^2(\Omega)$ for any $t > 0$. Taking the inner product of the equation with u and integrating over Ω we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \|Du\|^2 &= \int_\Omega f(u)u\,\mathrm{d}x \\ &\le \int_\Omega k - \alpha_2|u|^p\,\mathrm{d}x.\end{aligned}$$

Integrating both sides between 0 and t gives

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2\,\mathrm{d}s + \alpha_2\int_0^t\int_\Omega |u|^p\,\mathrm{d}x\,\mathrm{d}t \le kt|\Omega| + \frac{1}{2}\|u(0)\|^2,$$

showing that an initial condition in $L^2(\Omega)$ leads to a solution in $L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1)\cap L^p((0,T)\times\Omega)$ for any $T > 0$. Combined with estimates on $\mathrm{d}u/\mathrm{d}t$ one can in fact show that $u \in C^0([0,T];L^2(\Omega)$. These estimates form the basis of an *existence* proof, but do not guarantee uniqueness.

We get uniqueness and continuous dependence on initial conditions from the same argument: consider the difference of two solutions, u and v, with initial conditions u_0 and v_0 respectively, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u-v) - \Delta(u-v) = f(u) - f(v).$$

Taking the inner product with $u - v$ and integrating over Ω gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-v\|^2 + \|D(u-v)\|^2 = \int_\Omega (f(u(x)) - f(v(x)))(u(x)-v(x))\,\mathrm{d}x.$$

Now, if $u(x) > v(x)$ then

$$\begin{aligned}(f(u(x)) - f(v(x)))(u(x)-v(x)) &= \left(\int_{v(x)}^{u(x)} f'(s)\,\mathrm{d}s\right)(u(x)-v(x)) \\ &\le l|u(x)-v(x)|^2,\end{aligned}$$

with a similar argument giving the same bound if $v(x) > u(x)$. So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-v\|^2 + \|D(u-v)\|^2 \le l\int_\Omega |u(x)-v(x)|^2\,\mathrm{d}x,$$

from which it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u-v\|^2 \le l\|u-v\|^2$$

and so

$$\|u(t)-v(t)\| \le \mathrm{e}^{lt}\|u_0 - v_0\|.$$

We can therefore use this equation to define a semigroup on $L^2(\Omega)$.

With some further restrictions we could use the equation to define a semigroup on $H_0^1(\Omega)$, e.g. take $n = 1, 2$, $f(0) = 0$, and $|f'(s)| \leq C(1 + |s|^\gamma)$ for some $\gamma \geq 0$. But although the analysis most naturally employed to obtain a semigroup on $L^2(\Omega)$ is more cumbersome than the contraction mapping possibilities in $H_0^1(\Omega)$, the semigroup on $L^2(\Omega)$ is easier to analyse.

1.2.3 Navier-Stokes equations

The Navier-Stokes equations:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = g \qquad \text{and} \qquad \nabla \cdot u = 0.$$

We will take $x \in Q = [0, L]^d$ with $d = 2, 3$ with periodic boundary conditions. The function u is the d-component velocity, $u(x) = (u_1(x), u_2(x))$ or $(u_1(x), u_2(x), u_3(x))$, p is the scalar pressure, and g is a body force.

Define

$$\mathscr{V} = \{C^\infty \text{ functions } u \text{ that are periodic on } Q \text{ with } \nabla \cdot u = 0 \text{ and } \int_Q u = 0\}$$

and let H and V be the closure of $\mathscr{V}$ in $[L^2(Q)]^d$ and $[H^1(Q)]^d$ respectively. Heuristically, H and V are 'functions in L^2 or H^1 that are divergence free' (and have zero average on Q). From now on we drop the $\cdot^d$ in the notation for Sobolev spaces of vector-valued functions on Q.

The assumption that our functions have zero average on Q gives us a Poincaré inequality,

$$\|u\| \leq c\|Du\| \qquad \text{for all} \qquad u \in \dot{H}^1(Q), \tag{1.6}$$

where $\dot{H}^1(Q)$ denotes H^1 functions that are periodic on Q and have zero average.

It is easy to give a formal proof of existence of weak solutions (initial conditions in H, i.e. finite kinetic energy) in 2d and 3d. As before, we will just give formal estimates showing that $u_0 \in L^2$ implies that $u(t) \in L^2$ for all $t \geq 0$ (and some additional estimates too).

Simply take the inner product with u to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 - \nu\int_Q(\partial_i\partial_i u_j)u_j\,\mathrm{d}x + \int_Q[(u_i\partial_i)u_j]u_j\,\mathrm{d}x + \int_Q(\partial_i p)u_i\,\mathrm{d}x = \int_Q fu\,\mathrm{d}x.$$

Now, an integration by parts gives

$$-\int_Q(\partial_i\partial_i u_j)u_j\,\mathrm{d}x = \sum_{i,j}\int_Q(\partial_i u_j)^2\,\mathrm{d}x := \|Du\|^2;$$

for the nonlinear term note that

$$\int_Q[(v_i\partial_i)u_j]u_j\,\mathrm{d}x = -\int_Q[\partial_i(v_i u_j)]u_j\,\mathrm{d}x = -\int_Q(\partial_i v_i)|u|^2 - \int_Q[(v_i\partial_i)u_j]u_j\,\mathrm{d}x,$$

which implies, since $\nabla\cdot v = \partial_i v_i = 0$ that

$$\int_Q[(v_i\partial_i)u_j]u_j\,\mathrm{d}x = -\int_Q[(v_i\partial_i)u_j]u_j\,\mathrm{d}x,$$

i.e. that

$$\int_Q[(v\cdot\nabla)u]\cdot u\,\mathrm{d}x = 0 \tag{1.7}$$

and so the nonlinear term vanishes, as does the final term on the left-hand side since

$$\int_Q(\partial_i p)u_i,\mathrm{d}x = -\int_Q p(\partial_i u_i)\,\mathrm{d}x = 0 \tag{1.8}$$

since $\nabla\cdot u = 0$.

So in fact we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Du\|^2 = (f,u) \le \|f\|\|u\| \le \frac{\nu}{2c^2}\|u\|^2 + \frac{c^2\|f\|^2}{2\nu},$$

where c is the constant in (1.6), so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Du\|^2 \le \frac{c^2\|f\|^2}{\nu}.$$

Integrating from 0 to t it follows that

$$\|u(t)\|^2 + \nu\int_0^t\|Du(s)\|^2\,\mathrm{d}s \le \|u_0\|^2 + c^2 t\frac{\|f\|^2}{\nu}.$$

So the solution satisfies $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$. In the two-dimensional case there are sufficiently nice bounds on $\mathrm{d}u/\mathrm{d}t$ to guarantee that in fact $u \in C^0([0,T];H)$.

In 2d we can also obtain uniqueness (and continuous dependence) if we use Ladyzhenskaya's inequality

$$\|u\|_{L^4} \leq c\|u\|^{1/2}\|Du\|^{1/2} \tag{1.9}$$

to obtain an appropriate bound on the nonlinear term. If we use Hölder's inequality then

$$\begin{aligned}\left|\int_Q [(u\cdot\nabla)v]\cdot w\,\mathrm{d}x\right| &= \sum_{i,j}\left|\int_Q [u_j(\partial_j v_i)w_i\,\mathrm{d}x\right| \\ &\leq \sum_{i,j}\|u_j\|_{L^4}\|\partial_j v_i\|\|w_i\|_{L^4}, \\ &\leq \sum_{i,j}\|u_j\|^{1/2}\|Du_j\|^{1/2}\|\partial_j v_i\|\|w_i\|^{1/2}\|Dw_i\|^{1/2}\end{aligned}$$

using (1.9). Now use Hölder's inequality again:

$$\begin{aligned}\cdots &\leq \left(\sum_{i,j}\|u_j\|^2\|Du_j\|^2\right)^{1/4}\left(\sum_{i,j}\|\partial_j v_i\|^2\right)^{1/2}\left(\sum_{i,j}\|w_i\|^2\|Dw_i\|^2\right)^{1/4} \\ &\leq \|u\|^{1/2}\|Du\|^{1/2}\|Dv\|\|w\|^{1/2}\|Dw\|^{1/2},\end{aligned}$$

since if $a_k, b_k \geq 0$, $\sum_k a_k b_k \leq (\sum_k a_k)(\sum_k b_k)$. So

$$\left|\int_Q [(u\cdot\nabla)v]\cdot w\,\mathrm{d}x\right| \leq c\|u\|^{1/2}\|Du\|^{1/2}\|Dv\|\|w\|^{1/2}\|Dw\|^{1/2}. \tag{1.10}$$

If we consider $w = u - v$, the difference of two solutions u and v, then we have

$$w_t - \nu\Delta w + (u\cdot\nabla u) - (v\cdot\nabla v) + \nabla(p_u - p_v) = w_t - \nu\Delta w + (u\cdot\nabla w) + (w\cdot\nabla)v + \nabla\psi = 0.$$

Taking the inner product with w and using both (1.7) and (1.8) we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\|Dw\|^2 &= -\int_Q (w\cdot\nabla v)\cdot w\,\mathrm{d}x \\ &\leq \|w\|\|Dw\|\|Dv\| \\ &\leq \nu\|Dw\|^2 + \frac{1}{4\nu}\|w\|^2\|Dv\|^2,\end{aligned}$$

with the new inequality (1.10) used in the final line.

From this its follows that,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \leq \frac{1}{4\nu}\|Dv\|^2\|w\|^2,$$

and integrating this between zero and t gives

$$\|u(t) - v(t)\|^2 \leq \exp\left(\frac{1}{2\nu}\int_0^t \|Dv(s)\|^2\,\mathrm{d}s\right)\|u_0 - v_0\|^2.$$

Since $v \in L^2(0,T;H^1)$, the integral expression on the right-hand side is finite, which implies both continuous dependence on initial conditions and uniqueness.

The three-dimensional version of (1.10) is

$$\left|\int_Q [(u\cdot\nabla)v]\cdot w\,\mathrm{d}x\right| \leq c\|u\|^{1/4}\|Du\|^{3/4}\|Dv\|\|w\|^{1/4}\|Dw\|^{3/4} \tag{1.11}$$

which accounts for the fact that similar methods to do not suffice to prove the uniqueness of weak solutions of the three-dimensional equations.

To show the existence of strong solutions in 2d, in the periodic case we can make use of the useful orthogonality relation

$$\int_Q [(u\cdot\nabla)u]\cdot[\Delta u]\,\mathrm{d}x = 0 \tag{1.12}$$

for u divergence-free. The proof of this is based on component-wise cancellation and is tedious, not instructive, and does not work either for Dirichlet boundary conditions in 2d, nor for periodic boundary conditions in 3d.

Given (1.12) we take the inner product of the equations with $-\Delta u$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \nu\|\Delta u\|^2 = (f,\Delta u) \leq \frac{1}{2\nu}\|f\|^2 + \frac{\nu}{2}\|\Delta u\|^2,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \nu\|\Delta u\|^2 \leq \frac{\|f\|^2}{\nu}.$$

An integration shows that

$$\|Du(t)\|^2 + \int_0^t \|\Delta u(s)\|^2\,\mathrm{d}s \leq \|Du_0\|^2 + t\,\frac{\|f\|^2}{\nu}.$$

Giving a solution in $L^\infty(0,T;V)\cap L^2(0,T;H^2)$. Uniqueness follows in a similar way to before, using further estimates on the nonlinear term.

These results also enable one to define a semigroup on V in this case.

In the 3d case strong solutions (which are unique in the class of weak solutions) are known to exist on some finite time interval $[0, T)$, where T depends on $\|Du_0\|$. Given the topical interest of this problem, we remark here that rather than $\int[(u \cdot \nabla)u] \cdot \Delta u$ vanishing, we can only appeal to the bound

$$\left| \int [(u \cdot \nabla)u] \cdot \Delta u \, \mathrm{d}x \right| \leq c\|Du\|^{3/2}\|\Delta u\|^{3/2}.$$

Suppose for simplicity that $f = 0$. Then following the analysis above we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \nu\|\Delta u\|^2 \leq c\|Du\|^{3/2}\|\Delta u\|^{3/2}.$$

Using Hölder's inequality with $p = 4$ and $q = 4/3$ results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \nu\|\Delta u\|^2 \leq \frac{c_2}{\nu^3}\|Du\|^6.$$

One can drop the $\nu\|\Delta u\|^2$ term and integrating the resulting differential inequality to yield

$$\|Du(t)\|^2 \leq \frac{\|Du_0\|^2}{\sqrt{1 - ct\|Du_0\|^4/\nu^3}}.$$

So one can only guarantee that $\|Du(t)\|$ remains finite while $ct\|Du_0\|^4 < \nu^3$.

2

The global attractor I
Existence

If $(\mathscr{B}, S(t))$ is an infinite-dimensional dynamical system, a set $X \subset \mathscr{B}$ is said to be *invariant* if $S(t)X = X$ for all $t \geq 0$.

A set $X \subset \mathscr{B}$ is said to attract $B \subset \mathscr{B}$ if

$$\operatorname{dist}(S(t)B, X) \to 0 \qquad \text{as} \qquad t \to \infty,$$

where

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

A set $X \subset \mathscr{B}$ is said to be *attracting* if it attracts all bounded subsets of $\mathscr{B}$.

A set $\mathscr{A} \subset \mathscr{B}$ is said to be *the global attractor* if it is compact, invariant, and attracts all bounded subsets of $\mathscr{B}$.

Lemma 2.1 *The global attractor is unique.*

Proof Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two global attractors. Then, since $\mathscr{A}_2$ is bounded, it is attracted by $\mathscr{A}_1$,

$$\operatorname{dist}(S(t)\mathscr{A}_2, \mathscr{A}_1) \to 0 \qquad \text{as} \qquad t \to \infty.$$

But $\mathscr{A}_2$ is invariant, so $S(t)\mathscr{A}_2 = \mathscr{A}_2$, from which it follows that $\operatorname{dist}(\mathscr{A}_2, \mathscr{A}_1) = 0$. The argument is symmetric, so $\operatorname{dist}(\mathscr{A}_1, \mathscr{A}_2) = 0$, from which it follows that $\mathscr{A}_1 = \mathscr{A}_2$. □

Lemma 2.2 *Let K be a compact subset of a Banach space $\mathscr{B}$, and $x_n \in \mathscr{B}$ a sequence with $\lim_{n\to\infty} \operatorname{dist}(x_n, K) = 0$. Then (x_n) has a convergent subsequence, whose limit lies in K.*

Proof Write $x_n = k_n + z_n$, where $k_n \in K$ and $|z_n| \to 0$ as $n \to \infty$. Then there is a subsequence such that $k_{n_j} \to k^* \in K$, so $x_{n_j} \to k^*$ too. □

We now prove some properties of the omega limit set of a bounded set B, $\omega(B)$:

Proposition 2.3 *Suppose that there exists a compact attracting set K. Then for any bounded set B, the set*

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B} \tag{2.1}$$

$$= \{x \in \mathscr{B} : \ x = \lim_{n\to\infty} S(t_n)b_n \text{ for some } t_n \to \infty, \ b_n \in B\} \tag{2.2}$$

is compact, invariant, and attracts B. Furthermore, $\omega(B) \subseteq \omega(K)$.

Proof Suppose that $w_n \in \omega(B)$. Then there exist sequences $\{t_k^{(n)}\}$ with $t_k^{(n)} \to \infty$ as $k \to \infty$ and $\{b_k^{(n)}\}$ with $b_k^{(n)} \in B$ such that

$$w_n = \lim_{k\to\infty} S(t_k^{(n)})b_k^{(n)}.$$

It follows in particular that there exists a sequence $\{t_j\}$ with $t_j \to \infty$ and $b_j \in B$ such that

$$|w_j - S(t_j)b_j| < 1/j.$$

Since K must attract B by assumption, it follows from Lemma 2.2 that there is a subsequence such that $S(t_j)b_j$ converges. This limit must belong to $\omega(B)$. It follows that there is a subsequence of $\{w_j\}$ that converges to an element of $\omega(B)$, so $\omega(B)$ is (sequentially) compact.

