# Monotone operators in divergence form with $x$–dependent multivalued graphs

Gilles Francfort*, François Murat**, Luc Tartar***

## Abstract

We prove the existence of solutions to $-div\ a(x, grad\, u) = f$, together with appropriate boundary conditions, whenever $a(x, e)$ is a maximal monotone graph in $e$, for every fixed $x$. We propose an adequate setting for this problem, in particular as far as measurability is concerned. It consists in looking at the graph after a 45° rotation, for every fixed $x$; in other words, the graph $d \in a(x, e)$ is defined through $d - e = \varphi(x, d + e)$, where $\varphi$ is a Carathéodory contraction on $\mathbb{R}^N$. This definition is shown to be equivalent to the fact that $a(x, \cdot)$ is pointwise monotone and that, for any $g \in [L^{p'}(\Omega)]^N$ and any $\delta > 0$, the equation $d + \delta\, |e|^{p-2}e = g$ has a solution $(e, d)$ with $d \in a(x, e)$. Under additional coercivity and growth assumptions, the existence of solutions to $-div\ a(x, grad\, u) = f$ is then established.

Dimostriamo l'esistenza delle soluzioni per l'equazione $-div\ a(x, grad\, u) = f$ con opportune condizioni al bordo, nel caso in cui $a(x, e)$ sia un grafico massimale monotono in $e$ per ogni $x$ fissato. Innanzitutto proponiamo un quadro adeguato per questo problema, in particolare per quel che concerne la misurabilità. Queso consiste nel considerare il grafico dopo una rotazione di 45° per ogni $x$ fissato. In altre parole, il grafico $d \in a(x, e)$ e definito da $d - e = \varphi(x, d + e)$ dove $\varphi$ è una contrazione di Carathéodory su $\mathbb{R}^N$. Mostriamo che questa definizione è equivalente al fatto che $a(x, \cdot)$ è puntualmente monotono e che, per ogni $g \in [L^{p'}(\Omega)]^N$ ed ogni $\delta > 0$, l'equazione $d+\delta\, |e|^{p-2}e = g$ ha una soluzione $(e, d)$ con $d \in a(x, e)$. Si dimonstra poi l'esistenza delle soluzioni di $-div\ a(x, grad\, u) = f$ sotto ipotesi di crescita e coercitività.

## I. Introduction

Maximal monotone operators are hardly a new topic. Since the 1960's, an abundant literature has been produced in two main directions. A significant fraction of that literature is devoted to abstract maximal monotone operators from a Banach space into its dual. This is not our concern in the present study. An equally significant fraction examines "concrete"

* L.P.M.T.M., Université Paris-Nord, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France
** Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte courrier 187, 75252 Paris Cedex 05, France
*** Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213-3890, USA

operators, the model of which is

$$\begin{cases} -\, div\; a(x, grad\, u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Whenenever $a(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a (univalued) Carathéodory function which further satisfies appropriate growth and coercivity conditions, existence is classical (see e.g. [Le&Li], [Li]).

In this study we investigate equations of the form

$$-div\; d = f \quad \text{in } \mathcal{D}'(\Omega),$$

with $(grad\, u(x), d(x)) \in \mathcal{A}(x)$, where $\mathcal{A}(x)$ is, for each $x$, a maximal monotone graph. That so few results concerning this class of equations should be available came as a surprize to us. But, in all fairness, it is a near impossible task to survey the relevant literature, so that the results that are presented here might have been previously derived, unbeknownst to us. If such should be the case, we will gladly apologize for the oversight, and will duly acknowledge the anteriority of the ignored contribution.

Even more surprizing to us is the apparent absence (with the same caveat) of available results in the case where the graph is $x$–independent. Note that a very simplified version of the proofs developed below yields existence in such a setting.

To our knowledge, the only existence result available is to be found in [CP&DM&De], Theorem 2.7]. We reproduce the statement of that theorem for the reader's convenience.

**Theorem 1.1.** *Assume that $\mathcal{A}(x) \subset \mathbb{R}^N \times \mathbb{R}^N$ is an $x$–dependent graph with the following properties:*

(i) *$\{d \in \mathbb{R}^N : (e, d) \in \mathcal{A}(x)\}$ is closed for a.e. $x$ in $\Omega$ and every $e$ in $\mathbb{R}^N$;*

(ii) *$\mathcal{A}(x)$ is maximal monotone for a.e. $x$ in $\Omega$;*

(iii) *there exists $1 < p < +\infty$, $m(x) \geq 0$ in $L^1(\Omega)$, and $\alpha > 0$ such that, for a.e. $x$ in $\Omega$ and every $(e, d)$ in $\mathcal{A}(x)$*

$$d.e \geq -m(x) + \alpha(|e|^p + |d|^{p'})$$

*with and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$;*

(iv) *for any closed set $C$ of $\mathbb{R}^N$,*

$$\big\{(x, e) \in \Omega \times \mathbb{R}^N : \textit{there exists } d \in C \textit{ such that } (e, d) \in \mathcal{A}(x)\big\}$$

*is measurable with respect to the $\sigma$-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N)$, where $\mathcal{L}(\Omega)$ denotes the $\sigma$-algebra of Lebesgue measurable subsets of $\Omega$ and $\mathcal{B}(\mathbb{R}^N)$ that of all Borel subsets of $\mathbb{R}^N$.*

*Then, for every $f \in W^{-1,p'}(\Omega)$, there exists a solution $(u, d)$ to*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in [L^{p'}(\Omega)]^N, \\ -\, div\; d = f \quad \text{in } \mathcal{D}'(\Omega), \\ (grad\, u(x), d(x)) \in \mathcal{A}(x) \quad \text{a.e. in } \Omega. \end{cases}$$

The drawback of the above theorem is that assumption (iv), the measurability assumption, seems difficult to check in concrete cases. Furthermore, its proof uses delicate measurability selection theorems.

In this study, we propose a class of graphs for which measurability becomes obvious. Specifically, we investigate monotone graphs $\mathcal{A}_\varphi(x)$ of the form

$$\mathcal{A}_\varphi(x) = \left\{(e,d) \in \mathbb{R}^N \times \mathbb{R}^N : d - e = \varphi(x, d+e)\right\},$$

for a.e. $x$ in $\Omega$, where $\varphi(x,\lambda) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory contraction, i.e., is measurable in $x$ for every $\lambda$ and satisfies, for a.e. $x$ in $\Omega$,

$$|\varphi(x,\lambda) - \varphi(x,\lambda')| \leq |\lambda - \lambda'|, \quad \lambda, \lambda' \in \mathbb{R}^N.$$

Such graphs are easily seen to be maximal monotone (cf. Lemma 2.1 below); the idea of a 45° rotation of the graph as a useful tool for the study of monotone operators goes back to Minty, who proved an analogous result without $x$-dependence [Mi].

If we further assume coercivity and growth in the sense of (iii) of the above mentioned theorem, we then prove an existence result (Theorem 2.3 below). Note that the class we propose, although apparently different from the class of graphs considered in [CP&DM&De], is in fact identical. This was pointed out to us by G. Dal Maso, and his proof of the equivalence is given in Remark 2.2 below. In that respect, our existence result is not new. Our proof is however completely different, in particular because it eschews all the intricacies stemming from the measurability assumption (iv) of Theorem 1.1.

Section 2 details the setting and the main result (Theorem 2.3), while Section 3 proposes a first proof of that theorem with the help of a graph regularization in $\mathbb{R}^N$. Section 4 proves a similar existence result under different hypotheses, namely maximality and monotonicity in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ plus coercivity and growth (Theorem 4.4); note that in this theorem the operator is not assumed to be local. Section 5 reconciles the results of Theorems 2.3 and 4.4 by establishing a result of interest in its own right, essentially the equivalence between maximality in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ plus *pointwise* monotonicity and the existence of a monotone graph $\mathcal{A}_\varphi(x)$, where $\varphi$ is a Carathéodory contraction. At the end of that Section we observe (Remark 5.8) that subdifferentials of convex Carathéodory functions with appropriate coercivity and growth conditions are particular cases of the class under investigation. Section 6 investigates variants of the existence result when a zeroth order term is added or when $\varphi$, and therefore $a$, depends on the field $u$, a generalization of the so-called Leray-Lions operators.

Finally, we do not use the maximum principle at any point in this study, and identical results for systems or higher order equations with various variational boundary conditions could be similarly obtained.

## 2. The framework and the existence result

At the onset of this section we reestablish a simple equivalence lemma which justifies the standpoint adopted in this paper. By definition $\mathcal{A} \subset \mathbb{R}^N \times \mathbb{R}^N$ is a monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$ if and only if

$$\left.\begin{array}{r}(e,d)\in\mathcal{A}\\(e',d')\in\mathcal{A}\end{array}\right\}\Longrightarrow (d'-d).(e'-e)\geq 0,$$

where, from now onward $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^N$. Further, $\mathcal{A}$ is said to be maximal if and only if, whenever $(e,d)\in\mathbb{R}^N\times\mathbb{R}^N$ is such that

$$(d'-d).(e'-e)\geq 0, \quad \forall (e',d')\in\mathcal{A},$$

then $(e,d)\in\mathcal{A}$. In other words, there is no strict monotone extension of $\mathcal{A}$.

By definition a (possibly multivalued) function $\varphi$ defined on a subset of $\mathbb{R}^N$ – its domain $dom\,\varphi$ – with values in $\mathbb{R}^N$ is a contraction if and only if

$$\left.\begin{array}{r}(\lambda,\mu)\in graph\;\varphi\\(\lambda',\mu')\in graph\;\varphi\end{array}\right\}\Longrightarrow |\mu'-\mu|\leq|\lambda'-\lambda|,$$

in which case $\varphi$ is actually univalued.

To any $\mathcal{A}\subset\mathbb{R}^N\times\mathbb{R}^N$ we associate the multivalued function $\varphi_{\mathcal{A}}$ defined on a subset of $\mathbb{R}^N$ with values in $\mathbb{R}^N$ as follows:

$$(\lambda,\mu)\in graph\,\varphi_{\mathcal{A}} \quad\Longleftrightarrow\quad \exists(e,d)\in\mathcal{A},\quad \lambda=d+e,\quad \mu=d-e.$$

Conversely, to any multivalued function $\varphi$ defined on a subset of $\mathbb{R}^N$ with values in $\mathbb{R}^N$ we associate $\mathcal{A}_\varphi\subset\mathbb{R}^N\times\mathbb{R}^N$ defined as

$$\mathcal{A}_\varphi=\{(e,d)\in\mathbb{R}^N\times\mathbb{R}^N : d-e\in\varphi(d+e)\}. \tag{2.1}$$

Note that $\varphi_{\mathcal{A}_\varphi}=\varphi$ and $\mathcal{A}_{\varphi_{\mathcal{A}}}=\mathcal{A}$.

Then, the following lemma whose proof can also be found in [Mi], Lemma 3 and Theorem 4, or in [Al&Am], Proposition 1.1, holds true:

**Lemma 2.1.** *$\mathcal{A}\subset\mathbb{R}^N\times\mathbb{R}^N$ is a monotone graph if and only if $\varphi_{\mathcal{A}}$ is a contraction on its domain $dom\,\varphi_{\mathcal{A}}$. Furthermore, $\mathcal{A}$ is maximal if and only if $dom\,\varphi_{\mathcal{A}}=\mathbb{R}^N$.*

*Proof.* Let $(e,d),(e',d')\in\mathbb{R}^N\times\mathbb{R}^N$. Since

$$|(d'+e')-(d+e)|^2-|(d'-e')-(d-e)|^2=4(d'-d).(e'-e),$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$, the following equivalence holds:

$$(d'-d).(e'-e)\geq 0 \quad\Longleftrightarrow\quad |(d'-e')-(d-e)|^2\leq|(d'+e')-(d+e)|^2.$$

This proves the first part of the lemma.

