Stochastic Processes 2

Lesson 2

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In Lesson 1 we defined

- stochastic process
- finite-dimensional distribution
- mean value and autocovariance function
- strict and weak stationarity
- Gaussian process
- properties of autovariance function

Some important classes of stochastic processes

Markov processes

Definition:

We say that $\{X_t,\ t \in T\}$ is a Markov process with state space $(S, \mathcal{E})$, if for any $t_0, t_1, \dots, t_n,\ 0 \le t_0 < t_1 < \cdots < t_n$, it holds

$$P(X_{t_n} \le x | X_{t_{n-1}}, \dots, X_{t_0}) = P(X_{t_n} \le x | X_{t_{n-1}}) \ \text{ a.s.} \qquad (1)$$

for all $x \in \mathbb{R}$.

Relation (1) is called the Markov property. Simple cases: discrete state Markov processes, i.e., discrete and continuous time Markov chains.

Example:

Consider a Markov chain $\{X_t,\ t \geq 0\}$ with state space $S = \{0, 1\}$, initial distribution $P(X_0 = 0) = 1, P(X_0 = 1) = 0$ and the intensity matrix

$$\mathbf{Q} = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}, \quad \alpha > 0,\ \beta > 0$$

Let us treat the stationarity of this process.

We know:

$$\mathbf{p}(t)^T = \mathbf{p}(0)^T \mathbf{P}(t) = (1, 0)^T \mathbf{P}(t) = (p_{00}(t), p_{01}(t))^T$$

$$\mathbf{P}(t) = e^{\mathbf{Q}t} = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta + \alpha e^{-(\alpha+\beta)t} & \alpha - \alpha e^{-(\alpha+\beta)t} \\ \beta - \beta e^{-(\alpha+\beta)t} & \alpha + \beta e^{-(\alpha+\beta)t} \end{pmatrix}.$$

Example, continued.

With this initial distribution we have

$$\mathrm{E}X_t = 1 \cdot \mathsf{P}(X_t = 1) = p_{01}(t) = \frac{1}{\alpha+\beta} \cdot \left(\alpha - \alpha e^{-(\alpha+\beta)t}\right),$$

which depends on t, the process is neither strictly nor weakly stationary.

Now, let us suppose that the initial distribution is the stationary distribution, i.e., the probability distribution that satisfies $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \mathbf{P}(t)$
in our case:
$\pi_0 = \frac{\beta}{\alpha+\beta}, \pi_1 = \frac{\alpha}{\alpha+\beta}$
Then
$\mathbf{p}(t) = \boldsymbol{\pi} \Rightarrow \mathrm{E}X_t = 1 \cdot \mathsf{P}(X_t = 1) = \frac{\alpha}{\alpha+\beta}$

Example, continued.

for $t < s$,

$$\begin{aligned}
\mathrm{E}(X_t X_s) &= 1 \cdot P(X_t = 1, X_s = 1) \\
&= P(X_s = 1 | X_t = 1) P(X_t = 1) \\
&= p_{11}(s-t) P(X_t = 1) = p_{11}(s-t)\pi_1 \\
&= \frac{1}{\alpha+\beta}\left(\alpha + \beta e^{-(\alpha+\beta)(s-t)}\right) \cdot \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

$$\begin{aligned}
&\mathrm{E}(X_t X_s) - \mathrm{E}X_t \mathrm{E}X_s = \\
&= \frac{1}{\alpha+\beta}\left(\alpha + \beta e^{-(\alpha+\beta)(s-t)}\right) \cdot \frac{\alpha}{\alpha+\beta} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-(\alpha+\beta)(s-t)} = \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-(\alpha+\beta)|s-t|}
\end{aligned}$$

For $t > s$ we proceed in the same way.

Example, continued.

Summary:

If the initial distribution is the stationary distribution then $\mathrm{E}X_t = \frac{\alpha}{\alpha+\beta}$ and the autocovariance function

$$R(s,t) = \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-(\alpha+\beta)|s-t|}.$$

It means the process is weakly stationary.

It follows from the theory of Markov chains, that if the initial distribution is the stationary distribution (the probability invariant measure) then the process is also strictly stationary.

distinguish!!

- strict stationarity of a stochastic process
- weak stationarity of a stochastic process
- stationary distribution of a Markov process

Independent increments processes

Definition:

A process $\{X_t,\ t \in T\}$, where T is an interval, has independent increments, if for any $t_1, t_2, \dots, t_n \in T$ such that $t_1 < t_2 < \dots < t_n$, the random variables $X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

If for any $s, t \in T$, $s < t$, the distribution of the increments $X_t - X_s$ depends only on $t - s$, we say that $\{X_t,\ t \in T\}$ has stationary increments.

Example:

A Poisson process with intensity λ is a continuous time Markov chain $\{X_t,\ t \geq 0\}$ such that $X_0 = 0$ a.s. and for $t > 0$, X_t has the Poisson distribution with parameter λt. Increments of the Poisson process are independent and $X_t - X_s,\ s < t$ have the Poisson distribution with the parameter $\lambda(t - s)$. The Poisson process is neither strictly nor weakly stationary.