Now suppose that $x \in \omega(B)$. Then there exist sequences $\{t_n\}$ with $t_n \to \infty$ and $\{b_n\}$ with $b_n \in B$ such that $x = \lim_{n\to\infty} S(t_n)b_n$. Then, since $S(t)$ is continuous,

$$S(t)x = S(t)\left(\lim_{n\to\infty} S(t_n)b_n\right) = \lim_{n\to\infty} S(t + t_n)b_n,$$

and so $S(t)x \in \omega(B)$, i.e. $S(t)\omega(B) \subseteq \omega(B)$.

Now, if $y \in \omega(B)$ then $y = \lim_{n\to\infty} S(t_n)b_n$. For any fixed t, once $t_n \geq t$, we can write $S(t_n)b_n = S(t)[S(t_n - t)b_n]$. Using Lemma 2.2, we know that

$S(t_n-t)b_n$ has a convergent subsequence, which converges to some $\beta \in \omega(B)$. Taking the limit through this subsequence, it follows that $y = S(t)\beta$ with $\beta \in \omega(B)$, so $\omega(B) \subseteq S(t)\omega(B)$. It follows that $S(t)\omega(B) = \omega(B)$.

We now show that $\omega(B)$ attracts B. If not, then there exists a $\delta > 0$, and $t_n \to \infty$, $b_n \in B$ such that

$$\text{dist}(S(t_n)b_n, \omega(B)) > \delta.$$

But (by Lemma 2.2) $\{S(t_n)b_n\}$ has a convergent subsequence, whose limit must lie in $\omega(B)$, a contradiction.

Finally, to show that $\omega(B) \subseteq \omega(K)$, notice that for each n, there exists a δ_n such that

$$|x-k| < \delta_n \qquad \Rightarrow \qquad |S(n)x - S(n)k| < 1/n$$

for any $k \in K$. Now, if $\beta \in \omega(B)$ then $\beta = \lim_{j\to\infty} S(t_j)b_j$. For each n, consider the sequence $S(t_j - n)b_j$. Since K is attracting, there exists a j_n such that

$$\text{dist}(S(t_{j_n} - n)b_{j_n}, K) < \delta_n,$$

and so there exists a $k_n \in K$ with

$$|S(t_{j_n} - n)b_{j_n} - k_n| < \delta_n.$$

It follows that

$$|S(t_{j_n})b_{j_n} - S(n)k_n| < 1/n.$$

Since $S(t_{j_n})b_{j_n} \to \beta$, it follows also that $\beta = \lim_{n\to\infty} S(n)k_n$, and so $\beta \in \omega(K)$. □

The main theorem here is essentially an immediate corollary of the above.

Theorem 2.4 *There exists a global attractor $\mathscr{A}$ iff there exists a compact attracting set.*

Proof If $\mathscr{A}$ is an attractor then it is a compact attracting set. If K is a compact attracting set then $\omega(K)$ is compact and invariant; since $\omega(B) \subseteq \omega(K)$ for all B, and $\omega(B)$ attracts B, $\omega(K)$ attracts B. □

We now discuss some properties of the attractor in more detail.

Lemma 2.5 *$\mathscr{A}$ is the maximal compact invariant set, and the minimal set that attracts all bounded sets.*

Proof Let X be compact and invariant. Since X is compact it is bounded, so it is attracted to $\mathscr{A}$. Therefore

$$\text{dist}(S(t)X, \mathscr{A}) = \text{dist}(X, \mathscr{A}) \to 0 \qquad \text{as} \qquad t \to \infty,$$

i.e. $\text{dist}(X, \mathscr{A}) = 0$ so $X \subseteq \mathscr{A}$. Similarly, if Y attracts all bounded sets then Y attracts $\mathscr{A}$. The same argument, using invariance of $\mathscr{A}$, shows that $\mathscr{A} \subseteq Y$. □

We now make the following definition:

Definition 2.6 *We say that $S(t)$ has* the backwards uniqueness property *if $S(t)$ is injective for every $t \geq 0$, i.e.*

$$S(t)u_0 = S(t)v_0 \qquad \Rightarrow \qquad u_0 = v_0.$$

If $S(t)$ has the backwards uniqueness property then we have a standard dynamical system *on the attractor*.

Proposition 2.7 *If $S(t)$ has the backwards uniqueness property, then on $\mathscr{A}$ we can extend $S(t)$ to a group $\{S(t)\}_{t\in\mathbb{R}}$, and*

$$S(t)\mathscr{A} = \mathscr{A} \qquad \textit{for all} \qquad t \in \mathbb{R}.$$

Proof Take $t > 0$ and $x \in \mathscr{A}$. Since $\mathscr{A} = S(t)\mathscr{A}$, we have $x = S(t)y$ for some $y \in \mathscr{A}$. This y is unique by the backwards uniqueness property, and so we can define $S(-t)x = y$.

The map $S(-t) : \mathscr{A} \to \mathscr{A}$ is continuous, since it is the inverse of a continuous injective mapping on a compact set.

It is clear that defined this way $\{S(t)\}_{t\in\mathbb{R}}$ becomes a group of mappings. Since $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$ and $S(-t)$ is the inverse of $S(t)$, it follows that $S(t)\mathscr{A} = \mathscr{A}$ for all $t \in \mathbb{R}$. □

With another definition we will be in a position to give a non-dynamical characterisation of the attractor:

Definition 2.8 *A* complete orbit *of $S(\cdot)$ is a map $u : \mathbb{R} \to \mathscr{B}$ such that*

$$S(t)u(s) = u(t+s) \qquad \textit{for all} \qquad t \geq 0, s \in \mathbb{R},$$

i.e. a solution defined for all $t \in \mathbb{R}$*. Such an orbit is* bounded *if there exists an* $M > 0$ *such that*

$$\|u(t)\| \leq M \qquad \text{for all} \qquad t \in \mathbb{R}.$$

Theorem 2.9 *All complete bounded orbits lie in* $\mathscr{A}$*. If* $S(t)$ *has the backwards uniqueness property then* $\mathscr{A}$ *is precisely the union of all complete bounded orbits.*

Proof Let $\mathscr{O}$ be a complete bounded orbit and let $x \in \mathscr{O}$. Then for any $t > 0$ there exists a $y_t \in \mathscr{O}$ such that $x = S(t)y_t$. Since $\mathscr{O}$ is bounded, $\mathscr{A}$ attracts $\mathscr{O}$, and so

$$\text{dist}(x, \mathscr{A}) \leq \text{dist}(S(t)\mathscr{O}, \mathscr{A}) \to 0 \qquad \text{as} \qquad t \to \infty,$$

i.e. $x \in \mathscr{A}$. So $\mathscr{O} \subseteq \mathscr{A}$. Conversely if $x \in \mathscr{A}$ and $S(t)$ has the backwards uniqueness property then x lies on the complete bounded orbit $u(t) = S(t)x$. □

In many cases we can show something stronger than the existence of a compact attracting set, namely the existence of a compact *absorbing* set. We say that a set $X \subset \mathscr{B}$ is *absorbing* if for every bounded subset $B \subset \mathscr{B}$ there exists a time t_B such that

$$S(t)B \subseteq X \qquad \text{for all} \qquad t \geq t_B,$$

i.e. the orbits of all bounded sets eventually enter and do not leave X. Clearly the existence of a compact absorbing set implies the existence of a compact attracting set, which we know implies the existence of a global attractor.

2.1 Examples

We will continually use the following simple lemma (Gronwall's inequality) to integrate simple differential inequalities. The proof simply involves using the integrating factor $\exp[-A(t)]$.

Lemma 2.10 (Gronwall's inequality) *Suppose that*

$$\frac{\mathrm{d}X}{\mathrm{d}t} \leq a(t)x + b(t).$$

Then

$$X(t) \leq X(0)\exp[A(t)] + \int_0^t \exp[A(t) - A(s)]b(s)\,\mathrm{d}s,$$

where $A(t) = \int_0^t a(r)\,\mathrm{d}r$*. In particular if* $a(t) \equiv \alpha$ *and* $b(t) \equiv \beta$ *then*

$$X(t) \leq \left(X(0) + \frac{\beta}{\alpha}\right)\mathrm{e}^{\alpha t} - \frac{\beta}{\alpha}.$$

2.1.1 Reaction-diffusion equations

For both the reaction-diffusion equation and the Navier-Stokes equations we proceed in two steps: find an absorbing set in L^2, and use this to prove the existence of an absorbing set in H^1.

2.1.1.1 Absorbing set in L^2

If we take the inner product of the equation with u then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \|Du\|^2 = \int_\Omega f(u(x))u(x)\,\mathrm{d}x \leq \int_\Omega k - \alpha_2|u|^p\,\mathrm{d}x \leq k|\Omega|, \quad (2.3)$$

where $|\Omega|$ is the Lebesgue measure of Ω. Using the Poincaré inequality

$$\|u\| \leq \lambda_1^{-1/2}\|Du\|$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\lambda_1\|u\|^2 \leq 2k|\Omega|.$$

Gronwall's inequality (Lemma 2.10) then shows that

$$\|u(t)\|^2 \leq \|u_0\|^2\mathrm{e}^{-2\lambda_1 t} + \frac{k|\Omega|}{\lambda_1}(1 - \mathrm{e}^{-2\lambda_1 t}).$$

In particular, there exists a $t_0(\|u_0\|)$ such that

$$\|u(t)\|^2 \leq \rho_0^2 := \frac{2k|\Omega|}{\lambda_1} \qquad \text{for all} \qquad t \geq t_0. \quad (2.4)$$

Returning to (2.3) and integrating from t to $t+1$ for $t \geq t_0$ shows further that

$$\frac{1}{2}\|u(t+1)\|^2 + \int_t^{t+1} \|Du(s)\|^2\,\mathrm{d}s \leq k|\Omega| + \|u(t)\|^2,$$

and so in particular it follows that

$$\int_t^{t+1} \|Du(s)\|^2 \, \mathrm{d}s \le I_1 := k|\Omega| + \frac{2k|\Omega|}{\lambda_1}. \tag{2.5}$$

2.1.1.2 Absorbing set in H^1

We now use the integral bound in (2.5) to prove the existence of an absorbing set in H^1. The argument here can be formalised as the 'uniform Gronwall lemma', but this hides the simple idea behind it.

Taking the inner product with $-\Delta u$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \|\Delta u\|^2 = -\int_\Omega f(u)\Delta u \, \mathrm{d}x = \int_\Omega f'(u)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} \, \mathrm{d}x \le l\|Du\|^2.$$

Integrating this between s and t, where $t \le s \le t+1$, gives

$$\|Du(t+1)\|^2 + 2\int_s^{t+1} \|\Delta u(r)\|^2 \, \mathrm{d}r \le \|Du(s)\|^2 + 2l\int_s^{t+1} \|Du(r)\|^2 \, \mathrm{d}r,$$

which implies in particular that

$$\|Du(t)\|^2 \le \|Du(s)\|^2 + 2lI_1.$$

Integrating again with respect to s between t and $t+1$ now gives

$$\|Du(t+1)\|^2 \le \int_t^{t+1} \|Du(s)\|^2 \, \mathrm{d}s + 2lI_1 \le (1+2l)I_1 \qquad \text{for all} \qquad t \ge t_0(\|u_0\|). \tag{2.6}$$

So we have shown that a bounded set in H^1 is absorbing. Since H^1 is compactly embedded in L^2, this gives a compact subset of L^2 that is absorbing. So the RDE has a global attractor in $L^2(\Omega)$.

2.1.2 2d Navier-Stokes equations

For the 2d Navier-Stokes equations

$$u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \qquad \nabla\cdot u = 0$$

we will prove the existence of an attractor for the semigroup on $H \simeq L^2(Q)$. Recall that we have a Poincaré inequality

$$\|u\| \le \lambda^{1/2}\|Du\|$$

and the orthogonality condition

$$\int_Q [(u\cdot\nabla)u]\cdot u\,\mathrm{d}x = 0$$

when u is divergence-free ($\nabla\cdot u = 0$).

2.1.2.1 Absorbing set in $H \simeq L^2(Q)$

If we take the inner product of the equation with u in L^2 then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Du\|^2 = (f,u) \le \|f\|\|u\|. \tag{2.7}$$

Using the Poincaré inequality and Young's inequality ($2ab \le a^2 + b^2$) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\lambda\|u\|^2 \le \frac{\nu\lambda}{2}\|u\|^2 + \frac{1}{2\nu\lambda}\|f\|^2,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\lambda\|u\|^2 \le \frac{1}{\nu\lambda}\|f\|^2,$$

which after an application of Gronwall's inequality yields

$$\|u(t)\|^2 \le \mathrm{e}^{-\nu\lambda t}\|u_0\|^2 + \frac{\|f\|^2}{\nu^2\lambda^2}(1-\mathrm{e}^{-\nu\lambda t}).$$

This shows that there exists a time $t_0(\|u_0\|)$ such that

$$\|u(t)\|^2 \le \rho_0^2 := \frac{2\|f\|^2}{\nu^2\lambda^2} \qquad \text{for all} \qquad t \ge t_0. \tag{2.8}$$

If we return to (2.7) and integrate from t to $t+1$, with $t \ge t_0$, we now obtain

$$\frac{\|u(t+1)\|^2}{2} - \frac{\|u(t)\|^2}{2} + \nu\int_t^{t+1}\|Du(s)\|^2\,\mathrm{d}s \le \|f\|\rho_0,$$

so that

$$\nu\int_t^{t+1}\|Du(s)\|^2\,\mathrm{d}s \le I_0 := \|f\|\rho_0 + \frac{\rho_0^2}{2}.$$

2.1.2.2 Absorbing set in $V \simeq H^1(Q)$

We will take the inner product of the equations with $-\Delta u$. In the 2d case for periodic boundary conditions only we have the useful orthogonality relation

$$\int_Q [(u\cdot\nabla)u]\cdot\Delta u \,\mathrm{d}x = 0.$$

The proof is many cancellations using the divergence-free condition and is uninformative.

The analysis that follows here can be done in the Dirichlet case, but is a little messier since the nonlinear term does not vanish.

We obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 + \nu\|\Delta u\|^2 = (f,\Delta u) \le \|f\|\,\|\Delta u\| \le \frac{\|f\|^2}{2\nu} + \frac{\nu}{2}\|\Delta u\|^2,$$

and so in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Du\|^2 \le \frac{\|f\|^2}{\nu}.$$

Integrating between s and $t+1$, where $t \le s \le t+1$ gives

$$\|Du(t+1)\|^2 - \|Du(s)\|^2 \le \frac{\|f\|^2}{\nu},$$

and now integrating with respect to s between t and $t+1$ gives

$$\|Du(t+1)\|^2 \le \int_t^{t+1} \|Du(s)\|^2 + \frac{\|f\|^2}{\nu},$$

and so

$$\|Du(t+1)\|^2 \le \frac{I_0}{\nu} + \frac{\|f\|^2}{\nu} \qquad \text{for all} \qquad t \ge t_0(\|u_0\|),$$

i.e. a bounded set in V is absorbing.

Since V is compactly embedded in H (H^1 is compactly embedded in L^2) it follows that there is a compact absorbing set in H, and so the 2d Navier-Stokes equations have a global attractor in H.

With further analysis along similar lines one can show that there is an absorbing set in H^2. In particular, therefore, the global attractor is uniformly bounded in H^2, and we will use this fact from time to time later.

3

The global attractor II
Continuity & Structure

We now discuss the behaviour of attractors under perturbation of the semigroup. It is relatively easy to prove that the attractor cannot 'explode', but to show that it does not 'collapse' we need to some structural assumptions.

We will consider a family of semigroups indexed by $\eta \in [0, \eta_0)$, S_η, and assume that S_η converges to S_0 as $\eta \downarrow 0$ in the sense that

$$\lim_{\eta \downarrow 0} \left(\sup_{t \in [0,T]} \sup_{u_0 \in B} |S_\eta(t)u_0 - S_0(t)u_0| \right) = 0$$

for all $T > 0$ and bounded B, i.e. uniform convergence on bounded time intervals and bounded sets of initial conditions.

3.1 Upper semicontinuity – 'no explosion'

We now prove that attractors cannot explode under perturbation.