Consider a monotone graph $\mathcal{A}$. Assume that $dom\,\varphi_{\mathcal{A}}=\mathbb{R}^N$ and consider $(e,d)\in\mathbb{R}^N\times\mathbb{R}^N$ such that

$$(d'-d).(e'-e)\geq 0, \quad \forall(e',d')\in\mathcal{A}.$$

Set $\lambda := e + d$ and define $(e', d') \in \mathcal{A}$ by

$$\begin{cases} d' + e' = \lambda, \\ d' - e' = \varphi_{\mathcal{A}}(\lambda). \end{cases}$$

Since $(d' - d).(e' - e) \geq 0$,

$$|(d - e) - \varphi_{\mathcal{A}}(\lambda)|^2 = |(d - e) - (d' - e')|^2 \leq |(d + e) - (d' + e')|^2 = |\lambda - \lambda|^2 = 0.$$

Thus $d - e = \varphi_{\mathcal{A}}(\lambda) = \varphi_{\mathcal{A}}(d + e)$ and $(e, d) \in \mathcal{A}$, which proves the maximality of $\mathcal{A}$.

Conversely, assume that $dom\, \varphi_{\mathcal{A}} \not\subseteq \mathbb{R}^N$. Then, according to Kirszbraun's theorem (see e.g. [Fe], [Mi]), there exists an extension $\tilde{\varphi}_{\mathcal{A}}$ of $\varphi_{\mathcal{A}}$ which is a contraction on all of $\mathbb{R}^N$. Consider $\lambda \notin dom\, \varphi_{\mathcal{A}}$ and set

$$\begin{cases} d + e = \lambda, \\ d - e = \tilde{\varphi}_{\mathcal{A}}(\lambda). \end{cases}$$

Then, for any $(e', d') \in \mathcal{A}$,

$$\begin{aligned} |(d' - e') - (d - e)| &= |\varphi_{\mathcal{A}}(d' + e') - \tilde{\varphi}_{\mathcal{A}}(d + e)| \\ &= |\tilde{\varphi}_{\mathcal{A}}(d' + e') - \tilde{\varphi}_{\mathcal{A}}(d + e)| \leq |(d' + e') - (d + e)|. \end{aligned}$$

Thus $(d' - d).(e' - e) \geq 0$, i.e., $\mathcal{A}$ is not maximal. □

In the light of the previous lemma, maximal monotone graphs of $\mathbb{R}^N \times \mathbb{R}^N$ are equivalently defined through contractions *defined on all of* $\mathbb{R}^N$, which is the standpoint we adopt from now onward. An identical stanpoint is adopted in [Al&Am] who discuss the fine properties of monotone graphs of $\mathbb{R}^N \times \mathbb{R}^N$.

Throughout the remainder of this paper, $\Omega$ is a bounded open domain in $\mathbb{R}^N$, $p \in (1, \infty)$ and $p' = p/(p - 1)$ is its Hölder conjugate exponent, $m(x)$ a fixed non-negative function in $L^1(\Omega)$ and $\alpha$ is a strictly positive real number.

We define $\mathcal{M}(\alpha, m, p, \Omega)$ as the set of functions $\varphi(x, \lambda) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ with the following properties:

- $\varphi$ is Carathéodory; (2.2)
- $\varphi(x, \cdot)$ is a contraction for a.e. $x$ in $\Omega$; (2.3)
- if for any given $\lambda$ in $\mathbb{R}^N$, $e(x)$ and $d(x)$ are defined, for a.e. $x$ in $\Omega$, as

$$\begin{cases} d(x) + e(x) = \lambda, \\ d(x) - e(x) = \varphi(x, \lambda), \end{cases}$$

then, for a.e. $x \in \Omega$,

$$d(x).e(x) \geq -m(x) + \alpha(|e(x)|^p + |d(x)|^{p'}); \tag{2.4}$$

• $\varphi(x,0)=0$, for a.e. $x \in \Omega$. (2.5)

For a.e. $x$ in $\Omega$ we further denote by $\mathcal{A}_\varphi(x)$ the graph associated to $\varphi(x,\cdot)$ as in (2.1).

Note that, due to (2.5), $(0,0) \in \mathcal{A}_\varphi(x)$. Throughout the remainder of the study, we will always assume that $(0,0)$ belongs to the investigated graphs. When $\varphi$ satisfies (2.2)–(2.4), it is always possible to reduce to the case $\varphi(x,0)=0$ whenever there exists some $d_0 \in [L^{p'}(\Omega)]^N$ such that

$$\varphi(x, d_0(x)) = d_0(x) \quad \text{for a.e. } x \text{ in } \Omega,$$

or in other terms such that $(0, d_0(x)) \in \mathcal{A}_\varphi(x)$ ; indeed defining $\hat{\varphi}$, $\hat{e}$ and $\hat{d}$ by

$$\hat{\varphi}(x,\hat{\lambda}) = \varphi(x, \hat{\lambda} + d_0(x)) - d_0(x), \quad \hat{e}(x) = e(x), \quad \hat{d}(x) = d(x) - d_0(x),$$

it is easy to see that $\hat{\varphi}$ satisfies (2.2)–(2.5) for some $\hat{m}$ in $L^1(\Omega)$ and some $\hat{\alpha} > 0$.

Observe that for $\Omega, p$ and $m$ fixed, the set $\mathcal{M}(\alpha, m, p, \Omega)$ is non empty whenever $\alpha$ is sufficiently small: indeed the function $\varphi(x,\lambda) = \varphi(\lambda) = |e|^{p-2}e - e$, where $e$ is defined from $\lambda$ by $|e|^{p-2}e + e = \lambda$, belongs to $\mathcal{M}(\alpha, m, p, \Omega)$ when $\alpha \le 1/2$ and $m(x) \ge 0$ (the graph $\mathcal{A}_\varphi$ associated to this function is $\{(e,d) : d = |e|^{p-2}e\} \subset \mathbb{R}^N \times \mathbb{R}^N$ ). In contrast the set$\mathcal{M}(\alpha, m, p, \Omega)$ can be empty if $\alpha$ is too large, since when $\alpha > \sup((1/p),(1/p'))$, (2.4) and Young's inequality imply

$$|e(x)|^p + |d(x)|^{p'} \le C m(x),$$

in contradiction with the fact that $e(x)$ and $d(x)$ can be choosen such that $d(x) + e(x) = \lambda$ for every $\lambda \in \mathbb{R}^N$.

**Remark 2.2.** As already said in the introduction, $\{\mathcal{A}_\varphi(x) : \varphi \in \mathcal{M}(\alpha, m, p, \Omega)\}$ is precisely the set of $x$–dependent graphs considered in Theorem 1.1 which satisfy the additional condition $(0,0) \in \mathcal{A}(x)$, as communicated to us by G. Dal Maso whose proof we reproduce now.

Indeed, by Theorem 1.3 in [CP&DM&De], under the assumption (i), the measurability assumption (iv) of Theorem 1.1 is equivalent to the fact that

$$\mathcal{E} := \{(x,e,d) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N : (e,d) \in \mathcal{A}(x)\}$$

belongs to the $\sigma$-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^N)$. Since, by Lemma 2.1, we have

$$(e,d) \in \mathcal{A}(x) \qquad \Longleftrightarrow \qquad d - e = \varphi(x, d+e)\,,$$

we can write

$$\mathcal{E} = \{(x,e,d) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N : d - e = \varphi(x, d+e)\} = \Phi^{-1}(\mathcal{F})\,,$$

where $\Phi\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \Omega \times \mathbb{R}^N \times \mathbb{R}^N$ is defined by

$$\Phi(x,e,d) := (x, d+e, d-e)$$

and

$$\mathcal{F} := \{(x, \lambda, \mu) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N : \mu = \varphi(x, \lambda)\}\,.$$

Therefore, under the assumptions (i), (ii), and (iii) of Theorem 1.1, the measurability assumption (iv) is equivalent to the fact that $\mathcal{F}$ belongs to the $\sigma$-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N) \otimes \mathcal{B}(\mathbb{R}^N)$, which by Theorem 1.3 of [CP&DM&De] is equivalent to the fact that $\varphi\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable with respect to the $\sigma$-algebrae $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^N)$ and $\mathcal{B}(\mathbb{R}^N)$. Since $\varphi(x, \cdot)$ is a contraction for a.e. $x \in \Omega$, this measurability condition is equivalent to the fact that $\varphi$ is Carathéodory.

Therefore the graphs $\mathcal{A}_\varphi(x)$ considered here are exactly those graphs of Theorem 1.1 (Definition 2.1 of [CP&DM&De]) for which $(0, 0) \in \mathcal{A}(x)$.

Our main goal in this paper is to prove the following theorem, which in view of the above remark, is identical to Theorem 2.7 of [CP&DM&De], but with a completely different proof:

**Theorem 2.3.** *Consider $\varphi \in \mathcal{M}(\alpha, m, p, \Omega)$. For any $f \in W^{-1,p'}(\Omega)$ there exists $u$ and $d$ such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in [L^{p'}(\Omega)]^N, \\ -\,div\ d = f \quad in\ \mathcal{D}'(\Omega), \\ d(x) - grad\, u(x) = \varphi(x, d(x) + grad\, u(x)), \qquad for\ a.e.\ x\ in\ \Omega, \end{cases} \tag{2.6}$$

*(or equivalently $(grad\, u(x), d(x)) \in \mathcal{A}_\varphi(x)$ for a.e. $x$ in $\Omega$).* □

We present two separate proofs of Theorem 2.3 in Sections 3, 4 and 5, respectively. The first proof consists in a regularization of the graph of $\varphi$ in $\mathbb{R}^N$, which naturally leads us to a strongly monotone problem in $W_0^{1,2}(\Omega)$. The second proof boils down to a regularization of a monotone graph in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ associated to $\mathcal{A}_\varphi$. Both proofs use the abstract existence theorem for univalued, monotone, continuous, bounded and coercive operators on a reflexive Banach space stated in Theorem 2.4 below. Note that the first proof only uses it in a Hilbert space setting where it can be derived through application of Banach's fixed point theorem (cf. e.g. [Bre2], Theorem V.6, in the linear case), while the second proof is concerned with a Banach space in which case Brouwer's fixed point theorem is used (cf. e.g. [Li], Chapter 2, Theorem 2.1).

**Theorem 2.4.** *Let $V$ be a reflexive Banach space and $A : V \to V'$ be a univalued, monotone, continuous and bounded operator. Assume further that $A$ is coercive, i.e.,*

$$\lim_{\|v\|\uparrow+\infty} \frac{< A(v), v >}{\|v\|_V} = +\infty.$$

*Then, $A$ is onto.* □

Besides this, the proofs in the present paper only use elementary tools, except in Section 6, a short section devoted to variants of Theorem 2.3, where Schauder's fixed point theorem is also used.

Concerning notation, throughout the paper $\| \, \|_{L^p(\Omega)}$ denotes the norm in both $L^p(\Omega)$ or $[L^p(\Omega)]^N$ depending on its argument, $C$ denotes a generic positive constant in bounding estimates, so that, for example, $2C$ can be replaced by $C$.

For the reader's convenience, we should stress that from now onward the word "function" will always refer to univalued functions, while the expression "graph of $E \times F$" will refer to a subset of $E \times F$, seen as a multivalued function from a subset of $E$ into $F$. Also, the notation $(e, d)$ can be easily understood in reference to electric field and flux, respectively.

Finally, the calligraphic character $\mathcal{A}$ will always refer to a graph of $\mathbb{R}^N \times \mathbb{R}^N$ ($\mathcal{M}$, to a set of functions associated to graphs of that type), while the slanted character $A$ will refer to a graph of $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ ($M$, to a set of such graphs).

## 3. Proof of Theorem 2.3 by a graph regularization in $\mathbb{R}^N$

In this section, we propose a proof of Theorem 2.3 which uses a regularization of the pointwise graph of the operator.