Example:

Wiener process (Brownian motion process) is a Gaussian stochastic process $\{W_t, t \geq 0\}$ with the properties

1. $W_0 = 0$ a. s. and $\{W_t, t \geq 0\}$ has continuous trajectories
2. For any $0 \leq t_1 < t_2 < \cdots < t_n$, $W_{t_1}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent random variables (independent increments).
3. For any $0 \leq t < s$, the increments $W_s - W_t$ have normal distribution with zero mean and the variance $\sigma^2(s-t)$, where σ^2 is a positive constant. Especially, for any $t \geq 0$, $\mathrm{E} W_t = 0$ and $\mathrm{var}\, W_t = \sigma^2 t$.

The Wiener process is Gaussian but is neither strictly nor weakly stationary!

Computation of the autocovariance function of the Wiener process:
Since $\mathrm{E}W_t = 0$ for every $t \geq 0$ we have $\operatorname{cov}(W_s, W_t) = \mathrm{E}W_s W_t$.
Let $s < t$, then

$$\begin{aligned} \mathrm{E}W_s W_t &= \mathrm{E}(W_s - W_0)(W_s - W_0 + W_t - W_s) \\ &= \mathrm{E}(W_s - W_0)^2 + \mathrm{E}(W_s - W_0)(W_t - W_s) \\ &= \sigma^2 s = \sigma^2 \min(s, t) \end{aligned}$$

since for $s < t$ the increments $W_s - W_0$ and $W_t - W_s$ are independent and zero mean.
For $t < s$, $\mathrm{E}W_s W_t = \sigma^2 t = \sigma^2 \min(s, t)$.
We can see that the autocovariance function of the Wiener process is not a function of the difference of arguments, thus this process cannot be weakly stationary, though it is Gaussian.

Example:

Let $\{W_t, t \geq 0\}$ be the Wiener process defined above. The Ornstein-Uhlenbeck process $\{U_t, t \geq 0\}$ is defined by

$$U_t = e^{-\alpha t/2} W(e^{\alpha t}), \quad t \geq 0, \alpha > 0.$$

Then

- $\mathrm{E} U_t = 0$ for every $t \geq 0$,
- $\mathrm{var}\, U_t = \mathrm{E} U_t^2 = e^{-\alpha t} \mathrm{E} W^2(e^{\alpha t}) = \sigma^2$
- for $s < t$,

$$\begin{aligned} \mathrm{E} U_s U_t &= e^{-\alpha s/2} e^{-\alpha t/2} \sigma^2 \min(e^{\alpha s}, e^{\alpha t}) = \sigma^2 e^{-\alpha(t-s)/2} \\ &= \sigma^2 e^{-\alpha|t-s|/2} \end{aligned}$$

- for $s > t$,
  $\mathrm{E} U_s U_t = \sigma^2 e^{-\alpha(s-t)/2} = \sigma^2 e^{-\alpha|t-s|/2}$.

The process is weakly stationary and Gaussian, thus strictly stationary.

Notice:
It can appear that two different stochastic processes have the same form of the autocovariance function. It means that the autocovariance function does not determine the stochastic process uniquely.

Martingales

Definition:
Let $\{\Omega, \mathcal{A}, P\}$ be a probability space, $T \subset \mathbb{R}, T \neq \emptyset$. Let for any $t \in T$, $\mathcal{F}_t \subset \mathcal{A}$ be a $\sigma-$ algebra ($\sigma-$field). The system of σ-fields $\{\mathcal{F}_t, t \in T\}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for any $s, t \in T, s < t$ is called a filtration.

Definition:
Let $\{X_t, \ t \in T\}$ be a stochastic process defined on $\{\Omega, \mathcal{A}, P\}$, and let $\{\mathcal{F}_t, t \in T\}$ be a filtration. We say that $\{X_t, \ t \in T\}$ is adapted to $\{\mathcal{F}_t, t \in T\}$ if for any $t \in T$, X_t is $\mathcal{F}_t$ measurable.

Definition:
Let $\{X_t, \ t \in T\}$ be adapted to $\{\mathcal{F}_t, t \in T\}$ and $E|X_t| < \infty$ for all $t \in T$. Then $\{X_t, \ t \in T\}$ is said to be a martingale if $\mathrm{E}(X_t|\mathcal{F}_s) = X_s$ a.s. for any $s < t, s, t, \in T$.

Example:

Let $\{Y_n, n \in \mathbb{N}\}$ be a random sequence such that

$$Y_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}$$

where $X_1, X_2, \ldots$ are real-valued i.i.d. random variables with zero mean and finite variance δ^2.
Put $\mathcal{F}_n = \sigma\{Y_1, \ldots, Y_n\}$ (canonical filtration). Then

$$\begin{aligned} \mathrm{E}(Y_{n+1}|\mathcal{F}_n) &= \mathrm{E}(Y_{n+1}|Y_1, \ldots, Y_n) = \mathrm{E}(Y_{n+1}|X_1, \ldots X_n) \\ &= \mathrm{E}((Y_n + X_{n+1})|X_1, \ldots, X_n) = Y_n \ a.s. \end{aligned}$$

The sequence $\{Y_n, n \in \mathbb{N}\}$ is a martingale.
Further, $\mathrm{E}\, Y_n = 0, \mathrm{var}\, Y_n = n\delta^2, \mathrm{cov}\,(Y_m, Y_n) = \delta^2 \min(m, n)$.
The sequence $\{Y_n, n \in \mathbb{N}\}$ is not stationary.