Proposition 3.1 *Let $S_\eta : \mathscr{B} \to \mathscr{B}$ be a family of semigroups as above, and suppose that for all $\eta \in [0, \eta_0)$ there exists a global attractor $\mathscr{A}_\eta$. Then the following two statements are equivalent:*

(i) there exists an $\eta_1 > 0$ and a bounded set X such that

$$\mathscr{A}_\eta \subseteq X \qquad \textit{for all} \qquad \eta \in [0, \eta_1),$$

(ii)

$$\mathrm{dist}(\mathscr{A}_\eta, \mathscr{A}_0) \to 0 \qquad \textit{as} \qquad \eta \downarrow 0.$$

Proof It is clear that (ii) implies (i). So we only show that (i) implies (ii). Since $\mathscr{A}_0$ is an attractor, it attracts the bounded set X for S_0. So given $\epsilon > 0$ there exists a $T > 0$ such that

$$\text{dist}(S(T)X, \mathscr{A}_0) < \epsilon/2.$$

Now choose $\eta_1 > 0$ such that

$$\|S_\eta(T)u_0 - S_0(T)u_0\| < \epsilon/2 \qquad \text{for all} \qquad u_0 \in X.$$

Then

$$\mathscr{A}_\eta = S_\eta(T)\mathscr{A}_\eta \subseteq S_\eta(T)X \subseteq N(\mathscr{A}_0, \epsilon).$$

□

3.2 Connectedness of omega limit sets

In order to discuss the structure of the attractor further, we will require the following result on the connectedness of omega limit sets.

Lemma 3.2 *Assume that there exists a compact attracting set K, and that $\mathscr{B}$ is connected (i.e. the unit ball in $\mathscr{B}$ is connected). Then if X is a bounded connected set, $\omega(X)$ is connected, and the attractor $\mathscr{A}$ is connected.*

Proof First, we have already shown that if there is a compact attracting set K then $\omega(X) \subseteq \omega(K)$ for any bounded set X. In particular, if X is a bounded ball containing K then it follows from this and $\omega(X) \supseteq \omega(K)$ that $\mathscr{A} = \omega(X)$. Since one can therefore write the attractor as the omega limit set of a bounded connected set, we prove only that $\omega(X)$ is connected whenever X is bounded and connected.

Suppose not. Then there exist open sets O_1 and O_2 such that $O_1 \cap O_2 = \emptyset$, $\omega(X) \cap O_j \neq \emptyset$ for $j = 1, 2$, but $\omega(X) \subseteq O_1 \cup O_2$.

Now, since $S(t) : \mathscr{B} \to \mathscr{B}$ is continuous, $S(t)X$ is connected for each $t \geq 0$. But since $\omega(X)$ attracts X, for some t_0, $S(t)X \subset O_1 \cup O_2$ for all $t \geq t_0$, since $O_1 \cup O_2$ is an open neighbourhood of $\omega(X)$. It follows, then, that for all $t \geq t_0$ $S(t_0)X$ is contained entirely in O_1 or O_2, wlog O_1.

But since $S(\cdot)$ is continuous, $S(t)X$ cannot 'jump' from O_1 to O_2. It follows that $\omega(X) \subset O_1$, a contradiction. □

Note that this shows in particular that $\omega(u_0)$ is connected for any initial condition u_0.

3.3 Unstable manifolds of invariant sets

If X is an invariant set for $S(\cdot)$, we can define

$W^{\mathrm{u}}(X) = \{u_0 \in \mathscr{B} :\ u_0$ lies on a complete orbit u s.t. $\operatorname{dist}(u(t), X) \to 0$ as $t \to -\infty\}$.

In the case that X is a single point, one obtains the more familiar

$W^{\mathrm{u}}(x_0) = \{u_0 \in \mathscr{B} :\ u_0$ lies on a complete orbit u s.t. $u(t) \to x_0 \to 0$ as $t \to -\infty\}$.

All such unstable sets must lie in the attractor

Proposition 3.3 *For any bounded invariant set X, $W^{\mathrm{u}}(X) \subseteq \mathscr{A}$.*

Proof If $u_0 \in W^{\mathrm{u}}(X)$ then u_0 lies on a complete orbit $u : \mathbb{R} \to \mathscr{B}$. Since there is an attractor, $\operatorname{dist}(u(t), \mathscr{A}) \to 0$ as $t \to \infty$ and the forward portion of the orbit is bounded. Since u_0 lies in the unstable manifold of X, $\operatorname{dist}(u(t), X) \to 0$ as $t \to -\infty$, and the backward portion of the orbit is bounded. So u_0 lies on a complete bounded orbit, which implies that $u_0 \in \mathscr{A}$. □

3.4 Attractors for gradient semigroups

When the semigroup $S(\cdot)$ has a Lyapunov function, we can describe the structure of the attractor fairly completely.

We say that $S(\cdot)$ is gradient on a positively invariant subset $U \subset \mathscr{B}$ ($S(t)U \subseteq U$ for all $t \geq 0$) if there exists a Lyapunov function $\Phi : U \to \mathbb{R}$, i.e.

(i) $\Phi : U \to \mathbb{R}$ is continuous and bounded below,
(ii) $\Phi(S(t)u_0)$ is non-increasing in t, and
(iii) If $\Phi(S(T)u_0) = \Phi(u_0)$ for some $T > 0$ then u_0 is an equilibrium point (i.e. $S(t)u_0 = u_0$ for all $t \geq 0$).

Note that (iii) excludes the existence of periodic orbits.

Proposition 3.4 *Let $S(t)$ be gradient on U. Then for any $u_0 \in U$, $\omega(u_0) \subset \mathscr{E}$, where $\mathscr{E}$ is the set of equilibria. If $\mathscr{B}$ is connected and $\mathscr{E}$ is discrete then*

$\omega(u_0) \in \mathscr{E}$ (i.e. is one equilibrium), i.e. for every $u_0 \in U$ there exists an $e \in \mathscr{E}$ such that

$$S(t)u_0 \to e \qquad \text{as} \qquad t \to \infty.$$

Proof We want to prove that Φ is constant on $\omega(u_0)$. So note that

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{S(s)u_0 : \ s \geq t\}}.$$

Now,

$$\Phi_t := \sup\{\Phi(x) : \ x \in \overline{\{S(s)u_0 : \ s \geq t\}}$$

is a non-increasing function of t that is also bounded below since Φ is continuous, $\operatorname{dist}(S(s)u_0, \mathscr{A}) \to 0$, and $\mathscr{A}$ is compact. It follows that $\Phi^* = \lim_{t\to\infty} \Phi_t$ exists, and that $\Phi|_{\omega(u_0)} = \Phi^*$. Property (iii) implies that $\omega(u_0) \subseteq \mathscr{E}$. If $\mathscr{B}$ is connected then $\omega(u_0)$ is connected; if $\mathscr{E}$ is discrete then $\omega(u_0)$ must consist of a single point of $\mathscr{E}$. □

We know already that in general $W^{\mathrm{u}}(\mathscr{E}) \subseteq \mathscr{A}$. We now show that if the system is gradient and $S(\cdot)$ has the backwards uniqueness property on $\mathscr{A}$ then this is an equality.

Theorem 3.5 (Structure Theorem) *If $S(\cdot)$ has a Lyapunov function on $\mathscr{A}$, and the backwards uniqueness property on $\mathscr{A}$, then $\mathscr{A} = W^{\mathrm{u}}(\mathscr{E})$. If $\mathscr{B}$ is connected and $\mathscr{E}$ is discrete then*

$$\mathscr{A} = \bigcup_{e \in \mathscr{E}} W^{\mathrm{u}}(e).$$

Proof As remarked above, we already have $W^{\mathrm{u}}(\mathscr{E}) \subseteq \mathscr{A}$. Now, if $u_0 \in \mathscr{A}$ then since $S(\cdot)$ has the backwards uniqueness property on $\mathscr{A}$ we can define $u(t) = S(t)u_0$ for $t \leq 0$, and consider

$$\gamma = \cap_{t<0} \overline{\{u(s) : \ s \leq t\}}.$$

One can follow the argument in Proposition 2.3 to show that $\operatorname{dist}(S(t)u_0, \gamma) \to 0$ as $t \to -\infty$, so $u_0 \in W^{\mathrm{u}}(\gamma)$; while the argument of Proposition 3.4 shows that Φ is constant on γ, and so $\gamma \subseteq \mathscr{E}$. It follows that $u_0 \in W^{\mathrm{u}}(\mathscr{E})$, which gives the promised equality.

If $\mathscr{E}$ is discrete and $u_0 \in W^{\mathrm{u}}(\mathscr{E})$ then $S(t)u_0$ lies in a neighbourhood of $\mathscr{E}$ for all $t \leq t_0$, from which it follows that in fact $u_0 \in W^{\mathrm{u}}(e)$ for some element of e. □

3.5 Lower semicontinuity of the attractor

We now show that when the attractor has the structure in Theorem 3.5, it cannot collapse under perturbation. The proof as given here is taken from Stuart & Humphries (1996).

The main addition assumption, in addition to the structural one, is that the equilibira and their local unstable manifolds perturb continuously: more precisely, there exists a $\delta > 0$, such that for $0 < \epsilon < \delta/2$, one can find an η_0 such that

$$\|e_\eta - e_0\| < \epsilon$$

and

$$\mathrm{dist}_H(W^{\mathrm{u}}(e_0) \cap B_\delta(e_0), W^{\mathrm{u}}(e_\eta) \cap B_\delta(e_0)) < \epsilon.$$

Theorem 3.6 *Suppose that $\mathscr{E}$ is discrete,*

$$\mathscr{A}_0 = \bigcup_{e \in \mathscr{E}} W^{\mathrm{u}}(e),$$

and that the equilibria and their local unstable manifolds perturb continuously. Then

$$\lim_{\eta \downarrow 0} \mathrm{dist}(\mathscr{A}_0, \mathscr{A}_\eta) = 0. \tag{3.1}$$

Proof The conclusion (3.1) means that given any $\epsilon > 0$, for η sufficiently small, $\mathscr{A}_0$ is contained within an ϵ neighbourhood of $\mathscr{A}_\eta$. In other words, for every $u \in \mathscr{A}_0$, there must exist a $v(\eta) \in \mathscr{A}_\eta$ with $\|v(\eta) - u\| < \epsilon$.

Since $\mathscr{A}_0$ is compact, there exists a finite set of points $\{u_j\}_{j=1}^N$ such that for any $u \in \mathscr{A}_0$ there exists a $j \in \{1, \ldots, N\}$ such that $\|u - u_j\| < \epsilon/2$. So one need only show that for each u_j there exists a $v_j(\eta) \in \mathscr{A}_\eta$ with $\|u_j - v_j(\eta)\| < \epsilon$ for all η small enough.

First, note that there exists a $\zeta > 0$ such that if $u_0 \in \mathscr{A}_0$,

$$\|u_0 - u_1\| < \zeta \qquad \Rightarrow \qquad \sup_{t \in [0,T^*]} \|S(t)u_0 - S(t)u_1\| < \epsilon/4.$$

Now, given the form of the attractor, $u_k = S(T_k)z_k$ for some $z_k \in W^{\mathrm{u}}(e_j) \cap B_\delta(e_j)$. Set $T^* = \max_k T_k$, and choose $\eta_1 > 0$ small enough that for all $\eta \in [0, \eta_1)$:

(i)

$$\sup_{t \in [0,T^*]} \sup_{u_0 \in N_\delta(\mathscr{A}_0)} \|S_0(t)u_0 - S_\eta(t)u_0\| < \epsilon/4,$$

and

(ii) for each k, there exists a $z_k(\eta) \in W^{\mathrm{u}}(e_k(\eta))$ such that

$$\|z_k(\eta) - z_k\| < \zeta.$$

Now, consider $v_k(\eta) = S_\eta(T_k)z_k(\eta)$. Since $z_k(\eta) \in W^{\mathrm{u}}(e_k(\eta))$, $z_k(\eta) \in \mathscr{A}_\eta$, and since $\mathscr{A}_\eta$ is invariant for $S_\eta(\cdot)$, it follows that $v_k(\eta) \in \mathscr{A}_\eta$. Now it only remains to check that

$$\begin{aligned}
\|v_k(\eta) - u_k\| &= \|S_\eta(T_k)z_k(\eta) - S_0(T_k)z_k\| \\
&\leq \|S_\eta(T_k)z_k(\eta) - S_0(T_k)z_k(\eta)\| + \|S_0(T_k)z_k(\eta) - S_0(T_k)z_k\| \\
&< \epsilon/4 + \epsilon/4 = \epsilon/2,
\end{aligned}$$

as required. □

4

RDE as a gradient system: regularity

We now want to show that the RDE

$$u_t - \Delta u = f(u)$$

where

$$-k - \alpha_1 |s|^p \leq f(s)s \leq k - \alpha_2 |s|^p \qquad \text{and} \qquad f'(s) \leq l$$

gives rise to a gradient system, and so has an attractor of the form (3.5). In order to do this, we have to show that it has a Lyapunov function. In order to show that this Lyapunov function is continuous from $\mathscr{A}$ into $\mathbb{R}$, we will need some additional regularity for functions lying on the attractor.

We have already shown that the attractor is bounded in L^2 and in H^1. We will now show that it is also bounded in L^∞ and in H^2. (Of course, for the RDE on a one-dimensional domain, a bound in H^1 gives a bound in L^∞ with no additional work.)

4.1 A bound in $L^\infty(\Omega)$

In order to show this bound, we introduce

$$u_+(x) = \begin{cases} u(x) & u(x) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad u_-(x) = \begin{cases} u(x) & u(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that if $u \in L^2(\Omega)$ then $u_\pm \in L^2(\Omega)$ and $\|u_\pm\|_{L^2} \leq \|u\|_{L^2}$. We now show that the same is true if $u \in H^1(\Omega)$:

Lemma 4.1 *If $u \in H^1(\Omega)$ then $u_\pm \in H^1(\Omega)$, $\|u_\pm\|_{H^1} \leq \|u\|_{H^1}$, and*

$$Du_+(x) = \begin{cases} Du(x) & u(x) > 0 \\ 0 & \textit{otherwise} \end{cases}, \tag{4.1}$$

with a similar equality for Du_-.

Proof Define

$$u_\epsilon(x) = \begin{cases} (u(x)^2 + \epsilon^2)^{1/2} - \epsilon & u(x) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$u'_\epsilon(x) = \begin{cases} \frac{u(x)u'(x)}{(u(x)^2+\epsilon^2)^{1/2}} & u(x) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $\|u_\epsilon\|_{H^1} \leq \|u\|_{H^1}$, and that $u_\epsilon \to u_+$ in $L^2(\Omega)$. Since u_ϵ is bounded in H^1, it has a subsequence u_{ϵ_j} that converges weakly to some v, $u_{\epsilon_j} \rightharpoonup v$ in H^1. But weak convergence in H^1 implies strong convergence in L^2, so $u_{\epsilon_j} \to v$ in L^2. But we know that $u_\epsilon \to u_+$ in L^2, so $v = u_+$. It follows that $Du_+ \in H^1$ with $\|Du_+\|_{H^1} \leq \|Du\|_{H^1}$ as claimed, and taking limits of u'_ϵ as $\epsilon \to 0$ gives (4.1). The result for u_- follows since $u_- = -(-u)_+$. □

We can now prove that the attractor is bounded in L^∞.

Proposition 4.2 *The attractor for the RDE $u_t - \Delta u = f(u)$ is bounded in $L^\infty(\Omega)$ with*

$$\|u\|_\infty \leq \left(\frac{k}{\alpha_2}\right)^{1/p} \qquad \textit{for all} \qquad u \in \mathscr{A}.$$

Proof Set $M = (k/\alpha_2)^{1/p}$, so that for $s \geq M$, $f(s)s \leq 0$. Now multiply the equation by $(u(x) - M)_+$ and integrate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u(x)-M)_+\|^2 + \int_\Omega (-\Delta u)\cdot(u(x)-M)_+ \,\mathrm{d}x = \int_\Omega f(u)(u(x)-M)_+ \,\mathrm{d}x.$$

Now,

$$\int_\Omega (-\Delta u)\cdot(u(x)-M)_+ \,\mathrm{d}x = \int_\Omega Du\cdot D(u(x)-M)_+ \,\mathrm{d}x = \int_\Omega |D(u(x)-M)_+|^2 \,\mathrm{d}x,$$

using (4.1). We also have

$$\int_\Omega f(u)(u(x)-M)_+\,\mathrm{d}x = \int_{x:\ u(x)>M} f(u)u[1-(M/u)]\,\mathrm{d}x \le 0.$$

So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u-M)_+\|^2 + \|D(u-M)_+\|^2 \le 0.$$

Since $u=0$ on $\partial\Omega$, clearly $(u-M)_+=0$ on $\partial\Omega$, so we can use the Poincaré inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u-M)_+\|^2 \le -\lambda\|(u-M)_+\|^2.$$

Integrating from $-t$ to 0 it follows that

$$\|(u(0)-M)_+\|^2 \le \|(u(-t)-M)_+\|^2\mathrm{e}^{-2\lambda t}.$$

Since $\mathscr{A}$ is bounded in $L^2(\Omega)$, and any $u(0)$ lies on a complete bounded orbit, it follows letting $t\to\infty$ that $\|(u(0)-M)_+\|^2=0$. Therefore $u(x)\le M$ almost everywhere.