Consider a Carathéodory contraction $\varphi(x, \lambda)$, that is a $\varphi$ that satisfies (2.2), (2.3), and also assume that $\varphi(x, \cdot)$ is a strict contraction for a.e. $x$ in $\Omega$, i.e., that

$$|\varphi(x, \lambda) - \varphi(x, \lambda')| \leq \theta |\lambda - \lambda'|, \quad \lambda, \lambda' \in \mathbb{R}^N, \tag{3.1}$$

with $0 < \theta < 1$. Then,

**Lemma 3.1.** *If $\varphi$ is a strict Carathéodory contraction ((2.2), (3.1)), then for a.e. $x$ in $\Omega$, $\mathcal{A}_\varphi(x)$ is the graph of a strongly monotone (single valued) function $a(x, \cdot)$ defined on all of $\mathbb{R}^N$. Furthermore $a(x, e)$ is Carathéodory on $\Omega \times \mathbb{R}^N$ and Lipschitz in $e$, almost uniformly in $x \in \Omega$.* □

*Proof.* According to (2.1), if $(e, d), (e', d') \in \mathcal{A}_\varphi(x)$, then

$$|(d' - e') - (d - e)|^2 \leq \theta^2 |(d' + e') - (d + e)|^2,$$

that is, upon setting

$$C_\theta := \frac{1 - \theta^2}{2(1 + \theta^2)} > 0, \tag{3.2}$$

$$(d' - d).(e' - e) \geq C_\theta (|d' - d|^2 + |e' - e|^2), \tag{3.3}$$

which immediately proves that $\mathcal{A}_\varphi(x)$ is the graph of a single valued, strongly monotone, Lipschitz function denoted by $a(x, \cdot)$; in other words, the mapping $e \mapsto d$ is single valued, and $d = a(x, e)$.

The domain of $a(x, \cdot)$ is all of $\mathbb{R}^N$, as well as its range; indeed, for every $e \in \mathbb{R}^N$, $a(x, e)$ is the unique $d$ such that $d - e = \varphi(x, e + d)$, because it is the only fixed point of the mapping $d \mapsto e + \varphi(x, e + d)$, a strict contraction on $\mathbb{R}^N$ for almost every $x \in \Omega$; similarly, for every $d \in \mathbb{R}^N$, $a^{-1}(x, d)$ is the unique $e$ such that $d - e = \varphi(x, e + d)$, because it is the only fixed point of the mapping $e \mapsto d - \varphi(x, e+d)$, a strict contraction on $\mathbb{R}^N$ for almost every $x \in \Omega$.

It merely remains to prove that $a(x,e)$ is Carathéodory. Fix $e$ in $\mathbb{R}^N$, then $a(x,e)$ is the fixed point of the Carathéodory mapping

$$d \mapsto \Phi(x,d) := e + \varphi(x, d+e),$$

which is given as the almost pointwise limit of the sequence

$$\begin{cases} d^{n+1}(x) = \Phi(x, d^n(x)), \\ \quad d^0(x) = 0. \end{cases}$$

Since $\Phi$ is Carathéodory, each $d^n(x)$ is measurable on $\Omega$ hence $a(x,e)$, their almost pointwise limit. □

**Remark 3.2.** Note that the previous lemma does not use (2.4) nor (2.5).

Consider now $\varphi \in \mathcal{M}(\alpha, m, p, \Omega)$ and define, for $\eta > 0$,

$$\varphi_\eta(x,\lambda) := \frac{1}{1+\eta}\varphi(x,\lambda). \tag{3.4}$$

Then $\varphi_\eta$ satisfies (2.5) and (3.1) with $\theta = 1/(1+\eta)$.

According to Lemma 3.1, the associated $a_\eta(x,e)$ is Carathéodory, Lipschitz and strongly monotone since it satisfies, for the constant $c_\eta := C_{1/(1+\eta)}$ (cf. (3.2))

$$\begin{cases} (a_\eta(x,e') - a_\eta(x,e)).(e'-e) \geq c_\eta(|a_\eta(x,e') - a_\eta(x,e)|^2 + |e'-e|^2), \\ \text{with} \quad \dfrac{c_\eta}{\eta} \to \dfrac{1}{2} \quad \text{when} \quad \eta \to 0. \end{cases} \tag{3.5}$$

Furthermore, since $\varphi_\eta(x,0) = \varphi(x,0) = 0$, $a_\eta(x,0) = 0$. Consequently, $a_\eta(x,\cdot)$ is monotone, 2-coercive, and Lipschitz, almost uniformly in $\Omega$ for $\eta$ fixed. Then, the assumptions of Theorem 2.4 are trivially met by $\mathbf{A}_\eta : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ defined as $\mathbf{A}_\eta u := -div\ (a_\eta(x, grad\, u))$, so that, for any $g \in [L^\infty(\Omega)]^N$ there exists a (unique) solution to

$$\begin{cases} -\,div\ (a_\eta(x, grad\, u_\eta)) = -div\ g \quad \text{in } \mathcal{D}'(\Omega), \\ u_\eta \in W_0^{1,2}(\Omega). \end{cases} \tag{3.6}$$

Set

$$\begin{cases} e_\eta(x) = grad\, u_\eta(x), \\ d_\eta(x) = a_\eta(x, grad\, u_\eta(x)). \end{cases}$$

Then $e_\eta$ and $d_\eta$ are elements of $[L^2(\Omega)]^N$ and, a.e. in $\Omega$,

$$d_\eta(x) - e_\eta(x) = \varphi_\eta(x, d_\eta(x) + e_\eta(x)), \tag{3.7}$$

which means $(1+\eta)(d_\eta(x)-e_\eta(x)) = \varphi(x, d_\eta(x)+e_\eta(x))$. Defining $E_\eta(x)$ and $D_\eta(x)$ by

$$\begin{cases} E_\eta := \dfrac{(2+\eta)e_\eta - \eta\, d_\eta}{2}, \\ D_\eta := \dfrac{(2+\eta)d_\eta - \eta\, e_\eta}{2}, \end{cases} \tag{3.8}$$

we have

$$\begin{cases} D_\eta(x) - E_\eta(x) = (1+\eta)(d_\eta(x) - e_\eta(x)), \\ D_\eta(x) + E_\eta(x) = d_\eta(x) + e_\eta(x), \\ D_\eta(x) - E_\eta(x) = \varphi(x, D_\eta(x) + E_\eta(x)), \end{cases}$$

a.e. in $\Omega$ and recalling (2.4) we obtain

$$D_\eta(x).E_\eta(x) \geq -m(x) + \alpha(|E_\eta(x)|^p + |D_\eta(x)|^{p'}) \tag{3.9}$$

almost everywhere.

Integration of (3.9) over $\Omega$ yields

$$\begin{aligned} \alpha\left(\|E_\eta\|^p_{L^p(\Omega)} + \|D_\eta\|^{p'}_{L^{p'}(\Omega)}\right) + \frac{\eta}{2}\left(1+\frac{\eta}{2}\right)\left(\|e_\eta\|^2_{L^2(\Omega)} + \|d_\eta\|^2_{L^2(\Omega)}\right) \\ \leq \|m\|_{L^1(\Omega)} + \left(1+\eta+\frac{\eta^2}{2}\right)\int_\Omega d_\eta e_\eta\, dx, \end{aligned} \tag{3.10}$$

which establishes, along the way, that $E_\eta \in [L^p(\Omega)]^N$ and $D_\eta \in [L^{p'}(\Omega)]^N$. On the other hand, the use of $u_\eta$ as test fuction in (3.6) and the definition of $E_\eta$ yield

$$\int_\Omega d_\eta.e_\eta\, dx = \int_\Omega g e_\eta\, dx = \frac{2}{2+\eta}\int_\Omega g E_\eta\, dx + \frac{\eta}{2+\eta}\int_\Omega g d_\eta\, dx, \tag{3.11}$$

so that, recalling (3.10), we obtain

$$\begin{aligned} &\alpha\left(\|E_\eta\|^p_{L^p(\Omega)} + \|D_\eta\|^{p'}_{L^{p'}(\Omega)}\right) + \frac{\eta}{2}\left(1+\frac{\eta}{2}\right)\left(\|e_\eta\|^2_{L^2(\Omega)} + \|d_\eta\|^2_{L^2(\Omega)}\right) \\ &\leq \|m\|_{L^1(\Omega)} + \left(1+\eta+\frac{\eta^2}{2}\right)\left(\frac{2}{2+\eta}\|g\|_{L^{p'}(\Omega)}\|E_\eta\|_{L^p(\Omega)} + \frac{\eta}{2+\eta}\|g\|_{L^2(\Omega)}\|d_\eta\|_{L^2(\Omega)}\right), \end{aligned}$$

which implies, since $p > 1$, that (for $\eta$ bounded)

$$\|E_\eta\|_{L^p(\Omega)} + \|D_\eta\|_{L^{p'}(\Omega)} \leq C < +\infty, \tag{3.12}$$

$$\sqrt{\eta}\|e_\eta\|_{L^2(\Omega)} + \sqrt{\eta}\|d_\eta\|_{L^2(\Omega)} \leq C < +\infty. \tag{3.13}$$

At the possible expense of extracting subsequences (still indexed by $\eta$) we are thus at liberty to assume that, as $\eta \downarrow 0^+$,

$$\begin{cases} E_\eta \rightharpoonup e \quad \text{weakly in } [L^p(\Omega)]^N, \\ D_\eta \rightharpoonup d \quad \text{weakly in } [L^{p'}(\Omega)]^N. \end{cases} \tag{3.14}$$

Recall definition (3.8) of $E_\eta$ and $D_\eta$. Defining $\overline{p} := \min(p, 2)$ and $\overline{q} := \min(p', 2)$, (3.13), (3.14) imply that

$$\begin{cases} e_\eta \rightharpoonup e \quad \text{weakly in } [L^{\overline{p}}(\Omega)]^N, \\ d_\eta \rightharpoonup d \quad \text{weakly in } [L^{\overline{q}}(\Omega)]^N. \end{cases} \tag{3.15}$$

Since $e_\eta = grad\, u_\eta$ with $u_\eta \in W_0^{1,2}(\Omega)$, Poincaré's inequality yields the existence of $u \in W_0^{1,\overline{p}}(\Omega)$ such that

$$u_\eta \rightharpoonup u \quad \text{weakly in } W_0^{1,\overline{p}}(\Omega), \tag{3.16}$$

with $grad\, u = e$. Furthermore, in view of (3.6),

$$-div\ d = -div\ g \quad \text{in } \mathcal{D}'(\Omega). \tag{3.17}$$

Since $e \in [L^p(\Omega)]^N$ and $u \in W_0^{1,\overline{p}}(\Omega)$ with $grad\, u = e$, $u \in W_0^{1,p}(\Omega)$ whenever $\partial\Omega$ is smooth. At this point we assume that such is the case, but will remove the smoothness restriction later. Also, by (3.14), $d \in [L^{p'}(\Omega)]^N$.