A similar argument considering $(u+M)_-$ shows that $u(x)\ge -M$ everywhere, and so $\|u\|_{L^\infty}\le M$. □

It is worth emphasising again that if Ω is one-dimensional, we already know that $\mathscr{A}$ is bounded in L^∞ since it is bounded in H^1.

4.2 A bound in H^2

We show that the attractor is bounded in H^2. During the course of the proof we will also find the Lyapunov function for the semigroup on L^2.

Proposition 4.3 *The attractor for the RDE is bounded in $H^2(\Omega)$.*

Proof The plan is to rearrange the equation as $-\Delta u = f(u)-u_t$, show that the right-hand side is bounded in L^2, and use elliptic regularity to show that u must be in H^2: indeed, if $-\Delta u = g$, $u|_{\partial\Omega}=0$ and $g\in L^2$, it is known that $\|u\|_{H^2}\le c\|g\|_{L^2}$.

We already know that $f(u(\cdot))\in L^2$, since we know that $u(\cdot)\in L^\infty$ (this implies that $f(u(\cdot))\in L^\infty$, which immediately gives $f(u(\cdot))\in L^2$. So our task is to show that $u_t\in L^2$ for all solutions on $\mathscr{A}$.

First we take the inner product of the equation with u_t:

$$\|u_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|Du_t\|^2 = \int_\Omega f(u)u_t \,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathscr{F}(u(x))\,\mathrm{d}x,$$

where $\mathscr{F}(r) = \int_0^r f(s)\,\mathrm{d}s$. Note that this show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla u|^2 - \mathscr{F}(u)\,\mathrm{d}x = -|u_t|^2. \tag{4.2}$$

The quantity whose time derivative occurs on the left-hand side will be our Lyapunov function $V(u)$.

Integrating from 0 to t we obtain

$$\int_0^t \|u_t(s)\|^2\,\mathrm{d}s + \frac{1}{2}\|Du(t)\|^2 = \frac{1}{2}\|Du_0\|^2 + \int \mathscr{F}(u(t))\,\mathrm{d}x - \int \mathscr{F}(u(0))\,\mathrm{d}x.$$

Since $\mathscr{A}$ is bounded in L^∞ and H^1, this implies that

$$\int_0^t \|u_t(s)\|^2\,\mathrm{d}s \le C.$$

Now differentiate the governing equation with respect to t,

$$u_{tt} - \Delta u_t = f'(u)u_t$$

and take the inner product with t^2u_t:

$$(t^2u_t, u_{tt}) - (t^2u_t, \Delta u_t) = (t^2u_t, f'(u)u_t),$$

from whence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(t^2\|u_t\|^2) - t^2\|u_t\|^2 + t^2\|Du_t\|^2 = t^2\int_\Omega f'(u)|u_t|^2\,\mathrm{d}x \le lt^2\|u_t\|^2.$$

Integrating from 0 to t gives

$$t^2\|u_t(t)\|^2 \le (1+l)t^2\int_0^t \|u_t(s)\|^2\,\mathrm{d}s,$$

and so

$$\|u_t(t)\|^2 \le (1+l)C.$$

It follows that on the attractor, u_t is bounded in L^2. Since this implies that $-\Delta u$ is bounded uniformly in L^2 over the attractor, elliptic regularity theory gives a uniform bound in H^2 for all $u \in \mathscr{A}$. □

We can now show that

$$V(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \mathscr{F}(u)\,\mathrm{d}x$$

from (4.2) is a Lyapunov function. Equation (4.2) shows that it is non-increasing, and also that if $V(u(T)) = V(u_0)$ then $u_t = 0$ on $[0, T]$ and so u_0 is an equilibrium.

To show that V is continuous from $\mathscr{A}$ into $\mathbb{R}$, consider $V(u_1) - V(u_2)$:

$$\int_\Omega \frac{1}{2}(|\nabla u_1|^2 - |\nabla u_2|^2)\,\mathrm{d}x - \int_\Omega \mathscr{F}(u_1) - \mathscr{F}(u_2)\,\mathrm{d}x.$$

For the first term,

$$\begin{aligned} |(Du_1, Du_1) - (Du_2, Du_2)| &= |(Du_1 - Du_2, Du_1) + (Du_1, Du_1 - Du_2)| \\ &\le (\|Du_1\| + \|Du_2\|)\|Du_1 - Du_2\| \\ &\le C\|Du_1 - Du_2\|, \end{aligned}$$

since $\mathscr{A}$ is bounded in H^1. Now,

$$\|D(u_1 - u_2)\|^2 = (D(u_1 - u_2), D(u_1 - u_2)) = (u_1 - u_2, -\Delta(u_1 - u_2)) \le C'\|u_1 - u_2\|$$

since $\mathscr{A}$ is bounded in H^2. So $u \mapsto \|Du\|^2$ is continuous from $\mathscr{A}$ into $\mathbb{R}$.

For the second part,

$$\int \mathscr{F}(u_1) - \mathscr{F}(u_2)\,\mathrm{d}x = \int_\Omega \int_{u_2(x)}^{u_1(x)} f(s)\,\mathrm{d}s\,\mathrm{d}x,$$

and so

$$\left| \int \mathscr{F}(u_1) - \mathscr{F}(u_2)\,\mathrm{d}x \right| \le \int_\Omega C|u_1(x) - u_2(x)|\,\mathrm{d}x \le C'\|u_1 - u_2\|,$$

since $\mathscr{F}(u(x)) \le C$ on $\mathscr{A}$.

It follows that $V : \mathscr{A} \to \mathbb{R}$ is continuous, and so is a Lyapunov function. So the RDE generates a gradient dynamical system, and the attractor has the form (3.5).

4.2.1 The Chafee-Infante equation

For the particular example

$$u_t + \Delta u = \beta u - u^3 \qquad x \in [0, 1],$$

the 'Chafee-Infante' equation, the structure of the attractor is extremely well understood. The equation is gradient, so the attractor is the union of the unstable manifolds of the equilibria, and equilibria are all created in bifurcations from the zero solution.

If one considers the linearisation about zero,

$$U_t = (-\Delta + \beta I)U,$$

if the eigenvalues of $-\Delta$ are $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ then for $\beta < \lambda_1$ all solutions tend to the origin – there is only one equilibrium. For $\lambda_1 < \beta < \lambda_2$ there are three equilibria (there has been a pitchfork bifurcation about the origin at $\beta = \lambda_1$); for $\lambda_2 < \beta < \lambda_3$ there are five equilibira...

For $\lambda_n < \beta < \lambda_{n+1}$ there are $2n + 1$ equilibria, and the dimension of the attractor is n. Since in this case $\lambda_n \sim n^2$, it follows that the dimension of the attractor satisfies $d(\mathscr{A}) \sim \beta^{1/2}$. We now turn to the general problem of bounding the attractor dimension.

5

Bounding the dimension of the attractor

We will use the upper box-counting dimension. If $X \subset \mathscr{B}$ then

$$d_{\text{box}}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the number of balls of radius ϵ (in the norm of $\mathscr{B}$) required to cover X. Essentially this extracts the exponent d from $N(X,\epsilon) \simeq \epsilon^{-d}$.

The lim sup is necessary in general, but one can take

$$\limsup_{k \to \infty} \frac{\log N(X,\epsilon_k)}{-\log \epsilon_k},$$

if, for example, $\epsilon_k = \alpha^k \epsilon_0$ with $0 < \alpha < 1$.

It is a consequence of the definition that if one defines

$$V_\gamma(X,\epsilon) = \epsilon^\gamma N(X,\epsilon),$$

then if

$$\limsup_{\epsilon \to 0} V_\gamma(X,\epsilon) = 0$$

then $d_{\text{box}}(X) \le \gamma$. One can also take the limsup through a sequence $\epsilon_k = \alpha^k \epsilon_0$, as above.

We will now outline a method for proving that the box-counting dimension of an invariant set is finite. We will restrict to the case of dynamical systems on Hilbert spaces, and initially rather than dealing with continuous time it will be easier to deal with a fixed map $S : H \to H$.

The idea is to cover $\mathscr{A}$ by balls of radius ϵ:

$$\mathscr{A} \subset \bigcup_{j=1}^{N} B(x_j, \epsilon),$$

and then apply S to both sides:

$$S\mathscr{A} = \mathscr{A} \subset \bigcup_{j=1}^{N} SB(x_j, \epsilon).$$

If ϵ is sufficiently small then $SB(x_j, \epsilon)$ will be well approximated by $Sx_j + DS(x_j)[B(0, \epsilon)]$, where $DS(x_j)$ is the derivative of S at x_j. So the key to the argument will be that the image of a ball under a linear map is an ellipse, coupled with a result that allows one to cover an ellipse by a certain number of balls of specified radius.

5.1 The image of a ball under a linear map

Let H be a Hilbert space and $L : H \to H$ be a compact linear operator. Then L^*L is a compact self-adjoint linear operator that is positive; so it has a set a eigenvalues $\alpha_1^2 \geq \alpha_2^2 \geq \alpha_3^2 \cdots$ with corresponding orthonormal eigenvectors $\{e_j\}$ that form a basis for H.

Lemma 5.1 $LB(0, 1)$ *is an ellipse whose semi-axes are* $\{Le_j\}$, *and* $\|Le_j\| = \alpha_j$.

Proof First, we have

$$(Le_j, Le_i) = (L^*Le_j, e_i) = (\alpha_j^2 e_j, e_i) = \alpha_j^2 \delta_{ij},$$

so that the $\{Le_j\}$ are orthogonal with $\|Le_j\| = \alpha_j$.

Now if $u \in H$ with $u = \sum u_j e_j$ then

$$Lu = \sum_j u_j (Le_j) = \sum_{j:\ \alpha_j \neq 0} \underbrace{(u_j \alpha_j)}_{\xi_j} \frac{Le_j}{\alpha_j}.$$

So

$$u \in B(0, 1) \ \Leftrightarrow\ \sum |u_j|^2 = 1 \ \Leftrightarrow\ \sum_{j:\ \alpha_j \neq 0} \left(\frac{\xi_j}{\alpha_j}\right)^2 \leq 1 \text{ and } \xi_j = 0 \text{ if } \alpha_j = 0.$$

So $LB(0, 1)$ is an ellipse with semi-axes Le_j. □

As a result of this lemma, given a compact linear operator $L : H \to H$, define the linear expansion factors $\alpha_j(L)$ to be the square root of the eigenvalues of L^*L (equivalently, the eigenvalues of $(L^*L)^{1/2}$) in decreasing order,

$$\alpha_1(L) \geq \alpha_2(L) \geq \alpha_3(L) \geq \cdots$$

– these are the lengths of the semi-axes of the ellipse $LB(0,1)$ – and let $\omega_n(L)$ be the maximal expansion factor of n-dimensional volumes under L,

$$\omega_n(L) = \alpha_1(L)\alpha_2(L)\cdots\alpha_n(L).$$

We will need the following lemma on coverings of ellipsoids by balls – it says essentially that an ellipsoid can be covered by $\mathrm{Vol}_j(E)/\mathrm{Vol}_j(B_{r/7}(0))$ if $r \sim \alpha_j$, where Vol_j is (in some sense) the j dimensional volume. For a proof see Temam (1988).

Lemma 5.2 *Let E be an ellipsoid, with semi-axes $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdot$. Then for any $r < \alpha_1$, the number of balls of radius $\sqrt{2}r$ needed to cover E is less than*

$$7^j \frac{\omega_j}{r^j},$$

where $\omega_j = \alpha_1 \cdots \alpha_j$ and j is the largest integer such that $r \leq \alpha_j$.

5.2 Bounding the dimension of an invariant set of a map

Our result will treat invariant sets $\mathscr{A}$ of nonlinear maps $S : H \to H$ that are differentiable in the following sense: for every $u \in \mathscr{A}$ there exists a map $DS(u) : H \to H$ such that

$$\|S(u+h) - S(u) - DS(u)h\| \leq K\|h\|^{1+\alpha} \qquad \text{for all} \qquad u \in \mathscr{A}.$$

For the statement of the theorem, we choose $\bar{\omega}_n$ and $\bar{\alpha}_n$ such that

$$\alpha_n(DS(u)) \leq \bar{\alpha}_n \qquad \text{and} \qquad \omega_n(DS(u)) \leq \bar{\omega}_n \qquad \text{for all} \qquad u \in \mathscr{A},$$

respecting

$$\bar{\alpha}_1 \geq \bar{\alpha}_2 \geq \bar{\alpha}_3 \cdots \qquad \text{and} \qquad \bar{\alpha}_n^n \leq \bar{\omega}_n.$$

Theorem 5.3 *Assume that* $2\bar{\omega}_d^{1/d} < 1$. *Then for any* γ *such that*

$$(2\bar{\omega}_d^{1/d})^\gamma \max_{1\le j\le d} 7^j \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}} < 1,$$

$d_{\mathrm{box}}(\mathscr{A}) \le \gamma$.

Proof First choose $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$,

$$K\epsilon^{1+\alpha} < (2-\sqrt{2})\bar{\omega}_d^{1/d}\epsilon.$$

Cover $\mathscr{A}$ with balls of radius $\epsilon < \epsilon_0$,

$$\mathscr{A} \subset \bigcup_{i=1}^N B(u_i,\epsilon).$$

Then

$$S\mathscr{A} \subset \bigcup_{i=1}^N SB(u_i,\epsilon).$$

Now,

$$SB(u_i,\epsilon) \subseteq S(u_i) + DS[B(0,\epsilon)] + B(0, K\epsilon^{1+\alpha}).$$

We consider how to cover $SB(u_i,\epsilon)$ for each i:
(i) If $\alpha_1(DS(u_i)) < \bar{\omega}_d^{1/d}$ then

$$DS(u_i)B(0,\epsilon) \subseteq B(0,\alpha_1(DS(u_i))\epsilon) \subseteq B(0,\bar{\omega}_d^{1/d}\epsilon),$$

from which it follows that

$$SB(u_i,\epsilon) \subset B(S(u_i), 2\bar{\omega}_d^{1/d}\epsilon).$$

Thus it requires only one ball of radius $2\bar{\omega}_d^{1/d}\epsilon$ to cover $SB(u_i,\epsilon)$ in this case.

(ii) If $\bar{\omega}_d^{1/d} < \alpha_1(DS(u_i))$ then using Lemma 5.2 the number of balls of radius $\sqrt{2}\bar{\omega}_d^{1/d}$ needed to cover $DS(u_i)B(0,\epsilon)$ is bounded by

$$\frac{7^j\omega_j(DS(u_i))}{\bar{\omega}_d^{j/d}}$$

where j is the largest integer such that $\bar{\omega}_d^{j/d} \le \alpha_j$. Since $\bar{\omega}_d^{1/d} \ge \bar{\alpha}_d$, it follows that $j \le d$. So no more than

$$M := \max_{1\le j\le d} \frac{7^j\omega_j(DS(u_j))}{\bar{\omega}_d^{j/d}} \le \max_{1\le j\le d} 7^j \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}$$

balls of radius $\sqrt{2}\bar{\omega}_d^{1/d}$ are required to cover $DS(u_i)B(0,\epsilon)$.

It follows that $SB(u_j,\epsilon)$ can be covered by M balls of radius $2\bar{\omega}_d^{1/d}\epsilon$.

Combining these, if $\mathscr{A}$ is covered by $N(X,\epsilon)$ balls of radius ϵ, it can also be covered by

$$MN(X,\epsilon) \leq \max_{1\leq j\leq d} 7^j \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}} N(X,\epsilon)$$

balls of radius $2\bar{\omega}_d^{1/d}\epsilon$.

Thus

$$V_\gamma(X, 2\bar{\omega}_d^{1/d}\epsilon) \leq \left[(2\bar{\omega}_d^{1/d})^\gamma \max_{1\leq j\leq d} 7^j \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}}\right] V_\gamma(X,\epsilon).$$

If the expression in square brackets is less than 1, it therefore follows that $d_{\text{box}}(X) \leq \gamma$. □

In order to apply this to examples, note that we have

$$\bar{\omega}_d(S^n) \leq [\bar{\omega}_d(S)]^n.$$

So if

$$\bar{\omega}_d(S) < 1 \qquad \text{and} \qquad \bar{\omega}_d^{\gamma/d} \max_{1\leq j\leq d} \frac{\bar{\omega}_j}{\bar{\omega}_d^{j/d}} < 1 \tag{5.1}$$

we can find an n such that $\bar{\omega}_j(S^n)$ satisfies the conditions of Theorem 5.3, i.e. one can conclude that $d_{\text{box}}(\mathscr{A}) \leq \gamma$.

Now, write $\bar{q}_j = \log \bar{\omega}_j$. The conditions in (5.1) become

$$\bar{q}_d < 0 \qquad \text{and} \qquad \gamma > \max_{1\leq j\leq d} \frac{j\bar{q}_d - d\bar{q}_j}{\bar{q}_d}.$$

The following lemma is useful in applications:

Lemma 5.4 *Assume that* $\bar{q}_j \leq q_j$*, where* q_j *is a concave function of* j*. Then* $q_n < 0$ *implies that* $d_{\text{box}}(\mathscr{A}) \leq n$*.*

Proof Since q_j is concave, there exist α, β such that $q_j \leq -\alpha j + \beta$: choose α, β such that

$$0 < q_{n-1} = -\alpha(n-1) + \beta \qquad \text{and} \qquad 0 > q_n = -\alpha n + \beta.$$

Note that in particular it follows that $\beta/\alpha < n$.

The argument leading to the bound on γ uses only upper bounds on the $\bar{q}_j$s, so $d_{\rm box}(\mathscr{A}) \leq \gamma$ provided that

$$\gamma > \max_{1\leq j\leq d} \frac{j(-\alpha d+\beta)-d(-\alpha j+\beta)}{-\alpha d+\beta} = \max_{1\leq j\leq d} \frac{\beta(d-j)}{\alpha d-\beta} \leq \frac{\beta d}{\alpha d-\beta}.$$

Since d is arbitrary, one can let $d \to \infty$ and show that $d_{\rm box}(\mathscr{A}) \leq \gamma$ provided that $\gamma > \beta/\alpha$. But $\beta/\alpha < n$, so $d_{\rm box}(\mathscr{A}) \leq n$. □

5.3 Finding $\bar{\omega}_n$ for a flow

We consider the solution of

$$\dot{u} = F(u)$$

and the equation for the linearisation about $u(t)$,

$$\dot{U} = L(u(t))U.$$

We'll give an unconvincing and entirely spurious argument, which will give the right answer...

Write the solution of the linearised equation as

$$U(t) = \exp\left(\int_0^t L(u(s))\,{\rm d}s\right) U_0$$

(you can't do this - even for a two-dimensional linear equation $\dot{x} = A(t)x$ you can't write $x(t) = {\rm e}^{\int A(s)\,{\rm d}s}x_0$). So our expansion factor ω_n at time t will be related to the eigenvalues of

$$\exp\left(\int_0^t L(u(s))\,{\rm d}s\right)$$

(which isn't exactly true, since we have to take the eigenvalues of $(L^*L)^{1/2}$ in our rigorous argument). Since we are looking at $q_n(t) = \log\omega_n(t)$, this amounts to considering the eigenvalues of

$$I(t) = \int_0^t L(u(s))\,{\rm d}s$$

(not sure about the connection between eigenvalues of A and ${\rm e}^A$). The sum of the first n eigenvalues of this are given by

$$\sum_{j=1}^n (I(t)e_j, e_j)$$

where $\{e_j\}$ are the orthonormal eignevectors of $I(t)$ corresponding to the largest n eigenvalues (but $I(t)$ may not be self-adjoint). So then certainly

$$q_n(t) \leq \sup_{\text{orthonormal } \{\phi_j\}_{j=1}^n} (I(t)\phi_j, \phi_j).$$

Let's define the n-dimensional trace of a linear operator L by

$$\mathrm{Tr}_n(L) = \sup_{\text{orthonormal } \{\phi_j\}_{j=1}^n} (L\phi_j, \phi_j),$$

so we can write

$$q_n(t) \leq \mathrm{Tr}_n\left(\int_0^t L(u(s))\, ds\right).$$

Then clearly

$$q_n(t) \leq \int_0^t \mathrm{Tr}_n(L(u(s))\,\mathrm{d}s;$$

not only is this final step valid, but this result is actually true.

Since we are free to choose t, and what matters is that $q_n < 0$, it is sensible to consider

$$q_n' = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{Tr}_n(L(u(s))\,\mathrm{d}s := \langle \mathrm{Tr}_n(L(u(s))\,\mathrm{d}s\rangle$$

(so $\langle h\rangle$ denotes the long-time average of h). If $q_n' < 0$ then $q_n(t) < 0$ for some t. But, of course, we want this to hold all over the attractor, so finally we are lead to consider

$$\tilde{q}_n = \sup_{u_0\in\mathscr{A}} \langle \mathrm{Tr}_n(L(S(t)u_0))\rangle.$$

We make the following definition:

Definition 5.5 *$S(t)$ is* uniformly differentiable on the attractor*: for each $t > 0$ there exists a linear operator $\Lambda(t; u_0) : H \to H$ such that*

$$\sup_{u_0,v_0\in\mathscr{A}} \|[S(t)u_0 + \Lambda(t; u_0)(v_0 - u_0)] - S(t)v_0\| \leq K\|u_0 - v_0\|^{1+r}$$

for some $r > 0$, and $\sup_{u_0\in\mathscr{A}} \|\Lambda(t; u_0)\|_{\mathrm{op}} < \infty$.

We can now state the following theorem.

Theorem 5.6 *Suppose that $S(t)$ is uniformly differentiable on the attractor, and that $\Lambda(t; u_0)$ is compact for each $t > 0$ and is the solution of the equation*

$$\mathrm{d}U/\mathrm{d}t = L(S(t)u_0)U \qquad \textit{with} \qquad U(0) = \mathrm{Id}_H.$$

Set

$$\tilde{q}_j := \sup_{u_0 \in \mathscr{A}} \langle \mathrm{Tr}_n(L(S(t)u_0)) \rangle.$$

If the upper bounds on $\tilde{q}_j$ are concave then $\tilde{q}_n < 0$ implies that $d_{\mathrm{box}}(\mathscr{A}) \leq n$.

5.4 Example I: the reaction-diffusion equation

We return to

$$u_t - \Delta u = f(u) \qquad \text{with} \qquad f'(s) \leq l.$$

Note that for the example $f_\beta(s) = \beta s - s^3$ we considered in Section 4.2.1, $f'_\beta(s) \leq \beta$. We will assume that $S(t)$ is uniformly differentiable on the attractor (this is awkward to prove), and given this show:

Theorem 5.7 *The attractor of the RDE has*

$$d_{\mathrm{box}}(\mathscr{A}) \leq \left(\frac{cl}{\lambda_1} \right)^{d/2}.$$

Proof The linearised equation is

$$U_t = \Delta U + f'(u(t))U.$$

So we have to consider $\mathrm{Tr}_n(\Delta U + f'(u(t))I)$. Taking n orthonormal (in L^2) functions ϕ_j, we need to bound

$$\begin{aligned}\sum_{j=1}^n (\Delta\phi_j + f'(u)\phi_j, \phi_j) &= \sum_{j=1}^n (\Delta\phi_j, \phi_j) + \sum_{j=1}^n \int_\Omega f'(u)|\phi_j|^2 \,\mathrm{d}x \\ &\leq -c\lambda_1 n^{(d+2)/d} + \sum_{j=1}^n l \int_\Omega |\phi_j|^2 \,\mathrm{d}x \\ &= -c\lambda_1 n^{(d+2)/d} + nl,\end{aligned}$$

since $\|\phi_j\| = 1$. It follows that

$$\mathrm{Tr}_n(L(u)) \leq -c\lambda_1 n^{(d+2)/d} + nl,$$

which gives the same bound on $\tilde{q}_n$. This bound is concave, so if $\tilde{q}_n < 0$, $d_{\text{box}}(\mathscr{A}) \leq n$. This happens when $n^{2/d} > c(l/\lambda_1)$. □

5.5 Dimension bound for the 2d Navier–Stokes equations

We now return to the 2d Navier-Stokes equations. Here we will prove the uniform differentiability property:

Proposition 5.8 *The flow on the attractor of the 2d Navier-Stokes equations is uniformly differentiable, with $\Lambda(t; u_0)\xi$ the solution of*

$$\mathrm{d}U/\mathrm{d}t - \nu\Delta U + (u\cdot\nabla)U + (U\cdot\nabla)u + \nabla q = 0 \qquad \nabla\cdot U = 0. \tag{5.2}$$

Proof Let $u(t) = S(t)u_0$, $v(t) = S(t)v_0$, and let $U(t)$ be the solution of (5.2) with initial condition $\xi = v_0 - u_0$. We will consider

$$\theta(t) = u(t) + \xi(t) - v(t).$$

After some calculation, $\theta(t)$ satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} - \nu\Delta\theta + (u\cdot\nabla)\theta + (\theta\cdot\nabla)u + (w\cdot\nabla)w = 0.$$

Taking the inner product with θ gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 + \nu\|D\theta\|^2 &= -\int[(u\cdot\nabla)\theta]\cdot\theta - \int[\theta\cdot\nabla)u]\cdot\theta - \int[(w\cdot\nabla)w]\cdot\theta \\ &\leq k\|\theta\|\,\|D\theta\|\,\|Du\| + k\|w\|\,\|Dw\|\,\|D\theta\| \\ &\leq k\rho_1\|\theta\|\,\|D\theta\| + k\|w\|\,\|Dw\|\,\|D\theta\| \\ &\leq \frac{(k\rho_1)^2}{\nu}\|\theta\|^2 + \frac{\nu}{4}\|D\theta\|^2 + \frac{k^2}{\nu}\|w\|^2\,\|Dw\|^2 + \frac{\nu}{4}\|D\theta\|^2,\end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 + \nu\|D\theta\|^2 \leq c\|\theta\|^2 + c\|w\|^2\|Dw\|^2.$$

An application of Gronwall's inequality gives (since $\theta(0) = 0$)

$$\|\theta(t)\|^2 \leq c\mathrm{e}^{ct}\int_0^t \|w(s)\|^2\|Dw(s)\|^2\,\mathrm{d}s. \tag{5.3}$$

In our uniqueness proof (page 7) we obtained the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\|Dw\|^2 \leq c\|Du\|^2\|w\|^2 \tag{5.4}$$

which implies that

$$\|w(t)\|^2 \le \mathrm{e}^{c\rho_1^2 t}\|w(0)\|^2.$$

Multiplying (5.4) by $\|w\|^2$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^4 + \nu\|w\|^2\|Dw\|^2 \le c\rho_1^2\|w\|^4.$$

Integrating from 0 to t gives

$$\|w(t)\|^4 + 2\nu\int_0^t \|w(s)\|^2\|Dw(s)\|^2\,\mathrm{d}s \le 2\mathrm{e}^{c\rho_1^2 t}\|w(0)\|^4,$$

and so

$$\nu\int_0^t \|w(s)\|^2\|Dw(s)\|^2\,\mathrm{d}s \le K(t)\|w(0)\|^4.$$

Combining this with (5.3) gives

$$\|\theta(t)\|^2 \le K'(t)\|w(0)\|^4,$$

which proves the uniform differentiability on $\mathscr{A}$.

Compactness of Λ can be shown following the procedure we used to prove the existence of a compact absorbing set. Find a bound on $\|\Lambda(t)\xi\|$ and on $\int_0^t \|D(\Lambda(s)\xi)\|^2$, the use the uniform Gronwall approach to find a bound on $\|D(\Lambda(t)\xi)\|$ that depends only on $\|\xi\|$. This shows that $\Lambda(t)$ maps bounded sets in L^2 into bounded sets in H^1, i.e. that $\Lambda(t)$ is compact. □

Theorem 5.9 *The attractor of the 2d Navier-Stokes equations satisfies*

$$d_{\mathrm{box}}(\mathscr{A}) \le cG^2,$$

where G is the dimensionless Grashof number,

$$G = \frac{\|f\|}{\nu^2\lambda_1}.$$

Note that the best known bound in the Dirichlet case (at least in terms of G) is cG; in the periodic case it is $cG^{2/3}(1+\log G)^{1/3}$.

Proof We have to consider $\mathrm{Tr}_n(L(u))$ where

$$L(u)\phi = \nu\Delta\phi - (u\cdot\nabla)\phi - (\phi\cdot\nabla)u.$$

So we estimate

$$\begin{aligned}\sum_{j=1}^{n}(L(u)\phi_j,\phi_j) &= \sum_{j=1}^{n}(\nu\Delta\phi_j,\phi_j) - ((u\cdot\nabla)\phi_j,\phi_j) - ((\phi_j\cdot\nabla)u,\phi_j)\\ &= \nu\sum_j(\Delta\phi_j,\phi_j) - \sum_j((\phi_j\cdot\nabla)u,\phi_j)\\ &\le \nu\sum_j(\Delta\phi_j,\phi_j) + c\sum_{j=1}^{n}\|\phi_j\|\,\|D\phi_j\|\,\|Du\|\\ &\le \nu\sum_j(\Delta\phi_j,\phi_j) + \frac{\nu}{2}\sum_j\|D\phi_j\|^2 + \frac{c^2}{2\nu}\|Du\|^2\\ &= \frac{\nu}{2}\sum_j(\Delta\phi_j,\phi_j) + \sum_j\frac{c^2}{2\nu}\|Du\|^2\\ &\le -c\nu\lambda_1 n^2 + \frac{cn}{\nu}\|Du\|^2.\end{aligned}$$

Taking the time average gives

$$\tilde{q}_n \le -c\nu\lambda_1 n^2 + \frac{cn}{\nu}\langle\|Du\|^2\rangle.$$

This is a concave function of n, so we have

$$d_{\mathrm{box}}(\mathscr{A}) \le c\frac{\langle\|Du\|^2\rangle}{\nu^2\lambda_1}. \tag{5.5}$$

We can easily estimate the time average of $\|Du\|^2$. Taking the inner product of the original equation we obtain (we've done this already)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Du\|^2 = (f,u) \le \|f\|\|u\| \le \lambda_1^{-1/2}\|f\|\|Du\|$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|Du\|^2 \le \frac{\|f\|^2}{\nu\lambda_1}.$$

Integrating from 0 to t and dividing by t gives

$$\nu\frac{1}{t}\int_0^t \|Du(s)\|^2\,\mathrm{d}s \le \frac{1}{t}(\|u(t)\|^2 - \|u(0)\|^2) + \|f\|^2\nu\lambda_1,$$

and so

$$\langle\|Du\|^2\rangle \le \frac{\|f\|^2}{\nu^2\lambda_1}. \tag{5.6}$$

Using this in (5.5) gives

$$d_{\mathrm{box}}(\mathscr{A}) \le c \left(\frac{\|f\|}{\nu^2 \lambda_1} \right)^2 .$$

□

One can do better than this and obtain a bound linear in G if one uses the following Lieb-Thirring inequality (see Temam, 1988):

Theorem 5.10 *Let $\{\phi_j\}_{j=1}^n$ be a set of functions in H^1 that are orthonormal in L^2, and set*

$$\rho(x) = \sum_{j=1}^n |\phi_j(x)|^2 .$$

Then there exists a constant c independent of n such that

$$\left(\int_\Omega \rho(x)^2 \, \mathrm{d}x \right) = \|\rho\|_{L^2}^2 \le c \sum_{j=1}^n \|D\phi_j\|^2 .$$