It remains to prove that

$$d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega.$$

Recalling (3.7) and the contractive character of $\varphi_\eta$, we obtain, for every $\lambda \in [L^\infty(\Omega)]^N$,

$$|d_\eta(x) - e_\eta(x) - \varphi_\eta(x, \lambda(x))|^2 \leq |d_\eta(x) + e_\eta(x) - \lambda(x)|^2, \quad \text{a.e. in } \Omega, \tag{3.18}$$

or still

$$\begin{aligned} &|\varphi_\eta(x, \lambda(x))|^2 - 2d_\eta(x).\varphi_\eta(x, \lambda(x)) + 2e_\eta(x).\varphi_\eta(x, \lambda(x)) \\ &\leq |\lambda(x)|^2 - 2d_\eta(x).\lambda(x) - 2e_\eta(x).\lambda(x) + 4d_\eta(x).e_\eta(x), \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.19}$$

Since $\varphi_\eta$ is a Carathéodory contraction, $\varphi_\eta(x, \lambda(x)) \in [L^\infty(\Omega)]^N$ and the integration of (3.19) over $\Omega$ is licit. We obtain

$$\begin{aligned} &\int_\Omega |\varphi_\eta(x, \lambda(x))|^2\, dx - 2\int_\Omega d_\eta(x).\varphi_\eta(x, \lambda(x))dx + 2\int_\Omega e_\eta(x).\varphi_\eta(x, \lambda(x))dx \\ &\leq \int_\Omega |\lambda(x)|^2\, dx - 2\int_\Omega d_\eta(x).\lambda(x)dx - 2\int_\Omega e_\eta(x).\lambda(x)dx + 4\int_\Omega d_\eta(x).e_\eta(x)dx. \end{aligned}$$

Since $\varphi_\eta(x, \lambda(x))$ tends to $\varphi(x, \lambda(x))$ almost everywhere in $\Omega$ and remains bounded, it is immediate in view of (3.15) to pass to the limit in all terms of the previous inequality, except in the last one. But, by virtue of (3.6) and (3.15)

$$\int_\Omega d_\eta(x).e_\eta(x)dx = \int_\Omega g(x).e_\eta(x)dx \to \int_\Omega g(x).e(x)dx.$$

Since $d \in [L^{p'}(\Omega)]^N$, using in (3.17) the test function $u$ (which, as said before, belongs to $W_0^{1,p}(\Omega)$ when $\partial\Omega$ is smooth) yields

$$\int_\Omega d(x).e(x)dx = \int_\Omega g(x).e(x)dx,$$

so that

$$\int_\Omega d_\eta(x).e_\eta(x)\,dx \to \int_\Omega d(x).e(x)\,dx.$$

Collecting all limits yields

$$\int_\Omega \left\{|d(x) - e(x) - \varphi(x, \lambda(x))|^2 - |d(x) + e(x) - \lambda(x)|^2\right\} dx \le 0. \tag{3.20}$$

For any $r > 0$, choose

$$\lambda(x) = \begin{cases} d(x) + e(x) & \text{if} \quad |d(x) + e(x)| \le r, \\ 0 & \text{otherwise.} \end{cases}$$

Then (3.20) becomes

$$\int_{\{x:|d(x)+e(x)|\le r\}} |d(x) - e(x) - \varphi(x, d(x) + e(x))|^2\,dx - 4\int_{\{x:|d(x)+e(x)|>r\}} d(x).e(x)\,dx \le 0.$$

Since $d.e \in L^1(\Omega)$, we are at liberty to let $r$ tend to $+\infty$ in the previous inequality, which finally yields

$$\int_\Omega |d(x) - e(x) - \varphi(x, d(x) + e(x))|^2\,dx \le 0,$$

that is

$$d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega. \tag{3.21}$$

At this point, the existence of $u$, $e$ and $d$ such that

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in [L^{p'}(\Omega)]^N, \quad e = grad\,u, \\ -\,div\ d = -div\ g \quad \text{in } \mathcal{D}'(\Omega), \\ d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega, \end{cases}$$

has been established for any $g \in [L^\infty(\Omega)]^N$ when $\partial\Omega$ is smooth.

Consider now the case of a general bounded open set $\Omega$ and of a general $f \in W^{-1,p'}(\Omega)$. We approximate $\Omega$ by a sequence of open sets $\Omega_n$ with $\partial\Omega_n$ smooth and $\Omega_n \subset \Omega$ such that, for every compact $K$ of $\mathbb{R}^N$ with $K \subset \Omega$, $K \subset \Omega_n$ for every $n$ sufficiently large. Since $f = -div\ g$ with $g \in [L^{p'}(\Omega)]^N$, we approximate $f$ by

$$f_n := -div\ g_n, \quad g_n \in [L^\infty(\Omega)]^N, \quad g_n \to g \quad \text{strongly in } [L^{p'}(\Omega)]^N.$$

Let $u_n$, $e_n$ and $d_n$ be a solution to

$$\begin{cases} u_n \in W_0^{1,p}(\Omega_n), \quad d_n \in [L^{p'}(\Omega_n)]^N, \quad e_n = grad\, u_n, \\ -\, div\ d_n = f_n \quad \text{in } \mathcal{D}'(\Omega_n), \\ d_n(x) - e_n(x) = \varphi(x, d_n(x) + e_n(x)), \quad \text{a.e. in } \Omega_n. \end{cases} \tag{3.22}$$

For every $\psi$ in $L^p(\Omega_n)$ or $L^{p'}(\Omega_n)$, we define its extension $\tilde{\psi}$ to $L^p(\Omega)$ or $L^{p'}(\Omega)$ by

$$\tilde{\psi} = \psi \quad \text{in } \Omega_n, \quad \tilde{\psi} = 0 \quad \text{in } \Omega \backslash \Omega_n.$$

Then $\tilde{u}_n \in W_0^{1,p}(\Omega)$ with $\tilde{e}_n = (grad\, u_n)^\sim = grad\, \tilde{u}_n$. In view of (2.4), the use of $u_n$ as test function in (3.22) yields, at least for a subsequence (still indexed by $n$),

$$\begin{cases} \tilde{u}_n \rightharpoonup u \ \text{ weakly in } \ W_0^{1,p}(\Omega), \\ \tilde{e}_n \rightharpoonup e = grad\, u \ \text{ weakly in } \ [L^p(\Omega)]^N, \\ \tilde{d}_n \rightharpoonup d \ \text{ weakly in } \ [L^{p'}(\Omega)]^N, \end{cases}$$

while, by the above mentioned property of the sequence $\Omega_n$,

$$-div\ d = f \quad \text{in } \mathcal{D}'(\Omega).$$

It remains to prove that

$$d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega.$$

For every $\lambda \in [L^\infty(\Omega)]^N$ we have

$$|d_n(x) - e_n(x) - \varphi(x, \lambda(x))|^2 \leq |d_n(x) + e_n(x) - \lambda(x)|^2, \quad \text{a.e. in } \Omega,$$

and the rest of the proof is very similar to that which led from (3.18) to (3.21) since

$$\int_{\Omega_n} d_n(x).e_n(x)dx = \int_{\Omega_n} g_n(x).e_n(x)dx = \int_{\Omega} g_n(x).\tilde{e}_n(x)dx$$

$$\to \int_{\Omega} g(x).e(x)dx = \int_{\Omega} d(x).e(x)dx.$$

The proof of Theorem 2.3 is complete.

**Remark 3.3.** The previous proof easily extends to the case of equations (or even systems) of higher order, as well as to different boundary conditions for which a variational formulation holds.

## 4. Graph regularization in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$

In this section, we prove an existence result similar to Theorem 2.3, when the graph under consideration is now a monotone graph of $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ which is not necessarily pointwise monotone.

From now onward, $j$ and $\hat{\jmath} : \mathbb{R}^N \to \mathbb{R}^N$ are defined respectively as

$$j(e) = |e|^{p-2}e, \quad \hat{\jmath}(d) = |d|^{p'-2}d.$$

We consider, for $\alpha > 0, \mu \geq 0$, and $1 < p < +\infty$, the set $M(\alpha, \mu, p, \Omega)$ of graphs $A \subset [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ with the following properties:

• $A$ is monotone, that is, for any $(e, d)$, $(e', d')$ in $A$,

$$\int_\Omega (d' - d).(e' - e)\ dx \geq 0; \tag{4.1}$$

• $A$ is $\hat{\jmath}$-surjective, that is

$$\begin{cases} \text{for any} \quad \delta > 0 \quad \text{and any} \quad e \in [L^p(\Omega)]^N, \\ \text{there exists a (unique) element} \quad (e', d') \in A, \\ \text{such that} \quad e' + \delta\hat{\jmath}(d') = e; \end{cases} \tag{4.2}$$

• if $(e, d) \in A$, then

$$\int_\Omega d.e\ dx \geq -\mu + \alpha \int_\Omega (|e|^p + |d|^{p'})\ dx, \tag{4.3}$$

• $(0, 0) \in A$. (4.4)

**Remark 4.1.** The set $M(\alpha, \mu, p, \Omega)$ is equivalently defined as the set of all maximal monotone graphs $A \subset [L^p(\Omega)|^N \times [L^{p'}(\Omega)]^N$ such that (4.3), (4.4) hold; indeed, according to [Bre1], Proposition 2.2, in a Hilbert space setting, or to [Ba], Theorem 1.2, in a reflexive Banach space setting, the maximality of a monotone graph $A$ is equivalent to the surjectivity of either $A + \delta j$ or $A^{-1} + \delta\hat{\jmath}$ for any fixed $\delta > 0$. Our bias towards (4.2) is dictated by a wish not to appeal to any non-elementary result besides Theorem 2.4. The reader who is familiar with the theory of maximal monotone operators should thus feel at liberty to replace hypothesis (4.2) by the maximality of $A$.

**Remark 4.2.** If (4.2) is satisfied, then $A$ is maximal (this is the easy part of the result quoted in Remark 4.1). Indeed, take $B$ to be a monotone extension of $A$ and $(e, d) \in B$. Then, according to (4.2), there exists a unique $(e', d') \in A$ such that

$$e' + \delta\hat{\jmath}(d') = e + \delta\hat{\jmath}(d).$$

But, since $A \subset B$, $(e', d') \in B$ so that $\int_\Omega (d' - d).(e' - e)dx \geq 0$. Multiplying the latest equality by $d' - d$ and integrating on $\Omega$ yields, since $\delta > 0$

$$\int_\Omega (d' - d) \cdot (\hat{\jmath}(d) - \hat{\jmath}(d'))dx \leq 0,$$

which immediately implies that $d = d'$, hence that $e = e'$, and finally that $(e, d) \in A$. Thus $B = A$.

**Remark 4.3.** The graphs in $M(\alpha, \mu, p, \Omega)$ are not necessarily "pointwise monotone", but only "functionally monotone" in the sense of (4.1), in contrast to the graphs considered in Section 3. An example of a graph which satisfies (4.1)–(4.4) without being pointwise monotone is, in the case $p = 2$,

$$A := \{(e, e + \rho \int_\Omega \rho e \, dx) : e \in [L^2(\Omega)]^N\},$$

with $\rho \in L^2(\Omega)$.

However every graph of $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ defined pointwise through a function $\varphi$ of $\mathcal{M}(\alpha, m, p, \Omega)$ belongs to $M(\alpha, \mu, p, \Omega)$ with $\mu = \|m\|^p_{L^1(\Omega)}$. Conversely Corollary 5.3 below shows that every pointwise monotone graph of $M(\alpha, 0, p, \Omega)$ is associated to a function of $\mathcal{M}(\alpha, 0, p, \Omega)$.

We now prove the

**Theorem 4.4.** *Consider $A$ in $M(\alpha, \mu, p, \Omega)$. For any $f \in W^{-1,p'}(\Omega)$, there exists $u$ and $d$ such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in [L^{p'}(\Omega)]^N, \\ -\, div\ d = f \quad in\ \mathcal{D}'(\Omega), \\ (grad\, u, d) \in A. \end{cases}$$

*Proof.* Define, for any $\varepsilon > 0$, the mapping $A^\varepsilon : [L^p(\Omega)]^N \to [L^{p'}(\Omega)]^N$, by

$$A^\varepsilon(e) = d,$$

where

$$\begin{cases} e' + \varepsilon \hat{\jmath}(d) = e, \\ (e', d) \in A. \end{cases} \tag{4.5}$$

The mapping $A^\varepsilon$ is well defined in view of (4.2) and it is monotone; indeed, if $e$ and $\overline{e}$ are elements of $[L^p(\Omega)|^N$, with

$$\begin{cases} e' + \varepsilon \hat{\jmath}(d) = e, \quad (e', d) \in A, \\ \overline{e}' + \varepsilon \hat{\jmath}(\overline{d}) = \overline{e}, \quad (\overline{e}', \overline{d}) \in A, \end{cases}$$

then

$$\int_\Omega (d - \overline{d}).(e - \overline{e}) \, dx = \int_\Omega (d - \overline{d}).(e' - \overline{e}') \, dx + \varepsilon \int_\Omega (d - \overline{d}).(\hat{\jmath}(d) - \hat{\jmath}(\overline{d})) \, dx. \tag{4.6}$$

Therefore, by virtue of (4.1) together with the strictly monotone character of $\hat{\jmath}$, $A^\varepsilon$ is monotone. Further, $A^\varepsilon(0) = 0$ by (4.4).