To use this, note that

$$\sum_{j=1}^n ([\phi_j \cdot \nabla] u, \phi_j) = \int_Q \underbrace{\sum_{j=1}^n \sum_{i,k=1}^2 \phi_{ji}(x) \partial_i u_k(x) \phi_{jk}(x)}_{I(x)} \, \mathrm{d}x.$$

The integrand can be bounded pointwise by

$$\begin{aligned} |I(x)| &\le \sum_{j=1}^n \left[\left(\sum_{i,k=1}^2 |\partial_i u_k(x)|^2 \right)^{1/2} \left(\sum_{i,k=1}^2 |\phi_{ji}(x)\phi_{jk}(x)|^2 \right)^{1/2} \right] \\ &\le |Du(x)| \sum_{j=1}^n \left(\sum_{i,k=1}^2 |\phi_{ji}(x)\phi_{jk}(x)|^2 \right)^{1/2} \\ &= |Du(x)| \sum_{j=1}^n \sum_{i=1}^2 |\phi_{ji}(x)|^2 \\ &= |Du(x)| \sum_{j=1}^n |\phi_j(x)|^2, \end{aligned}$$

where the first equality follows from the identity

$$\sum_{i,k=1}^{2} |a_i a_k|^2 = \left(\sum_i |a_i|^2 \right)^2 .$$

So if we return to the second line on page 40, setting $\rho(x) = \sum_{j=1}^n |\phi_j(x)|^2$ as in the statement of Theorem 5.10, then

$$\begin{aligned}
\sum_{j=1}^{n} (L(u)\phi_j, \phi_j) &= \nu \sum_j (\Delta\phi_j, \phi_j) - \sum_j ((\phi_j \cdot \nabla)u, \phi_j) \\
&\leq \nu \sum_j (\Delta\phi_j, \phi_j) + \int_Q |Du(x)| \rho(x) \, \mathrm{d}x \\
&\leq \nu \sum_j (\Delta\phi_j, \phi_j) + \|Du\| \|\rho\|_{L^2} \\
&\leq \nu \sum_j (\Delta\phi_j, \phi_j) + \|Du\| \left(c \sum_{j=1}^{n} \|D\phi_j\|^2 \right)^{1/2} \\
&\leq \nu \sum_j (\Delta\phi_j, \phi_j) + \frac{\nu}{2} \left(\sum_{j=1}^{n} \|D\phi_j\|^2 \right) + \frac{c}{2\nu} \|Du\|^2 \\
&\leq \frac{\nu}{2} \sum_j (\Delta\phi_j, \phi_j) + \frac{c}{2\nu} \|Du\|^2 \\
&\leq -\frac{\nu}{2} \lambda_1 n^2 + \frac{c}{2\nu} \|Du\|^2.
\end{aligned}$$

Taking the time average we obtain, using (5.6),

$$\langle \mathrm{Tr}_n L(u) \rangle \leq -\frac{\nu}{2} \lambda_1 n^2 + \frac{c}{2\nu} \frac{\|f\|^2}{\nu^2 \lambda_1}$$

and hence $\langle \mathrm{Tr}_n L(u) \rangle$ is negative provided that $n > cG$.

6

Embedding finite-dimensional sets into $\mathbb{R}^k$

We will now show that many linear maps from H into $\mathbb{R}^k$ are injective on X provided that $k > 2d_{\text{box}}(X)$.

6.1 Auxiliary construction and estimates

For $\phi \in H$, let ϕ^* denote the element of H^* given by

$$\phi^*(u) = (\phi, u) \qquad \text{for all} \qquad u \in H.$$

Then $\|\phi^*\| = \|\phi\|$.

Now let V_j be a sequence of linear subspaces of H with $d_j := \dim(V_j) \to \infty$. We construct a set E_k of linear maps from H into $\mathbb{R}^k$ as follows:

$$E_k = \left\{ L = (L_1, \ldots, L_k) : \; L_n = \sum_{j=1}^{\infty} j^{-2} \phi_{nj}^*, \text{ with } \phi_{nj} \in S_j \right\},$$

where S_j is the unit ball in V_j. We define a measure μ on E_k to be what results from choosing each ϕ_{nj} at random from a uniform distribution on S_j; clearly $\mu(E_k) = 1$.

We want to show that for any $x \in H$,

$$\mu\{L \in E_k : \; |Lx| < \epsilon\} \leq c(\epsilon j^2 d_j^{1/2} \|P_j x\|^{-1})^k,$$

where P_j is the orthogonal projection onto V_j and c is a constant independent of x and j.

Lemma 6.1 *For any $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^m$,*

$$\lambda_m\{\phi \in S_m : \ |\alpha + (\phi \cdot x)| < \epsilon\} \leq \frac{\Omega_{m-1}}{\Omega_m} 2\epsilon |x|^{-1} = A\epsilon m^{1/2} |x|^{-1}.$$

Proof We have

$$\begin{aligned}\lambda_m\{\phi \in S_m : \ |\alpha + (\phi \cdot x)| < \epsilon\} &\leq \lambda_m\{\phi \in S_m : \ |\phi \cdot \hat{x}| < \epsilon |x|^{-1}\} \\ &\leq \frac{\Omega_{m-1}}{\Omega_m} 2\epsilon |x|^{-1},\end{aligned}$$

where $\Omega_j = \pi^{j/2}\Gamma((j/2)+1)$. □

Lemma 6.2 *If $x \in H$ then for every j,*

$$\mu\{L \in E_k : \ |Lx| < \epsilon\} \leq c(\epsilon j^2 d_j^{1/2} \|P_j x\|^{-1})^k, \tag{6.1}$$

where c is independent of j and x.

Proof Write $M(x,\epsilon)$ for the left-hand side of (6.1). First, note that

$$M(x,\epsilon) \leq \mu\{L \in E_k : \ |L_n x| < \epsilon, \ n = 1, \ldots, k\}.$$

Given $j \in \mathbb{N}$, fix ϕ_{ni} for all $i \neq j$. Consider

$$|L_m x| = |\alpha_m + j^{-2}(\phi_{mj}, x)|,$$

where

$$\alpha_m = \sum_{i \neq j} i^{-2}(\phi_{mi}, x).$$

It follows from Lemma 6.1 that for every choice of ϕ_{ni}, $i \neq j$,

$$\lambda_j(\{\phi_{mj} \in S_j : \ |L_m x| < \epsilon\}) \leq c\epsilon j^2 d_j^{1/2} \|P_j x\|^{-1},$$

from which (6.1) follows. □

6.2 Thickness exponent

Let $d(X,\epsilon)$ denote the smallest dimension of a linear subspace V such that every point of X lies within ϵ of V. Define the thickness exponent of X, $\tau(X)$, by

$$\tau(X) = \limsup_{\epsilon \to 0} \frac{\log d(X,\epsilon)}{-\log \epsilon}.$$

Note that $\tau(X) \le d_{\rm box}(X)$: clearly the linear space V spanned by the centres of $N(X,\epsilon)$ ϵ-balls that cover X has $\mathrm{dist}(X,V) \le \epsilon$.

6.3 Hunt & Kaloshin's embedding theorem

We can now prove the following result, due to Hunt & Kalsohin (1999):

Theorem 6.3 *Let X be a compact subset of a Hilbert space H, with $d_{\rm box}(X) < \infty$. Take $k > 2d_{\rm box}(X)$, and*

$$0 < \alpha < \frac{k-2d}{k(1+(\tau/2))}. \tag{6.2}$$

Then μ-almost every $L \in E_k$ (many linear maps $L : H \to \mathbb{R}^k$) is one-to-one between X and its image, and

$$|L^{-1}x - L^{-1}y| \le C|x-y|^{\alpha} \qquad \text{for all} \qquad x,y \in LX$$

for some $C > 0$.

Proof Let $d_j = d(X, 2^{-j\alpha}/3)$, and V_j a linear subspace such that $\mathrm{dist}(X,V_j) \le 2^{-j\alpha/3}$. Note that for any $\sigma > \tau$, $d_j \le C2^{j\alpha\sigma}$.

Now let E_k be the collection of linear maps from H into $\mathbb{R}^k$ define above, based on the subspaces V_j.

Define

$$Z_j = \{(x,y) \in X \times X : \ |x-y| \ge 2^{-\alpha j}\}$$

and

$$Q_j = \{L \in E_k : \ |Lx - Ly| \le 2^{-j} \text{ for some } (x,y) \in Z_j\}.$$

Cover Z_j with balls of radius 2^{-j}: it takes at most $C2^{2j\delta}$ for any $\delta > d_{\rm box}(X)$. Let Y be the intersection of Z_j with one these balls.

Then if $(x_0,y_0),(x,y) \in Y$, since $|(x,y)-(x_0,y_0)| \le 2^{-(j-1)}$ it follows that

$$|L(x_0-y_0)| \ge (2L+1)2^{-j} \quad \Rightarrow \quad |L(x-y)| \ge 2^{-j} \quad \text{for all} \quad (x,y) \in Y.$$

So

$$\begin{aligned}\mu\{L \in E_k : |L(x-y)| \le 2^{-j} \text{ for some } (x,y) \in Y\} \\ \le \mu\{L \in E_k : \ |L(x_0-y_0)| < (2L+1)2^{-j}\} \\ \le c[(2L+1)2^{-j} j^2 C 2^{j\alpha\sigma/2} 2^{\alpha j}]^k,\end{aligned}$$

from which it follows that

$$\begin{aligned}\mu(Q_j) &\leq C2^{2j\delta}c[(2L+1)2^{-j}j^2C2^{j\alpha\sigma/2}2^{\alpha j}]^k \\ &= Cc[(2L+1)C]^k j^{2k}2^{j[2\delta-k(1-\alpha(1+\sigma/2))]} \\ &= Mj^{2k}2^{j[2\delta-k(1-\alpha(1+\sigma/2))]}.\end{aligned}$$

It follows that if

$$2\delta - k(1-\alpha(1+\sigma/2)) < 0 \tag{6.3}$$

then

$$\sum_{j=1}^{\infty}\mu(Q_j) < \infty. \tag{6.4}$$

Now, if $k > 2d_{\mathrm{box}}(X)$, (6.3) can be satisfied for appropriate $\delta > d_{\mathrm{box}}(X)$ and $\sigma > \tau(X)$ provided that (6.2) holds.

In this case, the Borel-Cantelli lemma[1] implies that μ-almost every $L \in E_k$ can be in only a finite number of the Q_js. Thus, for μ-almost every $L \in E_k$, there exists a j_0 such that

$$|x-y| \geq 2^{-\alpha j} \qquad \Rightarrow \qquad |Lx-Ly| \geq 2^{-j} \qquad \text{for all} \qquad j \geq j_0.$$

It follows from this that L is injective on X, with α-Hölder continuous inverse as claimed. □

The same result holds for finite-dimensional subspaces of a Banach space $\mathscr{B}$ (Robinson, 2007), provided (6.2) is replaced by

$$0 < \alpha < \frac{k-2d}{k(1+\tau^*)}, \tag{6.5}$$

where $\tau^*(X)$ is the *dual thickness* of X, defined as follows:

For a given $\theta > 0$, let $d_\theta(X,\epsilon)$ denote the smallest dimension of a linear subspace V of $\mathscr{B}^*$ such that for every $x, y \in X$ with $\|x-y\| \geq \epsilon$, there exists a $\psi \in V$ such that $\|\psi\| = 1$ and

$$|\psi(x-y)| \geq \epsilon^{1+\theta}.$$

[1] Consider

$$\mathscr{Q} = \cap_{n=1}^{\infty} \cup_{j=n}^{\infty} Q_j.$$

Then $\mathscr{Q}$ consists precisely of those $L \in E_k$ for which $L \in Q_j$ for infinitely many values of j. Now, for any n we must have $\mu(\mathscr{Q}) \leq \mu(\cup_{j=n}^{\infty} Q_j) \leq \sum_{j=n}^{\infty}\mu(Q_j)$. Since $\sum_{j=1}^{\infty}\mu(Q_j) < \infty$, it follows that $\sum_{j=n}^{\infty}\mu(Q_j) \to 0$ as $n \to \infty$, and hence $\mu(\mathscr{Q}) = 0$.

Set

$$\tau_\theta^*(X) = \limsup_{\epsilon \to 0} \frac{\log d_\theta(X,\epsilon)}{-\log \epsilon},$$

and finally let

$$\tau^*(X) = \limsup_{\theta \to 0} \tau_\theta^*(X).$$

While in general there is no relation between the thickness and the dual thickness, one can show that $\tau(X) = 0$ implies that $\tau^*(X) = 0$.

Having a bound on the inverse of the linear maps L in Theorem 6.3 allows one to obtain a lower bound on the dimension of LX. This is a consequence of the following simple lemma:

Lemma 6.4 *Let $X \subset \mathscr{B}_1$, and let $f : \mathscr{B}_1 \to \mathscr{B}_2$ be a θ-Hölder map,*

$$\|f(x_1) - f(x_2)\|_{\mathscr{B}_2} \le c\|x_1 - x_2\|_{\mathscr{B}_1}^\theta$$

for some $0 < \theta \le 1$. Then $d_{\rm box}(f(X)) \le d_{\rm box}(X)/\theta$.

Proof A cover of X by $N(X,\epsilon)$ balls of radius ϵ leads to a cover of $f(X)$ by $N(X,\epsilon)$ balls of radius $C\epsilon^\theta$. So putting $\eta = C\epsilon^\theta$

$$\frac{N_{\mathscr{B}^2}(f(X),\eta)}{-\log \eta} \le \frac{N_{\mathscr{B}^1}(X,\epsilon)}{-\log C - \theta \log \epsilon},$$

and the result follows. □

7

Asymptotically determining nodes

As a prelude to the embedding result of the next chapter, we prove the following simple result due to Foias & Temam (1984):

Theorem 7.1 *For the 2d Navier-Stokes equations*

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \qquad \nabla \cdot u = 0$$

there exists a $\delta_0 = \delta_0(u, \|f\|)$*, such that if* $\{x_1, \ldots, x_k\}$ *are a collection of points in* Q *such that for any* $x \in Q$ *there exists a* j *such that* $|x - x_j| < \delta_0$*, then for any two solutions lying on the attractor,*

$$\max_j |u(x_j, t) - v(x_j, t)| \to 0 \qquad as \qquad t \to \infty$$

implies that

$$\|u(t) - v(t)\|_\infty \to 0 \qquad as \qquad t \to \infty.$$

In fact the solutions need not lie on the attractor for this to hold. We will use the fact that they do below to bound $\|Du\|$ and $\|\Delta u\|$ uniformly. But one can be more careful using integral estimates and obtain the same result for arbitrary solutions.

Proof First observe that $H^2(Q) \subset C^{0,1/2}$ (and in fact $C^{0,\alpha}$ for any $\alpha < 1$), so that

$$|u(x) - u(y)| \le c\|u\|_{H^2}|x - y|^{1/2}.$$

It follows that if $u \in H^2$ then

$$\|u\|_\infty \le \max_{x_j} |u(x_j| + c\delta_0^{1/2}\|\Delta u\|. \tag{7.1}$$

Given two solutions $u(t)$, $v(t)$, with $u_0, v_0 \in \mathscr{A}$, write

$$\eta(t) = \max_{x_j} |u(x_j,t) - v(x_j,t)|.$$

The assumption of the theorem is then that $\eta(t) \to 0$ as $t \to \infty$.

We will in fact prove that $\|Du(t) - Dv(t)\| \to 0$ as $t \to \infty$. Coupled with the bound on solutions in H^2 this leads, via an interpolation result for Sobolev spaces and the Sobolev embedding theorem, to the convergence of two solutions in L^∞.

Consider the difference of the two solutions, $w(t) = u(t) - v(t)$. This satisfies the equation

$$w_t - \nu\Delta w + (u\cdot\nabla)w + (w\cdot\nabla)u - (w\cdot\nabla)w + \nabla q = 0.$$

Taking the inner product with Δw and using the orthogonality property $((w\cdot\nabla)w, \Delta w) = 0$ leaves two nonlinear terms,

$$((u\cdot\nabla)w, \Delta w) + ((w\cdot\nabla)u, \Delta w).$$

Now, if one considers

$$([(w+\epsilon u)\cdot\nabla](w+\epsilon u), \Delta(w+\epsilon u)) = 0$$

it follows that

$$((u\cdot\nabla)w, \Delta w + ((w\cdot\nabla)u, \Delta w) + ((w\cdot\nabla)w, \Delta u) = 0.$$

So the equation for $\|Dw\|$ can be written

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|^2 + \nu\|\Delta w\|^2 = ((w\cdot\nabla)w, \Delta u).$$

One can bound the right-hand side by $c\|w\|_\infty\|Dw\|\|\Delta u\|$, and so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|^2 + \nu\|\Delta w\|^2 \le c\|w\|_\infty\|Dw\|\|\Delta u\|.$$

Now, we can use (7.1) to write

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|^2 + \nu\|\Delta w\|^2 &\le c\left\{\eta(t) + c\delta_0^{1/2}\|\Delta w\|\right\}\|Dw\|\|\Delta u\| \\ &\le c\eta(t)\|Dw\|\|\Delta u\| + c\delta_0^{1/2}\lambda_1^{-1/2}\|\Delta w\|^2\|\Delta u\| \\ &\le c\eta(t)\rho_1\rho_2 + c\delta_0^{1/2}\lambda_1^{-1/2}\|\Delta w\|^2\rho_2,\end{aligned}$$

using the fact that $\mathscr{A}$ is bounded in H^1 ($\|Du\| \le \rho_1$) and in H^2 ($\|\Delta u\| \le \rho_2$). Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|^2 + \left\{\nu - c\delta_0^{1/2}\lambda_1^{-1/2}\rho_2\right\}\|\Delta w\|^2 \le c\rho_1\rho_2\eta(t),$$

our after using the Poincaré inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Dw\|^2 + \left\{\nu - c\delta_0^{1/2}\lambda_1^{-1/2}\rho_2\right\}\lambda_1\|Dw\|^2 \leq c\rho_1\rho_2\eta(t).$$

Now, if δ_0 is sufficiently small that

$$\nu - c\delta_0^{1/2}\lambda_1^{-1/2}\rho_2 > 0$$

then we have an equation for $X(t) = \|Dw(t)\|^2$ of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} + aX \leq b(t),$$

where $a > 0$ and $b(t) \to 0$ as $t \to \infty$. We also know that $X(t) \leq 2\rho_1^2$, so $X(t)$ is bounded. It follows that $X(t) \to 0$ as $t \to \infty$: given $\epsilon > 0$, choose T such that $b(t) < \epsilon/2$ for all $t \geq T$, and then

$$\dot{X} + aX \leq (\epsilon/2) \qquad \Rightarrow \qquad X(T+s) \leq X(T)\mathrm{e}^{-at} + (\epsilon/2) \leq 2\rho_1^2\mathrm{e}^{-at} + (\epsilon/2)$$

so that for $t \geq T^*$ we have $X(t) \leq \epsilon$.

It follows that $\|Dw(t)\|^2 \to 0$ as $t \to \infty$. The L^∞ bound follows from the interpolation inequality

$$\|u\|_{H^{3/2}} \leq \|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2}$$

and the Sobolev embedding $H^{3/2} \subset L^\infty$. □

8

Gevrey regularity

One property of solutions lying on the attractor is that they are more regular than arbitrary solutions. In particular, if the forcing function f is real analytic then the functions lying in the attractor are real analytic, in a uniform way. Let $A = -\Delta$. Then we can interpret $A^{1/2}$ via the effect of A on the Fourier expansion of a function u: if

$$u = \sum_{j \in \mathbb{Z}^2} u_j \mathrm{e}^{\mathrm{i} \cdot jx}$$

then

$$Au = -\Delta u = \sum_{j \in \mathbb{Z}^2} u_j |j|^2 \mathrm{e}^{\mathrm{i} \cdot jx},$$

and $A^{1/2}$ is given by

$$A^{1/2} u = \sum_{j \in \mathbb{Z}^2} u_j |j| \mathrm{e}^{\mathrm{i} \cdot jx}.$$

A function $f(x)$ is real analytic [it can be represented locally by its Taylor series expansion] if and only if its derivatives satisfy

$$|D^\beta f| \le M |\beta|! \tau^{-|\beta|}$$

for some M and τ. This motivates the definition of the analytic Gevrey class $D(\mathrm{e}^{\tau A^{1/2}})$: this consists of functions such that

$$|\mathrm{e}^{\tau A^{1/2}} u| < +\infty,$$

where $e^{\tau^{1/2}A}$ is defined using the power series for exponentials,

$$e^{\tau A^{1/2}} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} A^{n/2}.$$

For more details see Friz & Robinson (2001).

If u has Fourier expansion as above, then

$$|e^{\tau A^{1/2}} u|^2 = \sum_{j \in \mathbb{Z}^2} e^{2\tau|j|} |u_j|^2.$$

In particular, therefore, if $u \in D(e^{\tau A^{1/2}})$ the Fourier coefficients of u must decay exponentially fast.

Foias & Temam (1989) showed that if $f \in D(e^{\sigma A^{1/2}})$ for some $\sigma > 0$ then $u(t)$ is bounded in $D(A^{1/2} e^{\tau A^{1/2}})$,

$$|A^{1/2} e^{\tau A^{1/2}} u| \leq K \qquad \text{for all} \qquad t \geq T,$$

T and K depend only on $|Du(0)|$.

We give the proof here, following Foias & Temam's paper closely. We assume the following result (which is lemma 2.1 in Foias & Temam (1989)):

Lemma 8.1 *If u, v, and $w \in D(Ae^{\tau A^{1/2}})$ for some $\tau > 0$ then $B(u,v) \in D(e^{\tau A^{1/2}})$ and*

$$|(e^{\tau A^{1/2}} B(u,v), e^{\tau A^{1/2}} Aw)| \\ \leq c|e^{\tau A^{1/2}} A^{1/2} u|^{1/2} |e^{\tau A^{1/2}} Au|^{1/2} |e^{\tau A^{1/2}} A^{1/2} v| |e^{\tau A^{1/2}} Aw|,$$

for some $c > 0$.

In order to make the notation more compact, we can write

$$(u,v)_\tau = (e^{\tau A^{1/2}} u, e^{\tau A^{1/2}} v)$$

and

$$((u,v))_\tau = (A^{1/2} e^{\tau A^{1/2}} u, A^{1/2} e^{\tau A^{1/2}} v)).$$

The result of lemma 8.1 is now

$$|(B(u,v), Aw)_\tau| \leq c|A^{1/2} u|_\tau^{1/2} |Au|_\tau^{1/2} |A^{1/2} v|_\tau |Aw|_\tau. \tag{8.1}$$

We now show:

Theorem 8.2 *If $f \in D(\mathrm{e}^{\sigma A^{1/2}})$ then for $t \le T(|f|_\sigma + |A^{1/2}u(0)|)$ we have*

$$|A^{1/2}\mathrm{e}^{\phi(t)A^{1/2}}u(t)| \le K(|f|_\sigma, |A^{1/2}u(0)|) \qquad \textit{for all} \qquad 0 \le t \le T,$$

where $\phi(t) = \min(\sigma, t)$.

Proof Taking the scalar product with Au (or u) in $D(\mathrm{e}^{\tau A^{1/2}})$ leads to an equation for $y = |A^{1/2}u|_\tau$ like $\dot{y} \le Ky^3$. Not only do the solutions of this equation blow up in a finite time, but also we need to control $|A^{1/2}u(0)|_\tau$ in order to control $|A^{1/2}u(t)|_\tau$: we would need to start with analyticity in order to prove it.

The trick to get round this is to define $\phi(t) = \min(t, \sigma)$, and take the scalar product of

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = f$$

with $\mathrm{e}^{2\phi(t)A^{1/2}}Au$ to obtain

$$\begin{aligned}
&\left(\frac{\mathrm{d}u}{\mathrm{d}t}, \mathrm{e}^{2\phi(t)A^{1/2}}Au\right) + \nu|\mathrm{e}^{\phi(t)A^{1/2}}Au|^2 \\
&\quad = (\mathrm{e}^{\phi(t)A^{1/2}}f, \mathrm{e}^{\phi(t)A^{1/2}}Au) - (\mathrm{e}^{\phi(t)A^{1/2}}B(u,u), \mathrm{e}^{\phi(t)A^{1/2}}Au) \\
&\quad = (f, Au)_\phi - (B(u,u), Au)_\phi \\
&\quad \le |f|_\phi|Au|_\phi + c|A^{1/2}u|_\phi^{3/2}|Au|_\phi^{3/2} \\
&\quad \le \frac{\nu}{4}|Au|_\phi^2 + c|A^{1/2}u|_\phi^{3/2}|Au|_\phi^{3/2} \\
&\quad \le \frac{\nu}{4}|Au|_\phi^2 + \frac{2}{\nu}|f|_\phi^2 + \frac{c}{\nu^3}|A^{1/2}u|_\phi^6.
\end{aligned}$$

The left hand side of the equation we can bound as

$$\begin{aligned}
&\left(\mathrm{e}^{\phi(t)A^{1/2}}\frac{\mathrm{d}u}{\mathrm{d}t}, \mathrm{e}^{\phi(t)A^{1/2}}Au\right) \\
&\quad = \left(A^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\phi(t)A^{1/2}}u(t)) - \frac{\mathrm{d}\phi}{\mathrm{d}t}A\mathrm{e}^{\phi(t)A^{1/2}}u(t), \mathrm{e}^{\phi(t)A^{1/2}}A^{1/2}u(t)\right) \\
&= \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^{1/2}\mathrm{e}^{\phi(t)A^{1/2}}u(t)|^2 - \frac{\mathrm{d}\phi}{\mathrm{d}t}(A\mathrm{e}^{\phi(t)A^{1/2}}u, \mathrm{e}^{\phi(t)A^{1/2}}A^{1/2}u) \\
&= \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\phi(t)}^2 - \frac{\mathrm{d}\phi}{\mathrm{d}t}(Au, A^{1/2}u)_{\phi(t)} \\
&\ge \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\phi(t)}^2 - |Au|_{\phi(t)}\|u\|_{\phi(t)} \\
&\ge \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\phi(t)}^2 - \frac{\nu}{4}|Au|_{\phi(t)}^2 - \frac{1}{\nu}\|u\|_{\phi(t)}^2.
\end{aligned}$$

We therefore have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|A^{1/2}u|_\phi + \nu|Au|_\phi^2 &\leq \frac{4}{\nu}|f|_\phi^2 + \frac{2}{\nu}|A^{1/2}u|_\phi^2 + \frac{c}{\nu^3}|A^{1/2}u|_\phi^6 \\ &\leq \frac{4}{\nu}|f|_\phi^2 + c + \frac{c}{\nu^3}|A^{1/2}u|_\phi^6.\end{aligned}$$

Now we can set

$$y(t) = 1 + |A^{1/2}u(t)|_{\phi(t)}^2,$$

and we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq Ky^3 \tag{8.2}$$

with

$$K = \frac{4}{\nu}|f|_\sigma^2 + c + \frac{c}{\nu}.$$

The solution of (8.2) is

$$y(t) \leq \frac{1}{y(0)^{-2} - 2Kt},$$

and so $y(t) \leq 2y(0)$ for $t \leq (4Ky(0)^2)^{-1}$. Since $\phi(0) = 0$, we have

$$y(0) = 1 + |A^{1/2}u(0)|^2,$$

and so for $t \leq T(|A^{1/2}u(0)|, |f|_\sigma)$, we have

$$|A^{1/2}u(t)| \leq K(|A^{1/2}u(0)|, |f|_\sigma).$$

□

It now follows that solutions on the attractor are uniformly bounded in $D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$:

Corollary 8.3 *The global attractor for the 2d Navier-Stokes evolution equation with periodic boundary conditions is uniformly bounded in* $D(A^{1/2}\mathrm{e}^{\tau A^{1/2}})$, *i.e. there exists a constant* K *such that if* $u \in \mathscr{A}$ *then*

$$|A^{1/2}\mathrm{e}^{\tau A^{1/2}}u| \leq K. \tag{8.3}$$

In particular the attractor consists of real analytic functions.

Proof The attractor is bounded in $H^1(Q)$, with $|A^{1/2}u| \leq M$ for all $u \in \mathscr{A}$. Set $T = T(M, |f|_\sigma)$. Since the attractor is invariant, if $u \in \mathscr{A}$ then there exists a $u_0 \in \mathscr{A}$ such that $u = S(T)u_0$. It follows from the above theorem that

$$|A^{1/2}\mathrm{e}^{\phi(T)}S(T)u_0| \leq K(M, |f|_\sigma).$$

Now set $\tau = \phi(T)$ and $K = K(M, |f|_\sigma)$, which gives (8.3). □

9

Instantaneous determining nodes

In this final chapter we will prove the following theorem, due to Kukavica & Robinson (2004). Where details are missing they can be found there.

Theorem 9.1 *Let $\mathscr{A} \subset L^2(Q,\mathbb{R}^d)$ have finite box-counting dimension. Suppose further that for each $r \in \mathbb{N}$, $\mathscr{A}$ is a bounded subset of $C^r(Q,\mathbb{R}^d)$, and that moreover for all $u, v \in \mathscr{A}$ with $u \neq v$, $u-v$ has finite order of vanishing.*

Then for $k \geq 16d_{\rm box}(\mathscr{A}) + 1$, Lebesgue almost every choice of k points in Q, $\mathbf{x} = (x_1, \ldots, x_k)$, makes the map $E_{\mathbf{x}} : \mathscr{A} \to \mathbb{R}^{dk}$ given by

$$u \mapsto (u(x_1), \ldots, u(x_k))$$

one-to-one between $\mathscr{A}$ and its image, i.e. if $u, v \in \mathscr{A}$ and $u(x_j) = v(x_j)$ for all $j = 1, \ldots, k$, then $u = v$.

We will have to assemble a number of preliminary results, that will all be combined in the final proof.

9.1 Order of vanishing

If $u \in C^\infty(\Omega,\mathbb{R}^d)$ (where Ω is an open subset in $\mathbb{R}^m$), the order of vanishing of u at $x \in \Omega$ is the smallest integer k such that $D^\alpha u(x) \neq 0$ for some multi-index α with $|\alpha| = k$. A function u is said to have finite order of vanishing in Ω if the order of vanishing of u at x is finite for every $x \in \Omega$.

Note that the order of vanishing of a function is uniformly bounded for all $x \in K$, whenever K is a compact subset of Ω. If not, then there would exist $x_j \in K$ such that the order of vanishing of u at x_j is at least j. But extracting a subsequence such that $x_{j_k} \to x^*$ produces a point x^* at which the order of vanishing of u is not finite, a contradiction.

Note also that real analytic functions have finite order of vanishing.

9.2 Zeros of functions with finite order of vanishing

Analytic functions cannot have 'too many' zeros, and the same is true of functions with finite order of vanishing. The following lemma will be sufficient for our purposes. The proof uses a tailored version of the implicit function theorem.

Lemma 9.2 *Let K be a compact connected subset of $\mathbb{R}^m$. Suppose that for all $p \in \Pi \subset \mathbb{R}^N$ the function $w = w(x;p)$, where $w : K \times \Pi \to \mathbb{R}^d$ has order of vanishing at most $M < \infty$, and that $\partial^\alpha w(x;p)$ depends on p in a θ-Hölder way for all $|\alpha| \leq M$.*

Then the zero set of $w(x;p)$,

$$\{(x,p) \in \mathbb{R}^m \times \mathbb{R}^N : \ w(x;p) = 0\}$$

is contained in a countable union of manifolds of the form

$$(x_i(x',p), x'; p)$$

where $x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ and x_i is a θ-Hölder functions of its arguments.

Note that in the case of the 2d Navier-Stokes equations, where w is a two-component function, this says that the manifolds containing the zeros are of the form

$$(x_1(x_2,p), x_2; p) \qquad \text{or} \qquad (x_1, x_2(x_1,p); p).$$

9.3 Hausdorff dimension

The proof will also use the Hausdorff dimension, since this definition is stable under countable unions (such as the countable union of manifolds that appear in Lemma 9.2).