Recalling (4.3), (4.5) and the definition of $\hat{\jmath}$, we have

$$\begin{aligned}\int_\Omega d.e\;dx &= \int_\Omega d.e'\;dx + \varepsilon\int_\Omega d.\hat{\jmath}(d)\;dx\\ &\geq -\mu + \alpha\int_\Omega (|e'|^p + |d|^{p'})\;dx + \varepsilon\int_\Omega |d|^{p'}\;dx\\ &= -\mu + \alpha\int_\Omega (|e-\varepsilon\hat{\jmath}(d)|^p + |d|^{p'})\;dx + \varepsilon\int_\Omega |\hat{\jmath}(d)|^p\;dx,\end{aligned}$$

which implies that

$$\int_\Omega d.e\;dx \geq -\mu + \alpha\int_\Omega |d|^{p'}\;dx + \beta\int_\Omega |e|^p\;dx \tag{4.7}$$

with $\beta$ depending only on $p$ and $\alpha$, provided that $\varepsilon \leq \varepsilon_0$ (for some $\varepsilon_0$ depending only on $p$ and $\alpha$) so that $\alpha|z-\varepsilon y|^p + \varepsilon|y|^p \geq \beta|z|^p$.

Therefore, $A^\varepsilon$ is coercive (uniformly in $\varepsilon$), i.e.,

$$\int_\Omega A^\varepsilon(e).e\,dx \geq \beta\|e\|^p_{L^p(\Omega)} - \mu,$$

and, upon application of Hölder's inequality, it also satisfies a growth condition (uniformly in $\varepsilon$), i.e.,

$$\|A^\varepsilon(e)\|_{L^{p'}(\Omega)} \leq C(\|e\|^{p-1}_{L^p(\Omega)} + 1).$$

Finally, $A^\varepsilon$ is continuous for every fixed $\varepsilon$. Indeed, let $e_n$ converge to $e$ in $[L^p(\Omega)]^N$; then $d_n = A^\varepsilon(e_n)$ is bounded in $[L^{p'}(\Omega)]^N$ in view of the latest inequality. Recalling (4.6), we have

$$\int_\Omega (d-d_n).(e-e_n)\;dx = \int_\Omega (d-d_n).(e'-\overline{e}'_n)\;dx + \varepsilon\int_\Omega (d-d_n).(\hat{\jmath}(d)-\hat{\jmath}(d_n))\;dx,$$

so that

$$\varepsilon\int_\Omega (d-d_n).(\hat{\jmath}(d)-\hat{\jmath}(d_n))\;dx \longrightarrow 0.$$

Since $\hat{\jmath}(d) = |d|^{p'-2}d$, it is well-known that the above limit implies in turn that $d_n$ converges (strongly) to $d$ in $[L^{p'}(\Omega)]^N$.

Summing up, we have shown that for $\varepsilon$ small enough, $A^\varepsilon$ is a monotone, bounded, continuous and coercive operator from $[L^p(\Omega)]^N$ into $[L^{p'}(\Omega)]^N$. Define

$$\begin{cases}\mathbf{A}^\varepsilon : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega),\\ \mathbf{A}^\varepsilon(u) = -div\;(A^\varepsilon(grad\,u)).\end{cases} \tag{4.8}$$

Theorem 2.4 applied to $\mathbf{A}^\varepsilon$ then implies, for any $f \in W^{-1,p'}(\Omega)$, the existence of $u$ with

$$\begin{cases}u^\varepsilon \in W^{1,p}_0(\Omega),\\ -\,div\;(A^\varepsilon(grad\,u^\varepsilon)) = f \quad \text{in } \mathcal{D}'(\Omega).\end{cases} \tag{4.9}$$

We now let $\varepsilon$ tend to 0 in (4.9). In view of the uniform coercivity and growth properties of $A^\varepsilon$ and of Poincaré's inequality,

$$\|u^\varepsilon\|_{W_0^{1,p}(\Omega)} \leq C < +\infty,$$
$$\|A^\varepsilon(grad\, u^\varepsilon)\|_{L^{p'}(\Omega)} \leq C < +\infty,$$

so that, at the possible expense of extracting a subsequence, still indexed by $\varepsilon$,

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ d^\varepsilon := A^\varepsilon(grad\, u^\varepsilon) \rightharpoonup d \quad \text{weakly in } [L^{p'}(\Omega)]^N, \end{cases}$$

with

$$-div\ d = f \quad \text{in } \mathcal{D}'(\Omega). \tag{4.10}$$

Consider, for any pair $(E, D) \in A$, the function $E^\varepsilon(x)$ defined as

$$E^\varepsilon(x) = E(x) + \varepsilon \hat{\jmath}(D(x)),$$

which converges to $E$ strongly in $[L^p(\Omega)]^N$, and remark that, by the very definition of $A^\varepsilon$,

$$A^\varepsilon(E^\varepsilon) = D.$$

Then,

$$\begin{cases} 0 \leq \displaystyle\int_\Omega (A^\varepsilon(grad\, u^\varepsilon) - A^\varepsilon(E^\varepsilon)).(grad\, u^\varepsilon - E^\varepsilon)dx \\ \quad = \displaystyle\int_\Omega (d^\varepsilon - D).(grad\, u^\varepsilon - E^\varepsilon)dx. \end{cases} \tag{4.11}$$

In view of (4.9), (4.10),

$$\int_\Omega d^\varepsilon.grad\, u^\varepsilon\ dx = < f, u^\varepsilon >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}$$

$$\longrightarrow \quad < f, u >_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega d.grad\, u\ dx,$$

which allows one to pass to the limit in (4.11) and yields

$$0 \leq \int_\Omega (d - D).(grad\, u - E)dx.$$

But, according to Remark 4.2, $A$ is maximal, so that $(d, grad\, u) \in A$.

The proof of Theorem 4.4 is complete. □

**Remark 4.5.** As stated in Remarks 4.1 and 4.2 above, the definition of $M(\alpha,\mu,p,\Omega)$ implies that every $A \in M(\alpha,\mu,p,\Omega)$ is a maximal monotone graph in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$. It would be tempting to prove Theorem 4.4 above by establishing that the graph $\mathbf{A} \subset W_0^{1,p}(\Omega) \times W^{-1,p'}(\Omega)$ defined as

$$(u,f) \in \mathbf{A} \iff f = -div\ d \quad \text{with} \quad (grad\,u, d) \in A,$$

is maximal monotone. However, we were unable to prove this assertion directly, although it is certainly true: indeed, a proof similar to that of Theorem 4.4 would show that the equation

$$\begin{cases} u \in W_0^{1,p}(\Omega), \quad d \in [L^{p'}(\Omega)]^N, \quad (grad\,u, d) \in A, \\ -\,div\,\left(d + \delta\,|grad\,u|^{p-2} grad\,u\right) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

has a (unique) solution $u$ for any $\delta > 0$ and any $f \in W^{-1,p'}(\Omega)$. This surjectivity property easily implies maximality as mentioned in Remark 4.2.

## 5. A characterization of pointwise monotone maximal graphs in $[L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$

In this section we compare the apparently disconnected assumptions of Theorems 2.3 and 4.4 and show that there are in fact identical when the graph $A \subset L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ considered in Theorem 4.4 is further assumed to be *pointwise monotone*, that is such that, for any $(e,d)$ and $(e',d')$ in $A$,

$$(d'(x) - d(x)).(e'(x) - e(x)) \geq 0, \text{ a.e. in } \Omega.$$

Specifically we prove the following

**Theorem 5.1.** *A graph $A \subset [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ with $(0,0) \in A$ is pointwise monotone and $\hat{\jmath}$-surjective in the sense of* (4.2) *if and only if there exists a Carathéodory contraction ((2.2), (2.3)) $\varphi$: $\Omega \times \mathbb{R}^N \to \mathbb{R}^N$ with $\varphi(x,0) = 0$ such that*

$$A = \{(e,d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N : \quad d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega\}. \tag{5.1}$$

□

**Remark 5.2.** In the spirit of Remark 4.1, Theorem 5.1 may be rephrased as follows:

A graph $A \subset [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ containing $(0,0)$ is pointwise monotone and maximal if and only if it is given by (5.1) for some Carathéodory contraction $\varphi : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that $\varphi(x,0) = 0$.

Adding coercivity and growth assumptions, Theorem 5.1 has the following immediate corollary:

**Corollary 5.3.** A graph $A \in M(\alpha,0,p,\Omega)$ *is pointwise monotone if and only if there exists $\varphi \in \mathcal{M}(\alpha,0,p,\Omega)$ such that*

$$A = \{(e,d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N : \quad d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega\}.$$

□

Note that in the above Corollary both the function $m$ and the constant $\mu$ are set to 0.

Corollary 5.3 immediately implies that Theorem 2.3 and 4.4 are identical, when $m(x) = \mu = 0$. The proof of Corollary 5.3 is based on the following locality lemma for pointwise monotone graphs.

**Lemma 5.4.** *If $A \subset [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ is a pointwise monotone graph in such that (4.2), (4.4) are satisfied, then $A$ is local, i.e.,*

$$(e, d) \in A \Longrightarrow (e\chi_B, d\chi_B) \in A, \text{ for any measurable set } B \subset \Omega,$$

where $\chi_B$ is the characteristic function of $B$. □

*Proof of Lemma 5.4.* By pointwise monotonicity, together with (4.4), if $(e, d) \in A$, then, for any $(e', d') \in A$ and for a.e. $x \in \Omega$, we have

$$\begin{cases} (d'(x) - d(x)).(e'(x) - e(x)) \geq 0, \\ \qquad\qquad d'(x).e'(x) \qquad\qquad \geq 0. \end{cases}$$

Thus, for any measurable $B \subset \Omega$, we get, by integration of the two inequalities above over $B$ and $\Omega \setminus B$ respectively,

$$\int_\Omega (d' - d\chi_B).(e' - e\chi_B)\, dx \geq 0.$$

But (4.2) is satisfied, so that $A$ is maximal according to Remark 4.2, and the previous inequality then implies that $(e\chi_B, d\chi_B) \in A$. □

*Proof of Corollary 5.3.* The proof of Corollary 5.3 is immediate. Indeed, according to Theorem 5.1, if $A \in M(\alpha, 0, p, \Omega)$, there exists a Carathéodory contraction $\varphi$: $\Omega \times \mathbb{R}^N \to \mathbb{R}$ with $\varphi(x, 0) = 0$ such that

$$A = \{(e, d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N : \quad d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega\}.$$

It remains to show that $\varphi$ satisfies (2.4) with $m(x) = 0$. But, since $A$ is pointwise monotone, $A$ is local in view of Lemma 5.4, so that (4.3) holds true with any measurable $B \subset \Omega$ in lieu of $\Omega$ itself. Since $\mu = 0$, (4.3) becomes

$$d(x).e(x) \geq \alpha(|e(x)|^p + |d(x)|^{p'}), \text{ for a.e. } x \in \Omega,$$

and $\varphi \in \mathcal{M}(\alpha, 0, p, \Omega)$. The converse is obvious, provided Theorem 5.1 holds true. □

The proof of Theorem 5.1 reduces to that of the two following lemmata:

**Lemma 5.5.** *If $\varphi$ is a Carathéodory contraction with $\varphi(x, 0) = 0$, then, for every $\delta > 0$ and every $f \in [L^{p'}(\Omega)]^N$ and $g \in [L^p(\Omega)]^N$, there exists a unique solution $(e, d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ to*

$$\begin{cases} d + \delta j(e) = f, \\ (e(x), d(x)) \in \mathcal{A}_\varphi(x), \quad \text{a.e. in } \Omega, \end{cases}$$

*and to*

$$\begin{cases} e + \delta \hat{\jmath}(d) = g, \\ (e(x), d(x)) \in \mathcal{A}_\varphi(x), \quad \text{a.e. in } \Omega, \end{cases}$$

*(see (2.1) for the definition of $\mathcal{A}_\varphi(x)$).* □

**Lemma 5.6.** *If $A \subset [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ is a pointwise monotone graph in such that* (4.2), (4.4) *are satisfied, then there exists a Carathéodory contraction $\varphi$ with $\varphi(x,0) = 0$ such that*

$$A = \{(e,d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N : \quad d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega\}.$$

□

*Proof of Lemma 5.5.* Upon changing $(e,d)$ into $(d,e)$, $p$ into $p'$, and $\varphi$ into $-\varphi$, the first result immediately yields the second one. Let us prove the first result.

We define $\varphi_\eta$ as in (3.4) as

$$\varphi_\eta(x,\lambda) = \frac{1}{1+\eta}\,\varphi(x,\lambda),$$

so that $\varphi_\eta$ is a strict Carathéodory contraction and that, according to (3.5), the associated $a_\eta(x,e)$ is Lipschitz and strongly monotone with associated constant $c_\eta \sim \eta/2$.

In a first step, we fix $f \in [L^\infty(\Omega)]^N$ and show that we can uniquely solve, for any $\delta \geq 0$,

$$a_\eta(x,e) + \delta j(e) = f,$$

with $e$ in $[L^\infty(\Omega)]^N$. In a second step, we impose $\delta > 0$, let $\eta$ tend to 0 and, in a third step, we consider $f$ in $[L^{p'}(\Omega)]^N$ in lieu of $[L^\infty(\Omega)]^N$.

*Step 1.* Let $(e_\eta, d_\eta)$ be the measurable pair defined, for a.e. $x$ in $\Omega$, as

$$\begin{cases} d_\eta(x) - e_\eta(x) = \varphi_\eta(x, f(x)) = \varphi_\eta(x, d_\eta(x) + e_\eta(x)), \\ d_\eta(x) + e_\eta(x) = f(x), \end{cases}$$

or equivalently as

$$\begin{cases} d_\eta(x) + e_\eta(x) = f(x), \\ d_\eta(x) = a_\eta(x, e_\eta(x)). \end{cases}$$

Note that $e_\eta$ and $d_\eta$ are elements of $[L^\infty(\Omega)]^N$. Therefore one can define a mapping $T_\eta : [L^\infty(\Omega)]^N \to [L^\infty(\Omega)]^N$ by

$$T_\eta(\overline{e}) = e,$$

with

$$a_\eta(x, e(x)) + e(x) = f(x) + \overline{e}(x), \quad \text{a.e. in } \Omega.$$

If $\overline{e}$ and $\overline{e}'$ are in $[L^\infty(\Omega)]^N$, then in view of (3.5)

$$|(T_\eta(\overline{e}) - T_\eta(\overline{e}'))(x)| \leq \frac{1}{c_\eta + 1}\,|(\overline{e} - \overline{e}')(x)|, \quad \text{a.e. in } \Omega,$$

so that $T_\eta$ is a strict contraction on $[L^\infty(\Omega)]^N$. The Banach fixed point theorem implies the existence of a unique $e \in [L^\infty(\Omega)]^N$ such that $T_\eta(e) = e$, or in other words of a unique $e \in [L^\infty(\Omega)]^N$ with

$$a_\eta(x, e(x)) = f(x), \quad \text{a.e. in } \Omega. \tag{5.2}$$

Let us now prove that, for any $\delta \geq 0$ and any $f \in [L^\infty(\Omega)]^N$, there exists a unique solution to

$$a_\eta(x, e) + \delta\, i(e) = f, \tag{5.3}$$

with $i : \mathbb{R}^N \to \mathbb{R}^N$, $i$ monotone, $i(0) = 0$ and $i$ Lipschitz, with Lipschitz constant $L_i$. To this effect, let us assume that, for some $\delta \geq 0$ and for any $f \in [L^\infty(\Omega)]^N$, we were able to prove the existence of a unique solution $e \in [L^\infty(\Omega)]^N$ to

$$a_\eta(x, e(x)) + \delta i(e(x)) = f(x). \tag{5.4}$$

A fixed point argument identical to that which proves the existence of a unique solution to (5.2) would then yield a unique solution $e \in [L^\infty(\Omega)]^N$ to

$$a_\eta(x, e(x)) + (\delta + \varepsilon) i(e(x)) = f(x), \quad \text{a.e. in } \Omega, \tag{5.5}$$

provided that $0 < \varepsilon < c_\eta / L_i$ (just look at the mapping $\overline{e} \to e$ with $a_\eta(x, e) + \delta\, i(e) = f - \varepsilon\, i(\overline{e})$). Since $\varepsilon$ is independent of $\delta$ and since, by virtue of (5.2) we know how to solve (5.4) with $\delta = 0$, we conclude to the existence of a unique solution to (5.3) for any $\delta \geq 0$ and any $f \in [L^\infty(\Omega)]^N$.

If $p \geq 2$, choose

$$j_R(e) = \begin{cases} j(e) = |e|^{p-2} e, & |e| \leq R, \\ R^{p-2} e, & |e| > R, \end{cases}$$

which is Lipschitz with $j_R(0) = 0$. Note that $j_R$ is monotone, as derivative of the $C^1$ convex function $\psi_R$ defined as

$$\psi_R(e) = \begin{cases} \dfrac{1}{p} |e|^p, & |e| \leq R, \\ \dfrac{1}{2} R^{p-2} |e|^2 + \left( \dfrac{1}{p} - \dfrac{1}{2} \right) R^p, & |e| > R. \end{cases}$$

According to (5.3), there exists a unique solution to

$$a_\eta(x, e) + \delta j_R(e) = f, \tag{5.6}$$

for any $f \in [L^\infty(\Omega)]^N$ and any $\delta \geq 0$.

Multiplication of (5.6) by $e$ yields

$$|e(x)| \leq \frac{\|f\|_{L^\infty(\Omega)}}{c_\eta}, \quad \text{a.e. in } \Omega,$$

so that if we choose $R > \|f\|_{L^\infty(\Omega)} / c_\eta$, then $j_R(e) = j(e)$ and we have solved

$$a_\eta(x, e) + \delta j(e) = f. \tag{5.7}$$

The solution to (5.7) is unique since $a_\eta$ is strictly monotone.

If now $p < 2$, choose

$$j_R(e) = \begin{cases} R^{p-2}e, & |e| < R, \\ j(e) = |e|^{p-2}e, & |e| > R, \end{cases}$$

which is once again a Lipschitz monotone function with $j_R(0) = 0$. According to (5.3), there exists a unique solution to

$$a_\eta(x, e) + \delta j_R(e) = f, \tag{5.8}$$

for any $f \in [L^\infty(\Omega)]^N$ and any $\delta \geq 0$.

Assume first that $f$ is piecewise constant, i.e.

$$f(x) = \Sigma\, \chi_k(x) f_k,$$

with $\chi_k$ the characteristic function of the set of points $x \in \Omega$ where $f(x) = f_k \in \mathbb{R}^N$. Then the solution $e(x)$ to (5.8) is of the form

$$e(x) = \Sigma\, \chi_k(x) e_k(x),$$

with

$$a_\eta(x, e_k(x)) + \delta j_R(e_k(x)) = f_k. \tag{5.9}$$

If $f_k = 0$, then clearly $e_k(x) = 0$; if $f_k \neq 0$, then, in view of (5.8), and because of the specific choice for $j_R$,

$$|f_k| \leq \frac{1}{c_\eta}|e_k(x)| + \delta|e_k(x)|^{p-1}, \quad \text{a.e. in } \Omega.$$

The mapping $t \in \mathbb{R}_+ \to \frac{1}{c_\eta}t + \delta t^{p-1}$ is monotone increasing; thus if we choose $R$ such that $|f_k| \geq \frac{1}{c_\eta}R + \delta R^{p-1}$ for every $k$ with $|f_k| \neq 0$, we have

$$|e_k(x)| \geq R,$$

so that we have solved

$$a_\eta(x, e(x)) + \delta j(e(x)) = f(x), \quad \text{a.e. in } \Omega, \tag{5.10}$$

for any piecewise constant function $f$ and any $\delta \geq 0$.

Consider now an arbitrary element $f \in [L^\infty(\Omega)]^N$. Let $f_n$ be a sequence of piecewise constant functions on $\Omega$ which converges a.e. to $f$ in $\Omega$, and define $e_n$ as the solution to (5.10) associated to $f_n$.

Fix $x$ to be a point in $\Omega$ such that $a_\eta(x, \cdot)$ is continuous, $f_n(x) \stackrel{n}{\to} f(x)$, and such that, for all $n$,

$$a_\eta(x, e_n(x)) + \delta j(e_n(x)) = f_n(x).$$

Then $e_n(x)$ and $a_\eta(x, e_n(x))$ are bounded sequences in $\mathbb{R}^N$. Thus there exists a subsequence $\{k(n)\}$ of $\{n\}$ with $k(n) \nearrow \infty$ (this subsequence depends on $x$) such that for some $e_x \in \mathbb{R}^N$

$$e_{k(n)}(x) \to e_x.$$

We now show that $e_x$ does not depend upon the subsequence $\{k(n)\}$ of $\{n\}$. Indeed, if $\{k'(n)\}$ is an other subsequence of $\{n\}$ with $k'(n) \nearrow \infty$ such that

$$e_{k'(n)}(x) \to e'_x,$$

then, since

$$a_\eta(x, e_{k'(n)}(x)) - a_\eta(x, e_{k(n)}(x)) + \delta\left(j(e_{k'(n)}(x)) - j(e_{k(n)}(x))\right) = f_{k'(n)}(x) - f_{k(n)}(x),$$

we obtain, having multiplied the above equality by $(e_{k'(n)}(x) - e_{k(n)}(x))$ and passed to the limit in $n$,

$$(a_\eta(x, e'_x) - a_\eta(x, e_x)).(e'_x - e_x) + \delta(j(e'_x) - j(e_x)).(e'_x - e_x) = 0;$$

since $a_\eta(x, e)$ is strictly monotone in $e$, we conclude that $e_x = e'_x$, hence the result.

Consequently the whole sequence $e_n(x)$ converges to $e_x$; since $x$ is an arbitrary point in $\Omega$ (up to a set of zero measure) we conclude that $e(x) := e_x$ is measurable (and actually belongs to $[L^\infty(\Omega)]^N$) and that we have solved

$$a_\eta(x, e(x)) + \delta j(e(x)) = f(x), \quad \text{a.e. in } \Omega, \tag{5.11}$$

for any $f \in [L^\infty(\Omega)]^N$ and any $\delta \geq 0$.

*Step 2.* We have thus established in (5.7) and (5.11) the existence and uniqueness of the solution $(e_\eta, d_\eta) \in [L^\infty(\Omega)]^N \times [L^\infty(\Omega)]^N$ to

$$\begin{cases} d_\eta + \delta j(e_\eta) = f, \\ d_\eta(x) - e_\eta(x) = \dfrac{1}{1+\eta}\,\varphi(x, d_\eta(x) + e_\eta(x)), \quad \text{a.e. in } \Omega, \end{cases}$$

for any $f \in [L^\infty(\Omega)]^N$ and any $\delta \geq 0$. Provided that $\delta > 0$, multiplication of the first equation by $e_\eta$ implies that $e_\eta$ is bounded in $[L^\infty(\Omega)]^N$, independently of $\eta$, because $j(e_\eta).e_\eta = |e_\eta|^p$. Then an argument identical to that which led to (5.11) shows the existence (and uniqueness) of $(e, d) \in [L^\infty(\Omega)]^N \times [L^\infty(\Omega)]^N$ with

$$\begin{cases} d + \delta j(e) = f, \\ d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega. \end{cases} \tag{5.12}$$

*Step 3.* Consider now $f \in [L^{p'}(\Omega)]^N$ and approximate it by $f_n \in [L^\infty(\Omega)]^N$. Once again the very same argument used to derive (5.11) implies the existence (and uniqueness) of $(e, d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N$ with

$$\begin{cases} d + \delta j(e) = f, \\ d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega, \end{cases}$$

provided that $\delta > 0$.

The proof of Lemma 5.5 is complete. □

**Remark 5.7.** We could have limited the result of Step 1 to piecewise constant functions without prejudice for the subsequent steps.