The d-dimensional Hausdorff measure is defined by

$$\mathscr{H}^d(X) = \liminf_{\epsilon \to 0} \left\{ \sum_i r_i^d : \ X \subset \cup_i U_i, \ \mathrm{diam}(U_i) = r_i \leq \epsilon \right\}.$$

The Hausdorff dimension is then

$$d_H(X) = \inf\{d \geq 0 : \ \mathscr{H}^d(X) = 0\}.$$

The Hausdorff dimension has the following properties

- $d_H(X) \leq d_{\mathrm{box}}(X)$;
- if $f : \mathscr{B}_1 \to \mathscr{B}_2$ is θ-Hölder then $d_H(f(X)) \leq d_H(X)/\theta$, so in particular $d_H(LX) \leq d_H(X)$ if L is a bounded linear map;
- if $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$ is θ-Hölder then

$$d_H(\{(x, f(x))\} \subset \mathbb{R}^{n+m}) \leq n + (1-\theta)m;$$

- stability under countable unions,

$$d_H\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_j d_H(X_j);$$

  and
- if $X \subset \mathbb{R}^n$ and $d_H(X) < n$ then X has (n-dimensional) Lebesgue measure zero.

9.4 Bounds on box-counting and thickness

In order to use the abstract embedding result of Theorem 6.3 we will need the following:

Lemma 9.3 *Let $Q \subset \mathbb{R}^m$, and let $X \subset L^2(Q, \mathbb{R}^d)$ be uniformly bounded in $C^r(Q, \mathbb{R}^d)$ for all $r \in \mathbb{N}$. Then for any $r \in \mathbb{N}$,*

(a) $d_{\mathrm{box}}(X; C^r) \leq d_{\mathrm{box}}(X, L^2)$ and

(b) $\tau^*(X; C^r) = 0$.

Proof We first note that if $\mathscr{B}_1$ and $\mathscr{B}_2$ are two Banach spaces such that $\mathscr{B}_1 \subset \mathscr{B}_2$ and

$$\|u\|_{\mathscr{B}_2} \leq c\|u\|_{\mathscr{B}^1}$$

then if $X \subset \mathscr{B}_1$,

$$d_{\text{box}}(X;\mathscr{B}_2) \leq d_{\text{box}}(X;\mathscr{B}_1) \qquad \text{and} \qquad \tau(X;\mathscr{B}_2) \leq \tau(X;\mathscr{B}_1).$$

Since we will show (b) by proving that $\tau(X;C^r)=0$, it is therefore sufficient – using the Sobolev embedding result

$$\|u\|_{C^r} \leq c\|u\|_{H^{(r/2)+m+1}}$$

– to show that for every $r \in \mathbb{N}$,

(a′) $d_{\text{box}}(X;H^r) \leq d_{\text{box}}(X;L^2)$ and
(b′) $\tau(X;H^r)=0$.

(a′) The Sobolev interpolation result that for $s>r$

$$\|u\|_{H^r} \leq \|u\|_{L^2}^{1-(r/s)}\|u\|_{H^s}^{r/s}$$

implies that the identity map on X from L^2 into H^r is $[1-(r/s)]$-Hölder. Using Lemma 6.4 this implies that $d_{\text{box}}(X;H^r) \leq d_{\text{box}}(X;L^2)/[1-(r/s)]$ for any $s \in \mathbb{N}$, which implies (a′).

(b′) Elliptic regularity implies that

$$\|u\|_{H^{2r}} \leq \|A^r u\|,$$

where $A=-\Delta$. Let $\{w_j\}$ be the eigenfunctions of $-\Delta$ on Q, ordered so that the corresponding eigenvalues satisfy $\lambda_{j+1} \geq \lambda_j$. Let P_n denote the orthogonal projection from L^2 onto the first n eigenvalues of A,

$$P_n u = \sum_{j=1}^{n}(u,w_j)w_j$$

and let $Q_n = I-P_n$ be its orthogonal complement. Then any $u \in X$ can be approximated by an element of P_nH to within an error in H^{2r} bounded by

$$\begin{aligned}\|u-P_nu\|_{H^{2r}} &\leq \|A^rQ_nu\| \\ &= \|A^{-s}Q_nA^{r+s}u\| \\ &\leq \|A^{-s}Q_n\|_{\text{op}}\|A^{r+s}u\| \\ &\leq \lambda_{n+1}^{-s}\|u\|_{H^{2(r+s)}} \\ &\leq \lambda_{n+1}^{-s}K_{2(r+s)},\end{aligned}$$

where K_j is the uniform bound on $\|u\|_{H^j}$ for $u \in X$. Since $\lambda_n \sim n^{2/m}$, it follows that $\tau(X;H^r) \leq m/2s$. But s is arbitrary, and so $\tau(X;H^r)=0$.

□

9.5 Proof of Theorem 9.1

Set $W = \mathscr{A} - \mathscr{A} \setminus \{0\}$, i.e.

$$W = \{a_1 - a_2 : \; a_1, a_2 \in \mathscr{A}, \; a_1 \neq a_2\}.$$

Decompose W as

$$W = \bigcup_{j=0}^{\infty} W_j$$

where $u \in W$ is in W_j if j is the smallest integer such that u has order of vanishing bounded by j throughout Q.

If for each j, $E_{\mathbf{x}}$ is non-zero on W_j for almost every choice of $\mathbf{x}$, then $E_{\mathbf{x}}$ is non-zero on W for almost every choice of $\mathbf{x}$.

So fix j. In follows from Lemma 9.3 that

$$d_{\text{box}}(W_j; C^r) \leq d_{\text{box}}(W_j; L^2) \leq d_{\text{box}}(\mathscr{A} - \mathscr{A}; L^2) \leq 2d_{\text{box}}(\mathscr{A}; L^2)$$

and that

$$\tau^*(W_j; C^r) = 0.$$

So, using the Banach space version of Theorem 6.3, for any $N > 4d_{\text{box}}(\mathscr{A})$ and

$$0 < \theta < 1 - \frac{4d_{\text{box}}(\mathscr{A})}{N}$$

there is a parametrisation $w(x;p)$ of W_j in terms of N coordinates $p \in \Pi \subset \mathbb{R}^N$, such that the map $p \mapsto w(x;p)$ is θ-Hölder into C^r. In other words, w and all its partial derivatives $\partial^\alpha w$ up to order $|\alpha| = j$ depend on p in a θ-Hölder way.

It follows from Lemma 9.2 that the zeros of $w(x;p)$ are contained in a countable collection of manifolds, given in the form

$$(x_i(x', p), x'; p)$$

where $x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ and x_i is a θ-Hölder functions of its arguments.

Collections of k such zeros are given by a collection of such manifolds. The parameter p is common to all these, so one obtains a countable collection of manifolds given as the graphs of θ-Hölder functions from $\mathbb{R}^{N+(m-1)k}$ into $\mathbb{R}^k$. So

$$d_H(k \text{ simultaneous zeros}) \leq N + (m-1)k + k(1-\theta).$$

These are 'bad choices' in $Q^k \times \Pi$. We want to exclude all bad choices in Q^k, so if we project these simultaneous zeros onto Q^k this does not increase the Hausdorff dimension:

$$d_H(\text{bad choices}) \leq N + (m-1)k + k(1-\theta).$$

If we can ensure that

$$N + (m-1)k + k(1-\theta) < \dim(Q^k) = mk$$

then the set of bad choices will have measure zero. This condition reduces to $N < k\theta$. Since θ can be chosen arbitrarily close to $1 - [4d(\mathscr{A})/N]$, it follows that we need

$$k > \frac{N^2}{N - 4d(\mathscr{A})}.$$

Choosing N to minimise this gives $k \geq 16 d_{\text{box}}(\mathscr{A}) + 1$.

Appendix: Sobolev spaces

Sobolev spaces are collections of spaces of functions whose (weak) derivatives are square integrable. Although spaces of continuous functions would seem to be more natural in the context of PDEs, Sobolev spaces are often provide a much more mathematically convenient setting in which to work.

On a smooth bounded domain $\Omega \subset \mathbb{R}^d$, $H^s(\Omega)$ consists of functions all of whose derivatives of order up to s are square integrable,

$$H^s(\Omega) = \{f \in L^2(\Omega) : \ D^\alpha f \in L^2(\Omega), \ |\alpha| \leq s\}.$$

The standard norm on H^s is

$$\|f\|^2_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \|D^\alpha f\|^2_{L^2(\Omega)}. \tag{9.1}$$

Here α is a multi-index ($\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$).

We will concentrate here on Sobolev spaces of periodic functions, for which the analysis can be done in terms of Fourier series. All the results here hold for Sobolev spaces of functions on bounded domains.

In this context, let $C^\infty_{\mathrm{p}}(Q)$ denote the collection of C^∞ functions that are periodic with period Q. Then $H^s_{\mathrm{p}}(Q)$ is the completion of $C^\infty_{\mathrm{p}}(Q)$ with respect to the norm (9.1), replacing Ω by Q throughout.

It is relatively easy to show that this definition is equivalent to the following characterisation in terms of Fourier series:

$$H^s_{\rm p}(Q) = \{u:\ u = \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i k\cdot x/L},\ \overline{c_k} = c_{-k},\ \sum_{k\in\mathbb{Z}^d} |k|^{2s}|c_k|^2 < \infty\}, \quad (9.2)$$

the main observation yielding this being that the $H^s_{\rm p}$ is equivalent to

$$\left(\sum_{k\in\mathbb{Z}^d} (1+|k|^{2s})|c_k|^2\right)^{1/2}$$

when u is given as a Fourier series as in (9.2).

Some standard results on Sobolev spaces that are not straightforward to prove in the bounded case are much simpler to prove in the periodic setting. For example, density of $C^\infty_{\rm p}(Q)$ in $H^s_{\rm p}(Q)$ is immediate from the definition – density of $C^\infty(\bar\Omega)$ in $H^s(\Omega)$ requires some non-trivial work (and requires sufficient smoothness of $\partial\Omega$).

The following results are all true in the case of bounded domains (remove the $\cdot_{\rm p}$ and replace Q by Ω) provided Ω is sufficiently smooth.

Theorem 9.4 *If $u \in H^s_{\rm p}(Q)$ with $s > d/2$ then $u \in C^0(\overline{Q})$ and*

$$\|u\|_\infty \le C_s \|u\|_{H^s}.$$

Proof For simplicity we take $L = 2\pi$. The general case follows with extra factors of L and 2π throughout as appropriate. All sums are over $\mathbb{Z}^d$.

If $u = \sum_k c_k e^{ik\cdot x}$ then if $\sum |c_k|$ is finite it follows that the Fourier series converges uniformly. Since each partial sum is continuous, from this it would follow that u itself is continuous, as clearly $\|u\|_\infty \le \sum_k |c_k|$. So we show that $\sum_k |c_k| \le C_s\|u\|_{H^s}$ to prove the result.

We have

$$\begin{aligned}\sum_k |c_k| &\le \sum_k (1+|k|^{2s})^{1/2}|c_k| \frac{1}{(1+|k|^{2s})^{1/2}} \\ &\le \left(\sum_k (1+|k|^{2s})|c_k|^2\right)^{1/2} \left(\sum_k \frac{1}{1+|k|^{2s}}\right)^{1/2}.\end{aligned}$$

Provided that $s > d/2$ the second sum is finite ($= C_s$, say), in which case

$$\sum_k |c_k| \le C_s \|u\|_{H^s},$$

and the result follows. □

The following corollary is straightforward:

Corollary 9.5 *If* $u \in H^s_{\rm p}(Q)$ *with* $s > d/2 + j$ *then* $u \in C^j(\overline{Q})$ *and*

$$\|u\|_{C^j} \le C_s \|u\|_{H^s}.$$

In particular, if $u \in H^s_{\rm p}(Q)$ *for all* $s \in \mathbb{N}$ *then* $u \in C^\infty(\overline{Q})$.

After the above result, it is natural to ask what happens if $u \in H^s_{\rm p}(Q)$ with $s < d/2$ (and in particular if $s = d/2$). The following result shows in particular that if $u \in H^{d/2}_{\rm p}(Q)$ then $u \in L^p(Q)$ for all $p \in [2, \infty)$ (but not necessarily ∞). The key is the following observation (a little weak for a result that requires some significant mathematical sophistication to prove): if $\underline{c} \in \ell^p$ and

$$u = \sum_k c_k \mathrm{e}^{\mathrm{i}k\cdot x}$$

then $u \in L^q(Q)$ (where p and q are conjugate) and

$$\|u\|_{L^q} \le \alpha_p \|\underline{c}\|_{\ell^p}.$$

Theorem 9.6 *If* $u \in H^s_{\rm p}(Q)$ *with* $s < d/2$ *then* $u \in L^p(Q)$ *for all*

$$p \in \left[2, \frac{d}{(d/2) - s}\right).$$

If $s = d/2$ *then the same is true for all* $p \in [2, \infty)$.

In fact the result is true when $s < d/2$ for all $p \in [2, d/((d/2) - s)]$, i.e. includes the right-hand end of the interval. So in particular in a three-dimensional domain, $H^1 \hookrightarrow L^6$, and in a two-dimensional domain $H^1 \hookrightarrow L^p$ for any $p \in [2, \infty)$.

Proof The argument follows that above, writing

$$\sum_k |c_k|^q = \sum_k [(1 + |k|^{2s})^{q/2} |c_k|^q](1 + |k|^{2s})^{-q/2}$$

and using Hölder's inequality appropriately. □

Finally, a key compactness property:

Theorem 9.7 $H^1_{\mathrm{p}}(Q)$ *is compactly embedded in* $L^2(Q)$*, i.e. any sequence* $\{u_n\}$ *that is bounded in* $H^1_{\mathrm{p}}(Q)$ *has a subsequence that converges in* $L^2(Q)$*.*

Proof Let $\{u^{(n)}\}$ with

$$u^{(n)} = \sum_k c_k^{(n)} \mathrm{e}^{\mathrm{i}k\cdot x}$$

be a sequence that is uniformly bounded in $H^1_{\mathrm{p}}(Q)$, i.e. for some $M > 0$,

$$\sum_k (1+|k|^2)|c_k^{(n)}|^2 \leq M$$

for each n. Let $\{k_j\}_{j\in\mathbb{N}}$ be an enumeration of the elements of $\mathbb{Z}^d$ (over which the sum is taken).

It is clear that each Fourier coefficient is uniformly bounded. One can therefore find a subsequence $u^{(n_{1,j})}$ such that $c_{k_1}^{(n_{1,j})}$ converges; and a subsequence of this, $u^{(n_{2,j})}$ such that $c_{k_1}^{(n_{1,j})}$ converges and $c_{k_2}^{(n_{1,j})}$ converges; and by continuing in this way one can find successive subsequences such that for $u^{(n_{m,j})}$, $c_{k_i}^{(n_{m,j})}$ converges for all $i = 1, \ldots, m$.

Now if one takes the 'diagonal subsequence' $u^{[j]} = u^{(n_{j,j})}$ it follows that $c_i^{[j]} := c_k^{(n_{j,j})}$ converges for every $k \in \mathbb{Z}^d$; call the limit c_k^*.

Now, for each finite K we have

$$\sum_{|k|\leq K} (1+|k|^2)|c_k^{[n]}|^2 \leq M,$$

and hence

$$\sum_{|k|\leq K} (1+|k|^2)|c_k^*|^2 \leq M.$$

Since this holds for all K,

$$\sum_k (1+|k|^2)|c_k^*|^2 \leq M,$$

so $u^* = \sum_k c_k^* \mathrm{e}^{\mathrm{i}k\cdot x}$ is an element of $H^1_{\mathrm{p}}(Q)$. To show that $u^{[n]} \to u$ in $L^2(Q)$, observe that

$$\sum_k (1+|k|^2)|c_k^{[n]} - c_k^*|^2 \leq 4M.$$

So now

$$\sum_k |c_k^{[n]} - c_k^*|^2 = \sum_{|k|\le K} |c_k^{[n]} - c_k^*|^2 + \frac{1}{K^2} \sum_{|k|\ge K} |k|^2 |c_k^{[n]} - c_k^*|^2$$
$$\le \frac{4M}{K^2} + \sum_{|k|\le K} |c_k^{[n]} - c_k^*|^2.$$

Given $\epsilon > 0$, choose K such that $4M/K^2 < \epsilon/2$, and then n sufficiently large that the second term is also less than $\epsilon/2$. It follows that $u^{[n]} \to u^*$ in $L^2(Q)$.

□

References

The material in Chapters 1–4 is all fairly standard, and you can find this covered in

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