*Proof of Lemma 5.6.* Since by assumption (4.2) is satisfied, we consider the mapping $A^\varepsilon : [L^p(\Omega)]^N \to [L^{p'}(\Omega)]^N$ defined as in the proof of Theorem 4.4 by

$$A^\varepsilon(e) = \overline{d}^\varepsilon,$$

with

$$\begin{cases} \overline{e}^\varepsilon + \varepsilon \hat{\jmath}(\overline{d}^\varepsilon) = e, \\ (\overline{e}^\varepsilon, \overline{d}^\varepsilon) \in A. \end{cases}$$

As already established at the onset of the proof of Theorem 4.4, $A^\varepsilon$ is a single-valued monotone continuous bounded operator from $[L^p(\Omega)]^N$ into $[L^{p'}(\Omega)]^N$.

In a first step we show that, for any $f \in [L^\infty(\Omega)]^N$, there exists a unique solution $e^\varepsilon \in [L^\infty(\Omega)]^N$ to

$$A^\varepsilon(e^\varepsilon) + e^\varepsilon = f.$$

In a second step we pass to the limit in $\varepsilon$ and conclude to the existence of a pair $(e, d) \in A$ such that for any $f \in [L^\infty(\Omega)]^N$

$$d + e = f. \tag{5.13}$$

The final step is devoted to the construction of $\varphi$, starting from (5.13).

*Step 1.* We define

$$\ell_R(e) = \begin{cases} e & \text{if } |e| < R, \\ \dfrac{|e|^{p-2} e}{R^{p-2}} & \text{if } |e| \geq R, \end{cases}$$

and note that it defines a strictly monotone, continuous and bounded operator from $[L^p(\Omega)]^N$ into $[L^{p'}(\Omega)]^N$, so that, $A^\varepsilon + \ell_R$ has the same properties. Application of Theorem 2.4 permits to conclude to the existence of a unique solution $e^\varepsilon \in [L^p(\Omega)]^N$ to

$$A^\varepsilon(e^\varepsilon)(x) + \ell_R(e^\varepsilon(x)) = f, \tag{5.14}$$

for any $f \in [L^{p'}(\Omega)]^N$. If $f \in [L^\infty(\Omega)^N]$, and if we choose $R \geq \|f\|_{L^\infty(\Omega)}$, (5.14) reduces to

$$A^\varepsilon(e^\varepsilon) + e^\varepsilon = f\,; \tag{5.15}$$

indeed multiplication of (5.14) by $e^\varepsilon$ immediately yields that, if $|e^\varepsilon(x)| > R$,

$$|e^\varepsilon(x)|^{p-1} \leq R^{p-2} |f(x)| \leq R^{p-1},$$

so that $|e^\varepsilon(x)| \leq R$, which is a contradiction. Moreover we have

$$\|e^\varepsilon\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}. \tag{5.16}$$

*Step 2.* We now pass to the limit in (5.15) as $\varepsilon \searrow 0^+$. By virtue of (5.16), and since $A^\varepsilon(e^\varepsilon) = f - e^\varepsilon$, there exists a subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ such that

$$\begin{cases} e^{\varepsilon'} \rightharpoonup e \quad \text{weak-}\star \text{ in } [L^\infty(\Omega)]^N, \\ A^{\varepsilon'}(e^{\varepsilon'}) \rightharpoonup d \quad \text{weak-}\star \text{ in } [L^\infty(\Omega)]^N, \end{cases} \tag{5.17}$$

with

$$d + e = f, \quad \text{a.e. in } \Omega. \tag{5.18}$$

Consider an arbitrary element $(\overline{e}, \overline{d}) \in A$ and set

$$\overline{e}^\varepsilon = \overline{e} + \varepsilon \hat{\jmath}(\overline{d}),$$

so that

$$\overline{e}^\varepsilon \to \overline{e} \quad \text{strongly in } [L^p(\Omega)]^N, \tag{5.19}$$

and

$$A^\varepsilon(\overline{e}^\varepsilon) = \overline{d}. \tag{5.20}$$

Then, since $A^\varepsilon$ is monotone,

$$\int_\Omega \Big(A^{\varepsilon'}(e^{\varepsilon'}) - A^{\varepsilon'}(\overline{e}^{\varepsilon'})\Big).(e^{\varepsilon'} - \overline{e}^{\varepsilon'})\, dx \geq 0. \tag{5.21}$$

But, in view of (5.15), the weak lower semicontinuity of the $L^2$-norm, and (5.18),

$$\overline{\lim} \int_\Omega A^{\varepsilon'}(e^{\varepsilon'}).e^{\varepsilon'}\, dx = \overline{\lim} \int_\Omega (f - e^{\varepsilon'}).e^{\varepsilon'}\, dx \leq \int_\Omega f.e \;\, dx - \int_\Omega |e|^2\, dx = \int_\Omega d.e \;\, dx,$$

while all the other terms in (5.21) pass to the limit by virtue of (5.19) and (5.20). We obtain

$$\int_\Omega (d - \overline{d}).(e - \overline{e}) dx \geq 0. \tag{5.22}$$

But, according to Remark 4.2, $A$ is maximal, so that (5.22) implies that $(e, d) \in A$.

Recalling (5.18), we have thus proved the existence of $(e, d) \in A \cap ([L^\infty(\Omega)]^N \times [L^\infty(\Omega)]^N)$ such that

$$e + d = f$$

for any $f$ in $[L^\infty(\Omega)]^N$.

*Step 3.* Define the mapping $T : [L^\infty(\Omega)]^N \to [L^\infty(\Omega)]^N$ by

$$T(f) = d - e. \tag{5.23}$$

The pointwise monotone character of $A$ immediately implies (see the beginning of the proof of Lemma 2.1) that, for any $f, g \in [L^\infty(\Omega)]^N$,

$$|T(f) - T(g)|(x) \leq |f - g|(x), \quad \text{a.e. in } \Omega. \tag{5.24}$$

Define, for $\lambda \in \mathbb{Q}^N$,

$$\varphi(x,\lambda) = T(\lambda)(x), \quad \text{a.e. in } \Omega.$$

Except maybe on a set of zero measure,

$$|\varphi(x,\lambda) - \varphi(x,\lambda')| \leq |\lambda - \lambda'|, \quad \lambda, \lambda' \in \mathbb{Q}^N,$$

so that, for a.e. $x$ in $\Omega$ and every $\lambda \in \mathbb{R}^N$, $\varphi(x,\lambda)$ is well defined as the (unique) limit of $\varphi(x,\lambda_n)$ with $\lambda_n \in \mathbb{Q}^N$ and $\lambda_n \to \lambda$. Further, $\varphi(x,\lambda)$ is clearly Carathéodory on $\Omega \times \mathbb{R}^N$.

For a fixed $f$ in $[L^\infty(\Omega)]^N$, consider a sequence $f_n$ of piecewise constant functions such that

$$f_n \to f, \quad \text{a.e. in } \Omega.$$

By virtue of (5.24)

$$T(f_n) \to T(f), \quad \text{a.e. in } \Omega. \tag{5.25}$$

But, if $f_n$ is constant on a measurable subset $\omega$ of $\Omega$, it is immediately seen, by the definition of $\varphi$, that

$$T(f_n)(x) = \varphi(x, f_n(x)), \quad \text{a.e. in } \omega.$$

Since $\varphi$ is a Carathéodory contraction

$$\varphi(x, f_n(x)) \to \varphi(x, f(x)), \quad \text{a.e. in } \Omega,$$

which, together with (5.25), implies that

$$T(f)(x) = \varphi(x, f(x)), \quad \text{a.e. in } \Omega.$$

We have thus shown so far that, for any $(e,d) \in A \cap ([L^\infty(\Omega)]^N \times [L^\infty(\Omega)]^N)$,

$$d(x) - e(x) = \varphi(x, d(x) + e(x)), \quad \text{a.e. in } \Omega. \tag{5.26}$$

Equality (5.26) remains true if $(e,d) \in A$. Indeed, according to Lemma 5.4, $A$ is local, so that, for any integer $n$,

$$(e_n, d_n) :=$$

$$(e\chi_{\{x\in\Omega:|d_n(x)|+|e_n(x)|\leq n\}}, d\chi_{\{x\in\Omega:|d_n(x)|+|e_n(x)|\leq n\}}) \in A \cap ([L^\infty(\Omega)]^N \times [L^\infty(\Omega)]^N).$$

Further,

$$(e_n(x), d_n(x)) \longrightarrow (e(x), d(x)), \text{ a.e. in } \Omega, \tag{5.27}$$

while, according to (5.26),

$$d_n(x) - e_n(x) = \varphi(x, d_n(x) + e_n(x)). \tag{5.28}$$

Passing to the limit in (5.28) is obvious in view of (5.27) because of the Carathéodory character of $\varphi$.

The proof of Lemma 5.6 is complete. □

We conclude this section by showing that a well-known class of monotone operators, namely, the subdifferentials of convex Carathéodory functions on $\mathbb{R}^N$ with appropriate coercivity and growth assumptions are associated to elements of $\mathcal{M}(\alpha, m, p, \Omega)$ for an adequate choice of $\alpha, m, p$. This is the object of the following

**Remark 5.8.** Consider $\Psi : \Omega \times \mathbb{R}^N \to \mathbb{R}$, Carathéodory, convex in its second argument, and such that, for some $m \geq 0 \in L^1(\Omega)$ and some $\alpha, \beta > 0$,

$$\alpha|e|^p \leq \Psi(x, e) \leq m(x) + \beta|e|^p.$$

Then the subdifferential $\partial_e\psi(x, e)$ belongs to the class $\mathcal{M}(\gamma, m, p, \Omega)$ for

$$\gamma = \inf(\alpha, \frac{1}{p'}(\frac{1}{\beta p})^{\frac{p'}{p}})$$

.

Indeed, if $\Psi^*(x, \cdot)$ denotes the Legendre transform of $\Psi(x, \cdot)$, then

$$\Psi^*(x, d) = \sup\{d.e' - \Psi(x, e')\} \geq \sup\{d.e' - m(x) - \beta|e'|^p\} = -m(x) + C|d|^{p'},$$

for $C = \frac{1}{p'}(\frac{1}{\beta p})^{\frac{p'}{p}}$. Since, for a.e. $x \in \Omega$,

$$d \in \partial_e\Psi(x, e) \Longleftrightarrow d.e = \Psi(x, e) + \Psi^*(x, d),$$

we obtain that for $d \in \partial_e\psi(x, e)$,

$$d.e \geq -m(x) + \alpha|e|^p + C|d|^{p'}. \tag{5.29}$$

Further, consider

$$J : e \in [L^p(\Omega)]^N \to \int_\Omega \Psi(x, e(x))\, dx.$$

The functional $J$ is convex and continuous. For any $f \in [L^{p'}(\Omega)]^N$ and any $\delta > 0$, the functional

$$I : e \in [L^p(\Omega)]^N \to J(e) + \frac{\delta}{p}\int_\Omega |e|^p\, dx - \int_\Omega f.u\, dx,$$

admits a unique minimum $e \in [L^p(\Omega)]^N$ that satisfies $0 \in \partial I(e)$; equivalently, there exists $d \in \partial J$ such that, for a.e. $x \in \Omega$

$$d(x) + \delta|e(x)|^{p-2}e(x) = f(x). \tag{5.30}$$

But $d \in \partial J$ if and only if $d(x) \in \partial_e\Psi(x, e(x))$, a.e. in $\Omega$, so that, by virtue of (5.30), there exists $d \in [L^{p'}(\Omega)]^N$ such that, for a.e. $x \in \Omega$,

$$\begin{cases} d(x) + \delta|e(x)|^{p-2}e(x) = f(x), \\ d(x) \in \partial_e\Psi(x, e(x)). \end{cases}$$

In other words, the graph

$$A_\Psi := \{(e,d) \in [L^p(\Omega)]^N \times [L^{p'}(\Omega)]^N : d(x) \in \partial_e \Psi(x, e(x)),\ \text{a.e. in}\ \ \Omega\}$$

meets the hypotheses of Theorem 5.1. In conclusion, there exists a Carathéodory contraction $\varphi_\Psi(x,e)$ such that, for a.e. any $(e,d) \in A_\Psi$,

$$e(x) - d(x) = \varphi_\Psi(x, e(x) + d(x)), \quad \text{a.e.}\ \ x \in \Omega,$$

which, together with (5.29), proves the result.

A classical computation moreover yields the following expression for $\varphi_\Psi$ in terms of $\partial_e\Psi(x,e)$:

$$\varphi_\Psi(x,\lambda) = \lambda - 2(I + \partial_e\Psi(x,\cdot))^{-1}(\lambda).$$

## 6. Miscellaneous results and extensions

The possible non-uniqueness of the solution $u$ to (2.6) in Theorem 2.3 can be cured by the addition of a zeroth order term in (2.6). Specifically, consider $j(\lambda) = |\lambda|^{p-2}\lambda$; then the following theorem holds true:

**Theorem 6.1** *Consider $\varphi \in \mathcal{M}(\alpha, m, p, \Omega)$. For any $f \in W^{-1,p'}(\Omega)$, there exists a unique $u \in W_0^{1,p}(\Omega)$ and a (possibly non-unique) $d \in [L^{p'}(\Omega)]^N$ such that*

$$\begin{cases} -\,div\ d + j(u) = f \quad \text{in}\ \mathcal{D}'(\Omega), \\ d(x) - grad\, u(x) = \varphi(x, d(x) + grad\, u(x)), \quad \text{a.e. in}\ \Omega. \end{cases} \tag{6.1}$$

*Proof.* We recall that the proof of Theorem 4.4 also provides a proof of Theorem 2.3 since any function of $\mathcal{M}(\alpha, m, p, \Omega)$ is associated to a graph of $M(\alpha, \mu, p, \Omega)$ (see Remark 4.3), and we use this proof. The operator

$$\begin{cases} \mathbf{J} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \\ \mathbf{J}(u) := j(u), \end{cases}$$

is clearly monotone, bounded, continuous, and compact, so that Theorem 2.4 applies to $\mathbf{A}^\varepsilon + \mathbf{J}$ with $\mathbf{A}^\varepsilon$ defined in (4.10) and yields the existence and uniqueness of $u^\varepsilon \in W_0^{1,p}(\Omega)$ with

$$-div\ (A^\varepsilon(grad\, u^\varepsilon)) + j(u^\varepsilon) = f \quad \text{in}\ \mathcal{D}'(\Omega). \tag{6.2}$$

Passing to the limit of (6.2) as $\varepsilon$ tends to 0 is performed exactly as at the end of the proof of Theorem 4.4 once it is observed that

$$f - j(u^\varepsilon) \to f - j(u) \quad \text{strongly in}\ W^{-1,p'}(\Omega).$$

The uniqueness of $u$ is immediate in view of the strict monotonicity of $j$. □

**Remark 6.2.** Note that the proof of Theorem 2.3 presented in Section 3 would be more technical to generalize to the above setting because the regularization used there introduces an $L^2$-setting which is ill suited to accomodate additional terms of the form $j(u)$ if $p \neq 2$.

**Remark 6.3.** Any continuous, bounded, strictly monotone mapping $h : [L^p(\Omega)]^N \to [L^{p'}(\Omega)]^N$ would do in lieu of $j$ in Theorem 6.1; in particular, $\delta j$, with any $\delta > 0$, is a valid candidate.

We now propose to extend the results of Theorem 2.3 and 6.1 to $u$ dependent graphs. Specifically we consider $\varphi : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ such that

- $\varphi$ is Carathéodory; (6.3)
-

$$\begin{cases} |\varphi(x,u,\lambda) - \varphi(x,u,\lambda')| \leq |\lambda - \lambda'|, \\ \lambda, \lambda' \in \mathbb{R}^N, \quad \text{for a.e. } x \text{ in } \Omega \text{ and any } u \in \mathbb{R}; \end{cases} \tag{6.4}$$

- if for any $u \in \mathbb{R}$ and any $\lambda \in \mathbb{R}^N$, $e_u(x)$ and $d_u(x)$ are defined, for a.e. $x$ in $\Omega$, as

$$\begin{cases} d_u(x) + e_u(x) = \lambda, \\ d_u(x) - e_u(x) = \varphi(x,u,\lambda) \end{cases}$$

then, for a.e. $x \in \Omega$,

$$d_u(x).e_u(x) \geq -m(x) + \alpha(|e_u(x)|^p + |d_u(x)|^{p'}); \tag{6.5}$$

- $\varphi(x,u,0) = 0$, for a.e. $x \in \Omega$ and any $u \in \mathbb{R}$. (6.6)

The following generalization of Theorems 6.1 and 2.3 holds true:

**Theorem 6.4.** *Assume that* $\varphi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ *satisfies* (6.3)-(6.6). *For any* $f \in W^{-1,p'}(\Omega)$, *there exists* $u$ *and* $d$ *a solution such that*

$$\begin{cases} u \in W_0^{1,p}(\Omega), d \in [L^{p'}(\Omega)]^N \\ -\,div\ d + j(u) = f \quad in\ \mathcal{D}'(\Omega), \\ d(x) - grad\, u(x) = \varphi(x, u(x), d(x) + grad\, u(x)), \quad \text{a.e. } in\ \Omega. \end{cases} \tag{6.7}$$

*Furthermore, the same result holds true if the zeroth order term* $j(u)$ *is dropped from the equation.*

*Proof.* Fix $f \in W^{-1,p'}(\Omega)$ and for any $v \in L^p(\Omega)$ define $\varphi_v : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\varphi_v(x,e) := \varphi(x, v(x), e).$$

Then $\varphi_v \in \mathcal{M}(\alpha, m, p, \Omega)$, so that Theorem 6.1 yields the existence of $u \in W_0^{1,p}(\Omega)$, $d \in [L^{p'}(\Omega)]^N$, with uniqueness for $u$, such that

$$\begin{cases} -\,div\ d + j(u) = f \quad \text{in } \mathcal{D}'(\Omega), \\ d(x) - grad\, u(x) = \varphi_v(x, d(x) + grad\, u(x)), \quad \text{a.e. in } \Omega. \end{cases} \tag{6.8}$$

We define the mapping $T : L^p(\Omega) \to L^p(\Omega)$ as

$$T(v) = u,$$

where $u$ is the unique solution to (6.8).

This mapping is continuous on $L^p(\Omega)$. Indeed, if $v_n$ tends strongly to $v$ in $L^p(\Omega)$, then a subsequence $v_{n_k}$ of $v_n$ is such that

$$v_{n_k}(x) \to v(x), \quad \text{a.e. in } \Omega.$$

Since $\varphi$ is Carathéodory,

$$\varphi_{v_{n_k}}(x, e) \to \varphi_v(x, e), \quad \text{a.e. in } \Omega,$$

for every $e \in \mathbb{R}^N$. If $u_{n_k}$ is the solution to (6.8) (with $\varphi_{v_{n_k}}$ replacing $\varphi_v$), the coercivity and growth condition (6.5) and Poincaré's inequality immediately imply the existence of a subsequence of $\{n_k\}$ (still denoted by $\{n_k\}$) such that

$$\begin{aligned} &u_{n_k} \rightharpoonup u, \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega), \\ &d_{n_k} \rightharpoonup d, \quad \text{weakly in } [L^{p'}(\Omega)]^N. \end{aligned} \tag{6.9}$$

Further, straightforward continuity yields

$$-div\ d + j(u) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{6.10}$$

Also we have

$$d_{n_k}(x) - grad\, u_{n_k}(x) = \varphi_{v_{n_k}}(x, d_{n_k}(x) + grad\, u_{n_k}(x)), \quad \text{a.e. in } \Omega,$$

and therefore, for any $\lambda \in [L^\infty(\Omega)]^N$,

$$|d_{n_k}(x) - grad\, u_{n_k}(x) - \varphi_{v_{n_k}}(x, \lambda(x))|^2 \le |d_{n_k}(x) + grad\, u_{n_k}(x) - \lambda(x)|^2, \quad \text{a.e. in } \Omega.$$

From here onward the argument is exactly that used in deriving (3.21) from (3.18) and it will not be repeated here. We obtain

$$d(x) - grad\, u(x) = \varphi_v(x, grad\, u(x)), \quad \text{a.e. in } \Omega,$$

which, together with (6.9), (6.10) implies that $T(v_{n_k}) \to T(v)$, strongly in $L^p(\Omega)$. But $T(v)$ does not depend upon the actual choice of subsequence of $v_n$, so that the whole sequence $T(v_n)$ converges to $T(v)$ in $L^p(\Omega)$.

Further, the mapping $T$ is compact. Indeed, as already used in (6.9), if $u$ satisfies (6.8), then because of the coercivity assumption (6.5),

$$\|u\|_{W_0^{1,p}(\Omega)} \le C < +\infty, \tag{6.11}$$

with $C$ only depending on $\|f\|_{W^{-1,p'}(\Omega)}$, $\|m\|_{L^1(\Omega)}$ and $\alpha$. Rellich's theorem immediately implies the result.

Finally note that $T$ sends all of $L^p(\Omega)$ into the closed convex compact subset of $L^p(\Omega)$ defined by (6.11). Appealing to Schauder's fixed point theorem, we conclude that $T$ admits a fixed point $u$ which thus satisfies (6.7).

Let us now consider the case without zeroth order term. Let $(u_n, d_n)$ be a solution to to

$$\begin{cases} u_n \in W_0^{1,p}(\Omega), \quad d_n \in [L^N(\Omega)]^N \\ -\,div\ d_n + \dfrac{1}{n} j(u_n) = f \quad \text{in } \mathcal{D}'(\Omega), \\ d_n(x) - grad\, u_n(x) = \varphi(x, u_n(x), grad\, u_n(x)), \quad \text{a.e. in } \Omega. \end{cases}$$

Such a solution exists according to (6.7) with $j$ replaced by $\frac{1}{n}j$ (see Remark 6.3). The same elementary estimates that were used before imply that, for a subsequence $\{n_k\}$ of $\{n\}$,

$$\begin{cases} u_{n_k} \rightharpoonup u, \quad \text{weakly in } W_0^{1,p}(\Omega), \text{ strongly in } L^p(\Omega), \text{ and a.e. in } \Omega, \\ d_{n_k} \rightharpoonup d, \quad \text{weakly in } [L^{p'}(\Omega)]^N, \end{cases}$$

with

$$-div\ d = f \quad \text{in } \mathcal{D}'(\Omega).$$

The result is obtained as for proving the continuity of $T$ upon noting that, for every $e$ in $\mathbb{R}^N$,

$$\varphi(x, u_{n_k}(x), e) \to \varphi(x, u(x), e), \quad \text{a.e. in } \Omega.$$

This completes the proof of Theorem 6.4. □

**Remark 6.5.** As a final note, we observe that all our results extend to the case of equations and systems of higher order, and to different sets of variational boundary conditions.

**Acknowledgements.** The authors express their gratitude to Gianni Dal Maso for many valuable remarks and in particular for contributing Remark 2.2.

## Bibliography

[Al&Am] Alberti, Giovanni; Ambrosio, Luigi, A geometrical approach to monotone functions in $\mathbb{R}^n$. Math. Z. **230** (2) 1999, 259–316.

[Ba] Barbu, Viorel, *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, Leiden, 1976. 352 pp.

[Bre1] Brezis, H., *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co. Inc., New York, 1973. vi+183 pp.

[Bre2] Brezis, Haïm, *Analyse fonctionnelle. Théorie et applications.* Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983. xiv+234 pp.

[Bro] Browder, Felix E., Nonlinear operators and nonlinear equations of evolution in Banach spaces. *Nonlinear functional analysis.* Proc. Sympos. Pure Math., Vol. XVIII, Part 2, Chicago, Ill., 1968, pp. 1–308. Amer. Math. Soc., Providence, R. I., 1976.

[CP&DM&De] Chiadò Piat, Valeria; Dal Maso, Gianni; Defranceschi, Anneliese, $G$-convergence of monotone operators, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (3) 1990, 123–160.

[Fe] Federer, Herbert, *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York 1969. xiv+676 pp.

[Le&Li] Leray, Jean; Lions, Jacques-Louis, Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France **93** 1965, 97–107.

[Li] Lions, J.-L., *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris 1969. xx+554 pp.

[Mi] Minty, George J., Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29** 1962, 341–346.